AN EXCELLENT SCHEME FOR THE COUPLE-HIGGS EQUATION

by

Zi-Ge WU and Chun-Fu WEI *

School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, China

Original scientific paper
https://doi.org/10.2298/TSCI2404523W

The couple-Higgs equation is an important non-linear evolution equation in the field of physics. In this paper, the couple-Higgs equation is investigated by employing the tanh function method, and the new solitary wave solution is successfully constructed. Finally, the physical properties of this solitary wave solution are elaborated by plotting the 3-D graphs with proper parameters.

Key words: *solitary wave solution, couple-Higgs equation, tanh function method*

Introduction

Most of complex non-linear phenomena are described by the non-linear evolution equations in the real world, [1-5]. Therefore, it is of great significance to study the analytical solutions of non-linear evolution equations. There are many efficient methods for deriving the solutions of non-linear evolution equations, such as sub-equation method [6], unified solver method [7], modified extended direct algebraic technique [8], sine-cosine method [9], Jacobi elliptic function method [10], extended tanh-coth expansion methods [11], exponential function method [12], variational method [13], and many others methods [14-21].

Consider the couple-Higgs equation:

$$\begin{aligned} &P_{tt} - P_{xx} + |P|^2 P - 2PQ = 0 \\ &Q_{tt} + Q_{xx} - \left(|P|^2\right)_{xx} = 0 \end{aligned} \tag{1}$$

where eq. (1) describes a system of conserved scalar nucleon interaction with a neutral scalar meson and Q represents a real scalar meson field and P represents a complex scalar nucleon field. For eq. (1), Wang [22] utilized the Hirota's bilinear method, the solutions was found by Hafez *et al.* [23] using the exp(– $\Phi(\xi)$)-expansion method, Seadawy *et al.* [24] obtained the solutions via the modified extend tanh-function method, the solutions was obtained by Jabbari, *et al.* [25] utilizing the He's semi-inverse method and the (G'/G)-expansion method, Ali *et al.* [26] utilized the Φ^6-model expansion method to get the solutions.

The tanh function method is a powerful mathematical tool to deal with different types of evolution equations. In this paper, the couple-Higgs equation is studied by using the tanh function method for the first time, and new type of solitary wave solution is successfully derived. Finally, some 3-D figures are given to describe the corresponding solitary wave solution with proper parameters.

* Corresponding author, e-mail: mathwcf@163.com

General properties of the method

In this section, the tanh function method is elaborated in detail.

Consider the non-linear evolution equation:

$$P(u,u_t,u_x,u_{tx},u_{tt},u_{xx},\cdots)=0 \tag{2}$$

Use the traveling wave transformation:

$$u(x,t)=\varphi(\xi),\ \ \xi=x-ct \tag{3}$$

Obtain the non-linear ODE:

$$O(\varphi,\varphi',\varphi'',\cdots)=0 \tag{4}$$

where O is the polynomial of $\varphi=\varphi(\xi)$.

Now, a new independent variable is defined:

$$Y(\xi)=\tanh(\xi) \tag{5}$$

We can find the following for some derivations:

$$\begin{gathered}\frac{\mathrm{d}}{\mathrm{d}\xi}\to\left(1-Y^2\right)\frac{\mathrm{d}}{\mathrm{d}Y}\\ \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\to\left(1-Y^2\right)\left[-2Y\frac{\mathrm{d}}{\mathrm{d}Y}+\left(1-Y^2\right)\frac{\mathrm{d}^2}{\mathrm{d}Y^2}\right]\\ \frac{\mathrm{d}^3}{\mathrm{d}\xi^3}=\cdots\\ \vdots\end{gathered} \tag{6}$$

We present the tanh series:

$$\varphi(\xi)=\sum_{i=0}^{n}a_iY^i=\sum_{i=0}^{n}a_i\tanh(\xi)^i \tag{7}$$

where n is the positive integer, and can be determined by using the balance principle.

Application of the method

In this section, the CHE is solved viaadopting the tanh function method.

