NEW EXACT SOLUTIONS OF THE LOCAL FRACTIONAL (3+1)-DIMENSIONAL KADOMSTEV-PETVIASHVILI EQUATION

by

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Aided by the local fractional derivative, we present a new local fractional (3+1)-dimensional Kadomstev-Petviashvili equation for describing the fractal water wave in this work. The non-differentiable transform is utilized to convert the local fractional equation into a local fractional ODE. On defining the Mittag-Leffler function on the Cantor sets, then a trial function based on the Mittag-Leffler function is proposed to seek for the non-differentiable exact solutions. The results reveal that the proposed method is a promising way to study the local fractional PDE arising in engineering and physics.

Key words: *local fractional derivative, Mittag-Leffler function, trial function, Cantor sets, non-differentiable exact solutions*

Introduction

A great deal of non-linear phenomena arising in mechanics, optics, plasma physics, magnetic field, vibration and so on can eventually be summarized as the non-linear PDE [1-8]. The study on their exact solutions has great significance for understanding the inner essence. In this work, we first consider the (3+1)-dimensional Kadomstev-Petviashvili equation (KPE):

$$\Pi_{xt} - 6\Pi_x^2 + 6\Pi\Pi_{xx} - \Pi_{xxxx} - \Pi_{yy} - \Pi_{zz} = 0 \tag{1}$$

Equation (1) acts an important role in the physics and is used widely to describe the water waves of long wavelength with weakly non-linear restoring forces. Recently, the fractal and fractional derivatives have got extensive attention in various fields such fluid [9, 10], optics [11, 12], circuits [13, 14], biomedical science [15], vibration [16, 17], and so on [18-20]. The advantage of fractional derivative is that it can describe various complex phenomena in extreme environment involving in non-smooth boundary, micro-gravity environment, porous medium, the phenomenon with memory effect and so on. In this study, we mainly focus on the local fractional derivative (LFD) [21-23], which is a new definition of the fractional derivative, has broad applications in different fields such as the diffusion [24], fractal media [25], oscillators [26], and porous media [27] and so on. Inspired by the latest research of the LFD, we adopt the LFD to the (3+1)-dimensional KPE to obtain its local fractional form that can model the fractal water wave:

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$$\frac{\partial^{\wp}}{\partial x^{\wp}}\left(\frac{\partial^{\wp}\Pi}{\partial t^{\wp}}\right)-6\left(\frac{\partial^{\wp}\Pi}{\partial x^{\wp}}\right)^{2}+6\Pi\frac{\partial^{2\wp}\Pi}{\partial x^{2\wp}}-\frac{\partial^{4\wp}\Pi}{\partial x^{4\wp}}-\frac{\partial^{2\wp}\Pi}{\partial y^{2\wp}}-\frac{\partial^{2\wp}\Pi}{\partial z^{2\wp}}=0 \tag{2}$$

where $\wp(0<\wp\leq 1)$ indicates the fractal dimension, $\Pi(x, y, z, t)$ is a fractal function that models the fractal water wave, and $\partial^{\wp}\Pi/\partial x^{\wp}$, $\partial^{\wp}\Pi/\partial y^{\wp}$, $\partial^{\wp}\Pi/\partial z^{\wp}$, and $\partial^{\wp}\Pi/\partial t^{\wp}$, are the local fractional derivatives and their definitions are given in section *Basic theory*.

Basic theory

The main target of this section is to introduce the definitions and some properties of the LFD and MLF.

Definition 1. For the function $\phi(x, t)$ of fractal order $\wp$, its LFD is defined [28]:

$$D^{(\wp)}\phi(x_0)=\frac{\mathrm{d}^{\wp}\phi(x)}{\mathrm{d}x^{\wp}}|\,x=x_0=\lim_{x\to x_0}\frac{\Delta^{\wp}\left[\phi(x)-\phi(x_0)\right]}{(x-x_0)^{\wp}} \tag{3}$$

where

$$\Delta^{\wp}\left[\phi(x)-\phi(x_0)\right]\cong\Gamma(1+\wp)\left[\phi(x)-\phi(x_0)\right]$$

