STUDY ON THE INTERACTION SOLUTION OF ZAKHAROV-KUZNETSOV EQUATION IN QUANTUM PLASMA

by

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The fundamental difference between quantum and traditional plasmas is the electron and ion composition, the former has a much higher density and extremely lower temperature, and it can be modelled by Zakharov-Kuznetsov (ZK) equation. In this paper, the Hirota bilinear method is used to study its solution properties.

Key words: *(2+1)-D variable coefficient ZK equation, Hirota bilinear method, plasma*

Introduction

Plasma [1-3] is electrically neutral, it is observed as a neutral gas on a large scale. When the temperature of a regular gas increases, the particles within the gas will strike each other with more force due to increased thermal movement, causing many electrons to be ejected from atoms and molecules. When the temperature reaches a certain level, all gas atoms become ionized, thus producing an equal amount of positive and negative charges, referred to as high ionization. With small concentrations, the plasma will keep its original behavior. However, when the concentration becomes large, it will experience an alteration in its dynamic behavior. At this time, the quantum effect in the plasma will play a significant role, which will change the dynamic behavior of the plasma. Quantum plasmas [4] are composed of electrons and ions, their density is very high, and temperature is extremely low, opposite to classical plasmas. Quantum plasma is widely used in different environments, including quantum diodes, ultra-small semiconductor devices, solid-density plasma interaction experiments, *etc*. Physicists have been involved in researching the phenomenon of quantum plasma, which has consequently driven mathematicians to study the quantum plasma equations concerning its physical components, it can be generally modelled by the magnetohydrodynamic model [5, 6], and it can be finally reduced to the ZK equation [7, 8].

This paper is to solve the ZK equation analytically. There are many analytical methods for this purpose. For examples, the variational method [9-16], the homogeneous balance method [17], the Backlund transformation method [18-20], the variational iteration method

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[21], the (G'/G) expansion method [22, 23], the hyperbolic function expansion method [24], the Hirota bilinear method [25, 26], the exp-function method [27, 28], the homotopy perturbation method [29, 30], and the mixed exponential method [31], and the Darboux transformation method [32]. The Hirota bilinear method [25, 26] will be used in this paper to solve the ZK equation with variable coefficients:

$$u_t + \delta(t)u_x + \alpha(t)uu_x + \beta(t)u_{xxx} + \varepsilon(t)u_{xyy} = 0 \tag{1}$$

where $\delta(t)$, $\alpha(t)$, $\beta(t)$, and $\varepsilon(t)$ are functions about t, and u is functions about x, y, and t. Equation (1) is often used to describe the motion of water wave in (2+1)-D space and the motion of plasma in magnetic field. The coefficient in the equation gives a clearer representation of the gradual change, the lack of symmetry of borders and external force.

When $\delta(t) = 1$, $\alpha(t) = 1$, $\beta(t) = 1$, and $\varepsilon(t) = 1$, eq. (1) becomes an extension of the standard constant coefficient ZK equation.

Equation (1) was widely studied by various methods [33-35], however the Hirota bilinear method might be more suitable for this problem.

Now we consider the constraint of $\alpha(t) = 2\varepsilon(t)$, $\beta(t) = \varepsilon(t)/9$, through the following dependent variable transformation:

$$u = 2(\ln f)_{xx} + 2(\ln f)_{xy} \tag{2}$$

We can convert eq. (1) into the following bilinear equation:

$$(3D_y^2 + D_x D_y - 2D_x^2) f \cdot f = 0 \tag{3}$$

$$[D_x D_t + D_y D_t + \delta(t) D_x D_y + \frac{\varepsilon(t)}{9} D_x^4 + \frac{\varepsilon(t)}{9} D_x^3 D_y + \varepsilon(t) D_x^2 D_y^2 + \varepsilon(t) D_x^3 D_y] f \cdot f = 0 \tag{4}$$

That is:

$$3 f_{yy} f - 3 f_y^2 + f_{xy} f - f_x f_y - 2 f_{xx} f + 2 f_x^2 = 0$$

$$f_{yt} - f_t f_x + f f_{xt} - f_t f_y + \delta(t)(f f_{xy} - f_x f_y) + \frac{\varepsilon(t)}{3}(f_{xx}^2 - f_x f_{xxy} + f_{xx} f_{xy}) -$$

$$-\frac{\varepsilon(t)}{9}(4 f_x f_{xxx} + f f f_{xxxx} + f f_{xxxy} - f_{xxx} f_y) +$$

$$+\varepsilon(t)(f f_{xxyy} + 2 f_{xy}^2 - 2 f_x f_{xyy} + f f_{xyyy} -$$

$$-2 f_{xxy} f_y - 3 f_{xyy} f_y + f_{xx} f_{yy} + 3 f_{xy} f_{yy} - f_x f_{yyy}) = 0$$

where $f = f(x, y, t)$, D-operator is defined as [25, 26]:

$$D_x^m \, D_y^n \, D_t^k \, f f' = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k f(x,\, y,\, t) f(x',\, y',\, t')\Big|_{x'=x, y'=y, t'=t} \tag{5}$$

