DYNAMIC BEHAVIORS OF THE NON-LINEAR LOCAL FRACTIONAL HEAT CONDUCTION EQUATION ON THE CANTOR SETS

by

Geng LI and Kang-Jia WANG *

[a] School of Physics and Electronic Information Engineering,
Henan Polytechnic University, Jiaozuo, China

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Based on the local fractional derivative, a fractal non-linear heat conduction equation, which can model the behavior of the heat transfer in the fractal medium, is extracted in this work. On defining the Mittag-Leffler function on the Cantor sets, two special functions namely the $TH_\upsilon(\mu_\upsilon)$ function and $CH_\upsilon(\mu_\upsilon)$ function are constructed, and then are employed along with Yang's non-differentiable transformation seek for the non-differentiable exact solutions. The obtained results confirm that the proposed method iseffective and powerful, and can provide a promising way to find the exact solutions of the fractal PDE.

Key words: *local fractional derivative, Mittag-Leffler function, Cantor sets, Yang's non-differentiable transformation, special functions*

Introduction

It is well known that the non-linear equations have been widely used in various fields involving in physics [1-5], vibration [6, 7], biology [8, 9], engineering technology [10, 11], economic [12] and so on.The solution of non-linear PDE is a major research content in non-linear problems. In this work, we first consider the non-linear heat conduction which reads [13]:

$$\frac{\partial\phi}{\partial t}-\alpha\frac{\partial^2\left(\phi^3\right)}{\partial x^2}-\phi+\phi^3=0 \tag{1}$$

where α is a non-zero constant and $\phi = \phi\ (x,\ t)$ represents the temperature. Due to the fact that fractional differential equations are more general than integer differential equations, and can more accurately describe objective laws and the nature of things than integer differential equations, the fractal and fractional calculus has been applied in many fields and has attracted increasing attention, such as the non-smooth boundary [14-17], microgravity space [18-20], fractal medium [21, 22] and so on, which all involve fractional derivatives. In addition, the fractional differential equations have greatly enriched the content of mathematical theory and permeated many fields of natural science. Inspired by recent research results on the fractional derivative, here in this work, we will apply local fractional derivative (LFD) to eq. (1) to extract its fractal form for describing the heat transfer in the fractal medium:

$$\frac{\partial_L^\upsilon\phi^\gamma\left(x,t\right)}{\partial_L t^\upsilon}-\alpha\frac{\partial_L^{2\upsilon}\left(\phi^3\right)}{\partial_L x^{2\upsilon}}-\phi+\phi^3=0 \tag{2}$$

* Corresponding author, e-mail: konka05@163.com

where $\upsilon(0 < \upsilon \le 1)$ is the fractal dimension, $\partial_L^\upsilon/\partial_L^\upsilon t^\upsilon$ and $\partial_L^\upsilon/\partial_L^\upsilon x^\upsilon$ are the LFD with respect to t and x, respectively. The definitions of the LFD can be seen in section *Preliminaries*. It is noted that eq. (2) becomes the classic model when $\vartheta = 1$. The aim of this work is extracting the non-differentiable (ND) exact solutions of eq. (2).

Preliminaries

Definition 1. The definition of the LFD for the function $\Pi_\upsilon(t)$ with fractal order $\upsilon(0 < \upsilon \le 1)$ is given [23]:

$$D^{(\upsilon)}\Pi_\upsilon(t) = \frac{d_L^\upsilon \Pi_\upsilon(t)}{d_L t^\upsilon} \mid t = t_0 = \lim_{t \to t_0} \frac{\Delta^\upsilon\left[\Pi_\upsilon(t) - \Pi_\upsilon(t_0)\right]}{(t - t_0)^\upsilon} \tag{3}$$

where

$$\Delta^\upsilon\left[\Pi(t) - \Pi(t_0)\right] \cong \Gamma(1+\upsilon)\left[\Pi(t) - \Pi(t_0)\right]$$

with the Euler's Gamma function given:

$$\Gamma(1+\upsilon) =: \int_0^\infty t^{\upsilon-1} \exp(-t) \mathrm{d}t$$

and there is:

$$\frac{\partial_L^{\Im\upsilon}\Pi_\upsilon(t)}{\partial_L t^{\Im\upsilon}} = \overbrace{\frac{\partial_L^\upsilon}{\partial_L t^\upsilon} \cdots \frac{\partial_L^\upsilon}{\partial_L t^\upsilon}}^{\Im \text{ times}} \Pi_\upsilon(t) \tag{4}$$