There exists the properties for the LFD [28]:

$$D^{(m\wp)}\phi(x_0)=\frac{\mathrm{d}^{m\wp}\phi(x_0)}{\mathrm{d}x^{m\wp}}=\overbrace{\frac{\mathrm{d}^{\wp}}{\mathrm{d}x^{\wp}}\cdots\frac{\mathrm{d}^{\wp}}{\mathrm{d}x^{\wp}}}^{m\ \text{times}}\phi(x_0) \tag{4}$$

Definition 2. The MLF of the fractal dimension $\wp$ on the Cantor sets (CS) is defined [28]:

$$\mathrm{MI}_{\wp}\left(u^{\wp}\right)=\sum_{\Im=0}^{\infty}\frac{u^{\Im\wp}}{\Gamma(1+\Im\wp)} \tag{5}$$

Property 1. For the LFD, there are [28]:

$$\frac{\partial^{\wp}}{\partial t^{\wp}}\left[p(t)\pm q(t)\right]=\frac{\partial^{\wp}}{\partial t^{\wp}}p(t)\pm\frac{\partial^{\wp}}{\partial t^{\wp}}q(t) \tag{6}$$

$$\frac{\partial^{\wp}}{\partial t^{\wp}}\left[p(t)q(t)\right]=p(t)\frac{\partial^{\wp}}{\partial t^{\wp}}q(t)+q(t)\frac{\partial^{\wp}}{\partial t^{\wp}}p(t) \tag{7}$$

$$\frac{\partial^{\wp}}{\partial t^{\wp}}\left[\frac{p(t)}{q(t)}\right]=\frac{\left[q(t)\frac{\partial^{\wp}}{\partial t^{\wp}}p(t)+p(t)\frac{\partial^{\wp}}{\partial t^{\wp}}q(t)\right]}{q(t)^{2}} \tag{8}$$

Property 2. For the MLF on CS, there exist the properties [28]:

$$D^{(\wp)}\mathrm{MI}_{\wp}\left(\Lambda u^{\wp}\right)=\Lambda\mathrm{MI}_{\wp}\left(u^{\wp}\right),\ \Lambda\ \text{is a constant} \tag{9}$$

$$\mathrm{MI}_{\wp}\left(u^{\wp}\right)\mathrm{MI}_{\wp}\left(v^{\wp}\right)=\mathrm{MI}_{\wp}\left(u^{\wp}+v^{\wp}\right) \tag{10}$$

$$\mathrm{MI}_{\wp}\left(u^{\wp}\right)\mathrm{MI}_{\wp}\left(-v^{\wp}\right)=\mathrm{MI}_{\wp}\left(u^{\wp}-v^{\wp}\right) \tag{11}$$

$$\mathrm{MI}_{\wp}\left(u^{\wp}\right)\mathrm{MI}_{\wp}\left(i^{\wp}v^{\wp}\right)=\mathrm{MI}_{\wp}\left(u^{\wp}+i^{\wp}v^{\wp}\right) \tag{12}$$

$$\mathrm{MI}_{\wp}\left(i^{\wp}u^{\wp}\right)\mathrm{MI}_{\wp}\left(i^{\wp}v^{\wp}\right)=\mathrm{MI}_{\wp}\left(i^{\wp}u^{\wp}+i^{\wp}v^{\wp}\right) \tag{13}$$

The exact solutions

For solving eq. (1), we consider the ND transformation [29, 30]:

$$\Pi\left(x^{\wp},t^{\wp}\right)=\Xi\left(\xi^{\wp}\right),\ \xi^{\wp}=m^{\wp}x^{\wp}+n^{\wp}y^{\wp}+k^{\wp}z^{\wp}+l^{\wp}t^{\wp} \tag{14}$$

By applying aforementioned transformation, we can convert eq. (2):

$$\left(m^{\wp}l^{\wp}-n^{2\wp}-k^{2\wp}\right)\frac{\mathrm{d}^{2\wp}\Xi}{\mathrm{d}\xi^{2\wp}}-6m^{2\wp}\left(\frac{\mathrm{d}^{\wp}\Xi}{\mathrm{d}\xi^{\wp}}\right)^{2}+6m^{2\wp}\Xi\frac{\mathrm{d}^{2\wp}\Xi}{\mathrm{d}\xi^{2\wp}}-m^{4\wp}\frac{\mathrm{d}^{4\wp}\Xi}{\mathrm{d}\xi^{4\wp}}=0 \tag{15}$$