Interaction between a breather wave and a water wave

To solve the rich interaction solutions of the (2+1)-D variable coefficient ZK equation, we consider the following test function:

$$f = k_1 e^{\xi_1} + k_2 e^{-\xi_2} + k_3 \cos(\xi_3) + a_{13} \tag{6}$$

where

$$\xi_1 = a_1 x + a_2 y + a_3(t) + a_4$$
$$\xi_2 = a_5 x + a_6 y + a_7(t) + a_8$$
$$\xi_3 = a_9 x + a_{10} y + a_{11}(t) + a_{12}$$

and a_1, a_2, a_4, a_5, a_6, a_8, a_9, a_{10}, and a_{12} are constants, and a_3, a_7, a_{11}, – the function of time t. Substituting eq. (5) into eq. (4) and making the coefficient of $e^{\xi_1}, e^{-\xi_2}, \cos(\xi_3), \sin(\xi_3)$ equal to zero, we can obtain a set of equations about $a_i(1 \le i \le 9)$. Solving this set of algebraic equations, we can extract the appropriate solutions of equations and evaluate the coefficients of the test function.

Case 1:

$$\delta(t) = -\frac{\begin{array}{c}9\varepsilon(t)^3 a_{11}'(t) - 18\varepsilon(t)^2 a_{11}'(t) + 104a_{10}^3\varepsilon(t)^4 - 376a_{10}^3\varepsilon(t)^3 + 432a_{10}^3\varepsilon(t)^2 - \\ -224a_{10}^3\varepsilon(t) + 64a_{10}^3\end{array}}{18a_{10}[\varepsilon(t)-1]\varepsilon(t)^2}$$

$$\varepsilon(t) = \varepsilon(t), \quad a_1 = 0, \quad a_3(t) = a_3(t), \quad a_7(t) = a_7(t), \quad a_9 = \int_a^t -\frac{2a_{10}[\varepsilon(s)-1]}{\varepsilon(s)}\mathrm{d}s$$
$$a_{11}(t) = a_{11}(t), \quad a_{13} = 0, \quad k_1 = 0, \quad k_2 = 0$$

and a_2, a_4, a_5, a_6, a_8, a_{10}, a_{12}, k_3 are arbitrary constants. By placing the previous coefficients into transformation eq. (2), the interaction solution of eq. (4) is obtained.

$$u_1(x,y,t) = \frac{\chi_1}{\zeta_1} \tag{7}$$

where

$$\chi_1 = 2a_{10}^2[\varepsilon(t)-1]\left(-4\sin^2\left\{a_{10}\left[\frac{2x}{\varepsilon(t)} - 2x + y\right] + a_{11}(t) + a_{12}\right\}\sec^2\left\{-\frac{2a_{10}^2 xy[\varepsilon(t)-1]}{\varepsilon(t)} + \right.\right.$$
$$\left.\left. + a_{11}(t) + a_{12}\right\}[\varepsilon(t)-1] + 2\tan^2\left\{a_{10}\left[\frac{2x}{\varepsilon(t)} - 2x + y\right] + a_{11}(t) + a_{12}\right\}\varepsilon(t) - 3\varepsilon(t) + 4\right)$$

$$\zeta_1 = \varepsilon(t)^2$$

Case 2:

$$\delta(t) = \frac{a_9^2}{9}, \quad \varepsilon(t) = 0, \quad a_1 = 0, \quad a_3(t) = a_3(t), \quad a_5 = 0, \quad a_7(t) = 0, \quad a_{10} = 0$$
$$a_{11}(t) = 0, \quad a_{13} = 0, \quad k_1 = 0$$

and a_2, a_4, a_6, a_8, a_9, a_{12}, k_2, k_3 are arbitrary constants. By placing the previous coefficients into transformation eq. (2), the interaction solution of eq. (4) is obtained.

$$u_2(x,y,t) = \frac{\chi_2}{\zeta_2} \tag{8}$$

where

$$\chi_2 = 2a_9 k_3 e^{a_6 y + a_8} \{a_9[k_2 \cos(a_9 x + a_{12}) + e^{a_6 y + a_8}]k_3 + a_6 k_2 \sin(a_9 x + a_{12})\}$$

$$\zeta_2 = [k_3 e^{a_6 y + a_8} \cos(a_9 x + a_{12}) + k_2]^2$$

Case 3:

$$\delta(t) = 0, \quad \varepsilon(t) = 2, \quad a_1 = 0, \quad a_3(t) = 4a_6 a_{10}^2 t, \quad a_{13} = 0, \quad a_7(t) = -4a_6 a_{10}^2 t$$

$$a_{11}(t) = \frac{1}{9} a_{10}(18a_6^2 - 19a_{10}^2)t$$

and a_2, a_4, a_5, a_6, a_8, a_9, a_{10}, a_{12}, k_i $(1 \le i \le 3)$ are arbitrary constants. The previous coefficients are substituted into transformation eq. (2) to obtain the interaction solution of eq. (4):