Property 1. There are the properties for the LFD [23]:

$$\frac{\partial_L^\upsilon}{\partial_L t^\upsilon}(\Lambda) = 0, \ \Lambda \text{ is the constant} \tag{5}$$

$$\frac{\partial_L^\upsilon}{\partial_L t^\upsilon}\left[\xi_\upsilon(t) \pm \zeta_\upsilon(t)\right] = \frac{\partial_L^\upsilon}{\partial_L t^\upsilon}\xi_\upsilon(t) \pm \frac{\partial_L^\upsilon}{\partial_L t^\upsilon}\zeta_\upsilon(t) \tag{6}$$

$$\frac{\partial_L^\upsilon}{\partial_L t^\upsilon}\left[\xi_\upsilon(t)\zeta_\upsilon(t)\right] = \xi_\upsilon(t)\frac{\partial_L^\upsilon}{\partial_L t^\upsilon}\zeta_\upsilon(t) + \zeta_\upsilon(t)\frac{\partial_L^\upsilon}{\partial_L t^\upsilon}\xi_\upsilon(t) \tag{7}$$

$$\frac{\partial_L^\upsilon}{\partial_L t^\upsilon}\left[\frac{\xi_\upsilon(t)}{\zeta_\upsilon(t)}\right] = \frac{\left[\zeta_\upsilon(t)\frac{\partial_L^\upsilon}{\partial_L t^\upsilon}\xi_\upsilon(t) - \xi_\upsilon(t)\frac{\partial_L^\upsilon}{\partial_L t^\upsilon}\zeta_\upsilon(t)\right]}{\zeta_\upsilon(t)^2} \tag{8}$$

Definition 2.The Mittag-Lefler finction (MLF) on the Cantor sets (CS) with the fractional dimension υ can be defined [23]:

$$\Theta_\upsilon(\mu^\upsilon) = \sum_{\Im=0}^\infty \frac{\mu^{\Im\upsilon}}{\Gamma(1+\Im\upsilon)} \tag{9}$$

Property 2. The MLF admits the properties [23]:

$$\Theta_\upsilon(u^\upsilon)\Theta_\upsilon(v^\upsilon) = \Theta_\upsilon(u^\upsilon + v^\upsilon) \tag{10}$$

$$\Theta_\upsilon\left(u^\upsilon\right)\Theta_\upsilon\left(-v^\upsilon\right)=\Theta_\upsilon\left(u^\upsilon-v^\upsilon\right) \tag{11}$$

$$\Theta_\upsilon\left(u^\upsilon\right)\Theta_\upsilon\left(i^\upsilon v^\upsilon\right)=\Theta_\upsilon\left(u^\upsilon+i^\upsilon v^\upsilon\right) \tag{12}$$

$$\Theta_\upsilon\left(i^\upsilon u^\upsilon\right)\Theta_\upsilon\left(i^\upsilon v^\upsilon\right)=\Theta_\upsilon\left(i^\upsilon u^\upsilon+i^\upsilon v^\upsilon\right) \tag{13}$$