Use the travelling wave transformation:

$$P(x,t)=U(\xi)\mathrm{e}^{i(\alpha x+\beta t)},\ \ \xi=x+\lambda t,\ \ Q(x,t)=V(\xi) \tag{8}$$

Plug eq. (8) into eq. (1), and get:

$$P_{tt}=\lambda^2U''e^{i(\alpha x+\beta t)}-2\lambda U'i\beta\mathrm{e}^{i(\alpha x+\beta t)}-U\beta^2\mathrm{e}^{i(\alpha x+\beta t)} \tag{9}$$

$$P_{xx}=U''\mathrm{e}^{i(\alpha x+\beta t)}+2U'i\alpha\mathrm{e}^{i(\alpha x+\beta t)}-U\alpha^2\mathrm{e}^{i(\alpha x+\beta t)} \tag{10}$$

$$\left|P\right|^2P=U^3\mathrm{e}^{i(\alpha x+\beta t)} \tag{11}$$

By utilizing the aforementioned transformation eq.(1), we have:

$$(\lambda^2-1)U''+U^3+(\alpha^2-\beta^2-2V)U-2i(\alpha-\lambda\beta)U'=0 \tag{12}$$

$$(\lambda^2+1)V''-2(U')^2-2UU''=0 \tag{13}$$

Integrate the second equality twice and put the constants of integration zero, get:

$$V=\frac{U^2}{\lambda^2+1} \tag{14}$$

Coefficient of the imaginary part in eq. (12) tends to zero, $\alpha = \lambda\beta$ is found. We have:

$$(\lambda^2 - 1)U'' + U^3 + (\alpha^2 - \beta^2 - 2V)U = 0 \tag{15}$$

Substituting eq. (14) into eq. (15) gives:

$$(\lambda^2 - 1)U'' + U^3 + (\alpha^2 - \beta^2)U - 2\frac{1}{\lambda^2 + 1}U^3 = 0 \tag{16}$$

Further we have:

$$(\lambda^4 - 1)U'' + (\lambda^2 - 1)U^3 + (\alpha^2 - \beta^2)(\lambda^2 + 1)U = 0 \tag{17}$$

According to the general properties of TFM, it can be considered:

$$U = \sum_{i=0}^{n} a_i \tanh(\xi)^i \tag{18}$$

Substituting eq. (18) into eq. (17) gives:

$$(\lambda^4 - 1)(1 - Y^2)\left[-2Y\frac{\mathrm{d}U}{\mathrm{d}Y} + (1 - Y^2)\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}\right] + (\lambda^2 - 1)U^3 + (\alpha^2 - \beta^2)(\lambda^2 + 1)U = 0 \tag{19}$$

By balancing the height order derivative term $\mathrm{d}^2U/\mathrm{d}Y^2$ with the non-linear term U^3, gives $n = 1$. We get:

$$U = a_0 + a_1 Y,\ \frac{\mathrm{d}U}{\mathrm{d}Y} = a_1,\ \frac{\mathrm{d}^2U}{\mathrm{d}Y^2} = 0 \tag{20}$$

Substituting eq. (20) into eq. (19), and have:

$$(\lambda^4 - 1)(1 - Y^2)(-2Ya_1) + (\lambda^2 - 1)(a_0 + a_1 Y)^3 + (\alpha^2 - \beta^2)(\lambda^2 + 1)(a_0 + a_1 Y) = 0 \tag{21}$$

This equation contains terms up to Y^3. We set the coefficients of all powers of $\tanh(\xi)^i$, $i = 1, 2, 3$ to equal to zero:

$$Y^0 : {a_0}^3\lambda^2 - {a_0}^3 + a_0\lambda^2\alpha^2 + a_0\alpha^2 - a_0\lambda^2\beta^2 - a_0\beta^2 = 0$$

$$Y^1 : -2a_1\lambda^4 + 2a_1 + 3{a_0}^2 a_1\lambda^2 - 3{a_0}^2 a_1 + a_1\lambda^2\alpha^2 + a_1\alpha^2 - a_1\lambda^2\beta^2 - a_1\beta^2 = 0$$

$$Y^2 : 3a_0{a_1}^2\lambda^2 - 3a_0{a_1}^2 = 0$$

$$Y^3 : \lambda^2{a_1}^3 - {a_1}^3 + 2a_1\lambda^4 \text{-} 2a_1 = 0$$

The result after solving the previous system:

$$P(x,t) = \mathrm{e}^{i(\alpha x + \beta t)}\left(-\alpha^2 + \beta^2 - 4\right)^{1/2}\tanh\left[x + \left(\frac{\alpha^2}{2} - \frac{\beta^2}{2} + 1\right)^{1/2} t\right] \tag{22}$$

$$Q(x,t) = \frac{1}{\left[\left(\frac{\alpha^2}{2} - \frac{\beta^2}{2} + 1\right)^{1/2}\right]^2 + 1}\left[\left(-\alpha^2 + \beta^2 - 4\right)^{1/2}\tanh(x + \left(\frac{\alpha^2}{2} - \frac{\beta^2}{2} + 1\right)^{1/2} t)\right]^2 \tag{23}$$

with

$$a_0 = 0,\ \lambda = \left(\frac{\alpha^2}{2} - \frac{\beta^2}{2} + 1\right)^{1/2}\ \text{ and }\ a_1 = \left(-\alpha^2 + \beta^2 - 4\right)^{1/2}$$