where

$$m^{\wp}l^{\wp}-n^{2\wp}-k^{2\wp}\neq 0$$

On the basis of MLF, we introduce the following trial function, which is assumed as the solution of eq. (15):

$$\Xi\left(\xi^{\wp}\right)=\frac{\sum_{\lambda=-q}^{p}\alpha_{\lambda}\mathrm{MI}_{\wp}\left(\lambda\xi^{\wp}\right)}{\sum_{\eta=-s}^{g}\beta_{\eta}\mathrm{MI}_{\wp}\left(\eta\xi^{\wp}\right)} \tag{16}$$

Substituting it into eq. (15), we obtain the highest order of

$$\frac{\mathrm{d}^{4\wp}\Xi}{\mathrm{d}\xi^{4\wp}}\ \text{and}\ \Xi\frac{\mathrm{d}^{2\wp}\Xi}{\mathrm{d}\xi^{2\wp}}$$

$$\frac{\mathrm{d}^{4\wp}\Xi}{\mathrm{d}\xi^{4\wp}}=\frac{\Delta_{1}\mathrm{MI}_{\wp}\left[\left(p+4g\right)\xi^{\wp}\right]+\ldots}{\Delta_{2}\mathrm{MI}_{\wp}\left(5g\xi^{\wp}\right)+\ldots} \tag{17}$$

$$\Xi\frac{\mathrm{d}^{2\wp}\Xi}{\mathrm{d}\xi^{2\wp}}=\frac{\Lambda_{1}\mathrm{MI}_{\wp}\left[\left(2p+2g\right)\xi^{\wp}\right]+\ldots.}{\Lambda_{2}\mathrm{MI}_{\wp}\left[\left(4g\right)\xi^{\wp}\right]+\ldots\ldots}=\frac{\Lambda_{1}\mathrm{MI}_{\wp}\left[\left(2p+3g\right)\xi^{\wp}\right]+\ldots.}{\Lambda_{2}\mathrm{MI}_{\wp}\left[\left(5g\right)\xi^{\wp}\right]+\ldots\ldots} \tag{18}$$

Balancing them yields:

$$p+4g=2p+3g \tag{19}$$

which gives:

$$p=g \tag{20}$$

In the same way, we get the lowest order of:

$$\frac{\mathrm{d}^{4\varepsilon}\Xi}{\mathrm{d}\xi^{4\varepsilon}}\ \text{and}\ \Xi\frac{\mathrm{d}^{2\varepsilon}\Xi}{\mathrm{d}\xi^{2\varepsilon}}$$

$$\frac{d^{4\wp}\Xi}{d\xi^{4\wp}} = \frac{....+\Lambda_3 \mathrm{MI}_\wp\left[-(q+4s)\xi^\wp\right]}{...+\Lambda_4 \mathrm{MI}_\wp\left[(-5s)\xi^\wp\right]} \tag{21}$$

$$\Xi\frac{d^{2\wp}\Xi}{d\xi^{2\wp}} = \frac{\Delta_3 \mathrm{MI}_\wp\left[-(2q+2s)\xi^\wp\right]}{...+\Delta_4 \mathrm{MI}_\wp\left[(-4s)\xi^\wp\right]} = \frac{...+\Delta_3 \mathrm{MI}_\wp\left[-(2q+3s)\xi^\wp\right]}{...+\Delta_4 \mathrm{MI}_\wp\left[(-5s)\xi^\wp\right]} \tag{22}$$

Balancing them, we have:

$$-(q+4s) = -(2q+3s) \tag{23}$$

From it, there is:

$$q = s \tag{24}$$

For simplicity, we can use $p = g = 1$, $q = s = 1$, then eq. (16) can be reduced:

$$\Xi\left(\xi^\wp\right) = \frac{\alpha_1 \mathrm{MI}_\wp\left(\xi^\wp\right)+\alpha_0+\alpha_{-1}\mathrm{MI}_\wp\left(-\xi^\wp\right)}{\mathrm{MI}_\wp\left(\xi^\wp\right)+\beta_0+\beta_{-1}\mathrm{MI}_\wp\left(-\xi^\wp\right)} \tag{25}$$