Definition 2.Two SF on the CS can be constructed based on the MLF [23]:

$$TH_\upsilon\left(\mu^\upsilon\right)=\frac{\Theta_\upsilon\left(\mu^\upsilon\right)-\Theta_\upsilon\left(-\mu^\upsilon\right)}{\Theta_\upsilon\left(\mu^\upsilon\right)+\Theta_\upsilon\left(-\mu^\upsilon\right)} \tag{14}$$

$$\mathrm{C}H_\upsilon\left(\mu^\upsilon\right)=\frac{\Theta_\upsilon\left(\mu^\upsilon\right)+\Theta_\upsilon\left(-\mu^\upsilon\right)}{\Theta_\upsilon\left(\mu^\upsilon\right)-\Theta_\upsilon\left(-\mu^\upsilon\right)} \tag{15}$$

Property 3. From the properties of the MLF, we get the properties of the two SF:

$$D^{(\upsilon)}TH_\upsilon\left(\mu^\upsilon\right)=1-TH_\upsilon^2\left(\mu^\upsilon\right) \tag{16}$$

$$D^{(\upsilon)}CH_\upsilon\left(\mu^\upsilon\right)=1-CH_\upsilon^2\left(\mu^\upsilon\right) \tag{17}$$

The ND exact solutions

For attaining the ND exact solutions of eq. (2), we take Yang's ND transformations [24, 25]:

$$\wp^\upsilon=\mu^\upsilon x^\upsilon+\omega^\upsilon t^\upsilon \tag{18}$$

It can convert eq. (2) into the form:

$$\omega^\upsilon\phi_L'-6\alpha\mu^{2\upsilon}\phi\left(\phi'\right)^2-3\alpha\mu^{2\upsilon}\phi^2\phi_L''-\phi+\phi^3=0 \tag{19}$$

where

$$\phi=\phi\left(\wp^\upsilon\right),\quad \phi_L'=\frac{d_L^\upsilon\varphi}{d_L\wp^\upsilon},\quad \phi_L''=\frac{d_L^{2\upsilon}\phi}{d_L\wp^{2\upsilon}}$$

Here we can assume the solution of eq. (19) is the form:

$$\phi\left(\Re^\upsilon\right)=\sum_{i=0}^{c}\varepsilon_i\aleph_\upsilon^i\left(\wp^\upsilon\right) \tag{20}$$

where $\varepsilon_i(i = 0, 1, 2,...,c)$ are constants to be determined later. There is:

$$\aleph_L'\left(\wp^\upsilon\right)=D_L^\upsilon\left[\aleph_\upsilon\left(\wp^\upsilon\right)\right]=1-\aleph_\upsilon^2\left(\wp^\upsilon\right) \tag{21}$$

where

$$\aleph_\upsilon\left(\wp^\upsilon\right)=TH_\upsilon\left(\wp^\upsilon\right)\ \text{or}\ \ CH_\upsilon\left(\wp^\upsilon\right) \tag{22}$$

Balance $\phi^2\phi''$ and ϕ' in eq. (19), we have:

$$2c+4+c-2=c+1 \tag{23}$$

So we have:

$$c=-\frac{1}{2} \tag{24}$$

The value of c is not the integer, so we introduce the transform:

$$\phi=\varphi^{-1/2} \tag{25}$$

which converts eq. (19) into the form:

$$6\alpha\mu^{2\upsilon}\varphi_L''\varphi-15\alpha\mu^{2\upsilon}\left(\varphi_L'\right)^2-2\omega^{\upsilon}\varphi^2\varphi_L'+4\varphi^2-4\varphi^3=0 \tag{26}$$

Similarly, it assumes the solution of eq. (3.9) as:

$$\varphi\left(\wp^{\upsilon}\right)=\sum_{i=0}^{c}\varepsilon_i\aleph^i\left(\wp^{\upsilon}\right) \tag{27}$$

Now we balance the terms $\varphi''\varphi$ and $\varphi^2\varphi'$ in eq. (26), there is:

$$2c+2=3c+1 \tag{28}$$

which yields

$$c=1 \tag{29}$$

Thus eq. (27) can be re-written:

$$\varphi\left(\wp^{\upsilon}\right)=\varepsilon_0+\varepsilon_1\aleph_{\upsilon}\left(\wp^{\upsilon}\right) \tag{30}$$