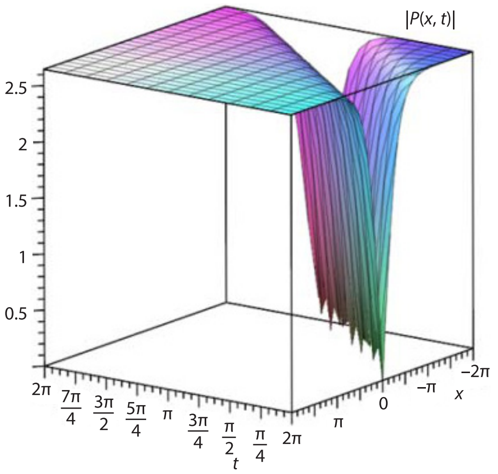

Figure 1. The 3-D graphs of eq. (22) for the values of $\alpha = 2$ and $\beta = 1$

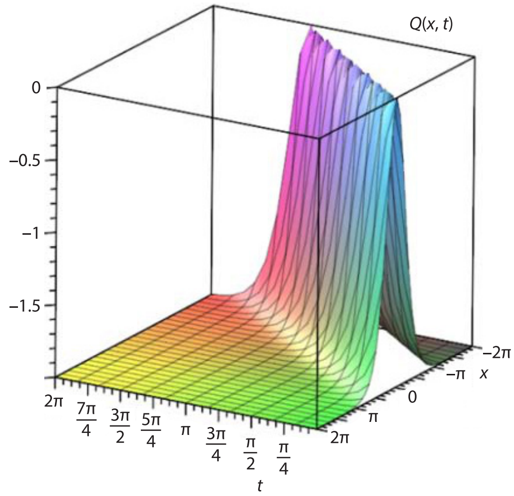

Figure 2. The 3-D graphs of eq. (23) for the values of $\alpha = 2$ and $\beta = 1$

The corresponding 3-D graphs of eqs. (22) and (23) are plotting in figs. 1 and 2 with proper parameters.

Conclusion

In this paper, we successfully apply the tanh function method to construct new solitary wave solution for the couple-Higgs equation. Some 3-D graphs are presented with suitable parameters. These new results are helpful to further understanding of the corresponding physical phenomena. Furthermore, this proposed method will be used to investigate other non-linear evolution equations in future work.

References

[1] Chavanis, P. H., Non-linear Mean-Field Fokker-Planck Equations and Their Applications in Physics, *Astrophysics and Biology, Comptes Rendus Physique*, *7* (2006), 2, pp. 318-330
[2] Yang, X. J., Alsolami, A. A., On a Subtrigonometric Series Solution for the Scaling-Law Wave Motion for the Special Vibrating String, *International Journal of Geometric Methods in Modern Physics*, On-line first, https://doi.org/10.1142/S0219887824400358, 2024
[3] Kumar, S., *et al*., Specific Wave Profiles and Closed-Form Soliton Solutions for Generalized Non-Linear Wave Equation in (3+1)-Dimensions with Gas Bubbles in Hydrodynamics and Fluids, *Journal of Ocean Engineering and Science*, *8* (2023), 2, pp. 91-102
[4] Wang, K. L., Wei, C. F., Novel Optical Soliton Solutions to Nonlinear Paraxial Wave Model, *Modern Physics Letters B*, *2024* (2024), ID2450469
[5] Yang, X. J., The Scaling-Law Flows: An Attempt at Scaling-Law Vector Calculus, *Fractals*, *33* (2024), 4, ID2340126
[6] Rehman, H. U., *et al*.,Analysis of Brownian Motion in Stochastic Schrodinger Wave Equation Using Sardar Sub-Equation Method, *Optik*, *289* (2023), 2, ID171305
[7] Chou, D., *et al*., Analyzing Optical Solition Propagation in Perturbed Non-Linear Schrodinger Equation: a Multi-Technique Study, *Optik*, *302* (2024), 2, ID 171714
[8] Samir, I., *et al*., Construction of Wave Solutions for Stochastic Radhakrishnan-Kundu-Lakshmanan Equation Using Modified Extended Direct Algebraic Technique, *Results in Physics*, *55* (2023), 3, ID107191
[9] Wazwaz, A. M., The Tanh Method and the Sine-Cosine Method for Solving the KP-MEW Equation, *International Journal of Computer Mathematics*, *82* (2005), 2, pp. 235-246
[10] Hussain, A., *et al.*, The Jacobi Elliptic Function Method and Its Application for the Stochastic NNV System, *Alexandria Engineering Journal*, *81* (2023), 3, pp. 347-359