Insetting eq. (25) into eq. (15), collecting the same power of the $\mathrm{MI}_\wp(\rho\xi^\wp)$ and setting their coefficients as zero leads to a set of algebraic equations. Solving them, we have:

Family one:

$$\alpha_1 = \alpha_1,\ \alpha_0 = \frac{\alpha_{-1}+\alpha_1\beta_0^2}{\beta_0},\ \ \alpha_{-1} = \alpha_{-1},\ \beta_0 = \beta_0,\ \beta_{-1} = 0,\ l^\wp = \frac{k^{2\wp}-6\alpha_1 m^{2\wp}+m^{4\wp}+n^{2\wp}}{m^\wp}$$

Thus eq. (25) can be written:

$$\Xi\left(\xi^\wp\right) = \frac{\alpha_1 \mathrm{MI}_\wp\left(\xi^\wp\right)+\dfrac{\alpha_{-1}+\alpha_1\beta_0^2}{\beta_0}+\alpha_{-1}\mathrm{MI}_\wp\left(-\xi^\wp\right)}{\mathrm{MI}_\varepsilon\left(\xi^\wp\right)+\beta_0} \tag{26}$$

So the exact solution of eq. (2) on the CS is obtained:

$$\Pi(x,y,z,t) = \alpha_1+\alpha_{-1}\frac{\dfrac{1}{\beta_0}+\mathrm{MI}_\wp\left(-m^\wp x^\wp - n^\wp y^\wp - k^\wp z^\wp - \dfrac{k^{2\wp}-6\alpha_1 m^{2\wp}+m^{4\wp}+n^{2\wp}}{m^\varepsilon}t^\wp\right)}{\mathrm{MI}_\wp\left(m^\wp x^\wp + n^\wp y^\wp + k^\wp z^\wp + \dfrac{k^{2\wp}-6\alpha_1 m^{2\wp}+m^{4\wp}+n^{2\wp}}{m^\varepsilon}t^\wp\right)+\beta_0} \tag{27}$$

Family two:

$$\alpha_1 = \alpha_1,\ \alpha_0 = \alpha_0,\ \ \alpha_{-1} = -\beta_0\left(-\alpha_0+\alpha_1\beta_0\right),\ \beta_0 = \beta_0,\ \beta_{-1} - 0,\ \ l^\wp = \frac{k^{2\wp}-6\alpha_1 m^{2\wp}+m^{4\wp}+n^{2\wp}}{m^\varepsilon}$$

With this, we can get the expression of eq. (25):

$$\Xi\left(\xi^\wp\right) = \frac{\alpha_1 \mathrm{MI}_\wp\left(\xi^\wp\right)+\alpha_0-\beta_0\left(-\alpha_0+\alpha_1\beta_0\right)\mathrm{MI}_\wp\left(-\xi^\wp\right)}{\mathrm{MI}_\wp\left(\xi^\wp\right)+\beta_0} \tag{29}$$

So the second ND exact solution of eq. (2) on the CS is attained:

$$\Pi(x,y,z,t)=\alpha_1+ \\ +\frac{\alpha_0-\alpha_1\beta_0-\beta_0(-\alpha_0+\alpha_1\beta_0)\mathrm{MI}_\wp\left(-m^\wp x^\wp-n^\wp y^\wp-k^\wp z^\wp-\dfrac{k^{2\wp}-6\alpha_1 m^{2\wp}+m^{4\wp}+n^{2\wp}}{m^\wp}t^\wp\right)}{\mathrm{MI}_\wp\left(m^\wp x^\wp+n^\wp y^\wp+k^\wp z^\wp+\dfrac{k^{2\wp}-6\alpha_1 m^{2\wp}+m^{4\wp}+n^{2\wp}}{m^\wp}t^\wp\right)+\beta_0} \quad (29)$$

where we note that the solutions of eqs. (27) and (29) become the exact solution of the classic (3+1)-dimensional KPE for $\wp = 1$.

Conclusion

With the aid of the LFD, a new local fractinoal (3+1)-dimensional KPE is derived in this study. By defining the MLF on the CS, a trial function is proposed to search for the ND exact solutions. By means of this method, two sets of the ND exact solutions are attained. As one can see, the proposed method is straightforward, powerful and can be applied to study the other local fractional PDEs in engineering and physics.

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