Taking eq. (30) together with eq. (22) into eq. (26), setting the coefficients of all powers of $\aleph(\wp^{\upsilon})$ yields a set of algebraic equations. On solving them, we have:

Set 1:

$$\varepsilon_0=\frac{1}{2},\ \ \varepsilon_1=-\frac{1}{2},\ \ \omega^{\upsilon}=-\frac{1}{3},\ \ \mu^{\upsilon}=\pm\frac{1}{3\sqrt{\alpha}}$$

Thus we have:

$$\varphi\left(\wp^{\upsilon}\right)=\frac{1}{2}-\frac{1}{2}TH_{\upsilon}\left(\wp^{\upsilon}\right) \tag{31}$$

or

$$\varphi\left(\wp^{\upsilon}\right)=\frac{1}{2}-\frac{1}{2}CH_{\upsilon}\left(\wp^{\upsilon}\right) \tag{32}$$

In the view of eqs. (18) and (25), we get attain the ND exact solutions of eq. (2) on the CS as:

$$\phi_1^{\pm}\left(x^{\upsilon},t^{\upsilon}\right)=\left[\frac{1}{2}-\frac{1}{2}TH_{\upsilon}\left(\pm\frac{1}{3\sqrt{\alpha}}x^{\upsilon}+\frac{1}{3}t^{\upsilon}\right)\right]^{-1/2} \tag{33}$$

and

$$\phi_2^{\pm}\left(x^{\upsilon},t^{\upsilon}\right)=\left[\frac{1}{2}-\frac{1}{2}CH_{\upsilon}\left(\pm\frac{1}{3\sqrt{\alpha}}x^{\upsilon}+\frac{1}{3}t^{\upsilon}\right)\right]^{-1/2} \tag{34}$$

Set 2:

$$\varepsilon_0=\frac{1}{2},\ \ \varepsilon_1=\frac{1}{2},\ \ \omega^{\upsilon}=\frac{1}{3},\ \ \mu^{\upsilon}=\pm\frac{1}{3\sqrt{\alpha}}$$

Taking aforementioned results, we have:

$$\varphi\left(\wp^{\upsilon}\right)=\frac{1}{2}+\frac{1}{2}TH_{\upsilon}\left(\wp^{\upsilon}\right) \tag{35}$$

and

$$\varphi\left(\wp^{\upsilon}\right)=\frac{1}{2}+\frac{1}{2}CH_{\upsilon}\left(\wp^{\upsilon}\right) \tag{36}$$

By eqs. (18) and (25), the solutions of eq. (2) on the CS can be acquired:

$$\phi\left(x^{\upsilon},t^{\upsilon}\right)=\left[\frac{1}{2}+\frac{1}{2}TH_{\upsilon}\left(\pm\frac{1}{3\sqrt{\alpha}}x^{\upsilon}+\frac{1}{3}t^{\upsilon}\right)\right]^{-1/2} \tag{37}$$

and

$$\phi\left(x^{\upsilon},t^{\upsilon}\right)=\left[\frac{1}{2}+\frac{1}{2}CH_{\upsilon}\left(\pm\frac{1}{3\sqrt{\alpha}}x^{\upsilon}+\frac{1}{3}t^{\upsilon}\right)\right]^{-1/2} \tag{38}$$

Conclusion

In this study, a new fractal non-linear heat conduction equation that can model the behavior of the heat transfer in the fractal medium is derived within the LFD. By defining the MLF on the CS, we successfully construct two SF namely, the $TH_{\upsilon}(\mu^{\upsilon})$ function and $CH_{\upsilon}(\mu^{\upsilon})$ function. Then the two special functions, together with Yang's ND transformation, are employed to extract the ND exact solutions. Four ND exact solutions are obtained. It is confirmed the method proposed in this work are simple and powerful, and can provide a promising way to construct the exact solutions of the fractal PDE with the LFD arising in the nature science.

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