[11] Ali, M., *et al.*, A Variety of New Periodic Solutions to the Damped (2+1)-Dimensional Schrodinger Equation via the Novel Modified Rational Sine-Cosine Functions and the Extended Tanh-Coth Expansion Methods, *Results in Physics*, *37* (2022),3, 105462
[12] Ahmad, J., *et al.*, Solitary Wave Structures for the Stochastic Nizhnik-Novikov-Veselov System Via Modified Generalized Rational Exponential Function Method, *Results in Physics*, *52* (2023), 3, ID106776
[13] Wang, K. L., New Optical Solitons for Nonlinear Fractional Schrödinger Equation via Different Analytical Approaches, *Fractals*, *32* (2024), 5, ID 2450077
[14] Liu, J.-G., *et al*., On the Generalized Time Fractional Diffusion Equation: Symmetry Analysis, Conservation Laws, Optimal System and Exact Solutions, *International Journal of Geometric Methods in Modern Physics*, *17* (2020), 1, ID2050013
[15] Wang, K. L., Novel Solitary Wave and Periodic Solutions for the Non-Linear Kaup-Newell Equation in Optical Fibers, *Optical and Quantum Electronics*, *56* (2024), 2, ID514
[16] Wang, K. L., Novel Soliton Solutions of Fractional Long and Short-Wave Interaction System, *Fractals*, *32* (2024), 1, ID2450023
[17] Wei, C. F., A New Fractal Modelling for the Nerve Impulses Based on Local Fractional Derivative, *Fractals*, *32* (2024), 4, ID2440027
[18] Wang, K. L., Novel Perspective to the Fractional Schrodinger Equation Arising in Optical Fibers, *Fractals*, *32* (2024), 2, ID2450034
[19] Liu, J.-G., *et al*., Symmetry Analysis of the Generalized Space and Time Fractional Korteweg-de Vries Equation, *International Journal of Geometric Methods in Modern Physics*, *18* (2021), 14, ID2150235
[20] Wang, K. L., New Mathematical Approaches to Non-Linear Coupled Davey-Stewartson Fokas System Arising in Optical Fibers, *Mathematical Methods in the Applied Sciences*, On-line first, https://doi.org/10.1002/mma.10175, 2024
[21] Wang, K. L., Novel Analytical Approach to Modified Fractal Gas Dynamics Model with the Variable Coefficients, *ZAMM-Zeitschrift Fur Angewandte Mathematik Und Mechanik*, *103* (2023), 6, ID 202100391
[22] Wang, T., Soliton Dynamics to the Higgs Equation and Its Multi-Component Generalization, *Wave Motion*, *120* (2023), 2, ID103144
[23] Hafez, M. G., *et al.*, Traveling Wave Solutions for Some Important Coupled Non-Linear Physical Models via the Coupled Higgs Equation and the Maccari System, *Journal of King Saud University-Science*, *12* (2015) 27, pp. 105-112
[24] Seadawy, A. R., *et al.*, Bifurcations of Traveling Wave Solutions for Dodd-Bullough-Mikhailov Equation and Couple Higgs Equation and Their Applications, *Chinese Journal of Physics*, *55* (2017), 4, pp. 1310-1318
[25] Jabbari, A., *et al*., Exact Solutions of the Couple Higgs Equation and the Maccari System Using He's Semi-Inverse Method and (G′/G)-Expansion Method, *Computers and Mathematics with Applications*, *62* (2011), 3, pp. 2177-2186
[26] Ali, K. K., *et al*., Closed Form Wave Profiles of the Coupled-Higgs Equation Via the φ^6-Model Expansion Method, *International Journal of Modern Physics*, *37* (2023), 4, ID2350070

Paper submitted: March 2, 2024
Paper revised: March 14, 2024
Paper accepted: May 10, 2024