$$u_3(x, y, t) = \frac{\chi_3}{\zeta_3} \tag{9}$$

where

$$\chi_3 = 2a_6 a_{10} k_2 k_3 \sin\left[a_{10}(-2a_6^2 t + x - y) + \frac{19a_{10}^3 t}{9} - a_{12}\right] e^{a_6(4a_{10}^2 t + y) + a_8}$$

$$\zeta_3 = \left\{k_3 e^{a_6 y + a_8} \cos\left[a_{10}(-2a_6^2 t + x - y) + \frac{19a_{10}^3 t}{9} - a_{12}\right] + k_1 e^{a_6(4a_{10}^2 t + y) + a_4 + a_8} + k_2 e^{4a_6 a_{10}^2 t}\right\}^2$$

Case 4:

$$\delta(t) = -\frac{9a_5 a_3'(t) + 9a_5 a_7'(t) + 9a_2 a_7'(t) + 9a_2 a_{11}'(t) + a_2 a_5^3}{9a_2 a_5}, \quad \varepsilon(t) = 0, \quad a_1 = 0$$

$$a_3(t) = a_3(t), \quad a_7(t) = a_7(t), \quad a_{11}(t) = a_{11}(t), \quad a_{13}(t) = 0, \quad k_3 = 0$$

and a_2, a_5, a_6, a_8, a_9, a_{10}, a_{12}, k_1, k_2 are arbitrary constants. Utilizing the previously listed, transformation eq. (2) can be used to derive the interaction solution of eq. (4):

$$u_4(x, y, t) = 2a_5 k_2 \left\{a_5\left[\chi_4 - \frac{a_2 k_1 \zeta_4}{(k_1 \zeta_4 + k_2)^2}\right]\right\} \tag{10}$$

where

$$\chi_4 = -\frac{k_2}{[k_1 e^{a_3(t) + a_7(t) + a_2 y + a_5 y + a_4 + a_8} + k_3]^2} - \frac{1}{k_1 e^{a_3(t) + a_7(t) + a_2 y + a_5 y + a_4 + a_8} + k_2}$$

$$\zeta_4 = e^{a_3(t) + a_7(t) + a_2 y + a_5 y + a_4 + a_8}$$

Case 5:

$$\delta(t) = \frac{-9\varepsilon(t)^3 a_7'(t) + 18\varepsilon(t)^2 a_7'(t) + 26a_6^3 \varepsilon(t)^4 + 94a_6^3 \varepsilon(t)^3 + 108a_6^3 \varepsilon(t)^2 - 56a_6^3 \varepsilon(t) + 16a_6^3}{18a_6[\varepsilon(t) - 1]\varepsilon(t)^2}$$

$$\varepsilon(t) = 0, \quad a_1 = 0, \quad a_3(t) = a_3(t), \quad a_5 = \frac{2a_6[\varepsilon(t) - 1]}{\varepsilon(t)}, \quad a_7(t) = a_7(t), \quad a_9 = 0, \quad a_{10} = 0$$

$$\text{a \_\{11\}(t)} = 0, \quad a_{13} = 0, \quad k_1 = 0, \quad k_3 = 0$$

and a_2, a_6, a_8, a_9, $a_{12,}$ k_2 are arbitrary constants. Plugging the coefficients from earlier into transformation eq. (2) renders the interaction solution of eq. (4):

$$u_5(x,y,t)=\frac{\chi_5}{\zeta_5} \tag{11}$$

where

$$\chi_5=4a_6^2k_2k_3e^{\frac{\varepsilon(t)[-a_7(t)+a_6(2x-y)-a_8]-2a_6x}{\varepsilon(t)}}\cos(a_{12})[\varepsilon(t)-2][\varepsilon(t)-1]$$

$$\zeta_5=\varepsilon(t)^2\left\{k_2e^{\frac{\varepsilon(t)[-a_7(t)+a_6(2x-y)-a_8]-2a_6x}{\varepsilon(t)}}+k_3\cos(a_{12})\right\}^2$$

Case 6:

$$\delta(t)=-\frac{a_5^2}{9},\quad \varepsilon(t)=0,\quad a_1=0,\quad a_3(t)=a_3(t),\quad a_6=0,\quad a_7(t)=0,\quad a_9=0,\quad a_{11}(t)=0$$
$$a_{12}=a_{12},\quad a_{13}=0,\quad k_1=0$$

and a_3, a_5, a_8, a_{10}, $a_{12,}$ $k_{2,}$ k_3 are arbitrary constants. By placing the previous coefficients into transformation eq. (2), the interaction solution of eq. (4) is obtained:

$$u_6(x,y,t)=-\frac{\chi_6}{\zeta_6} \tag{12}$$

where

$$\chi_6=2a_5k_2k_3e^{a_5(-x)-a_8}[a_5\cos(a_{10}y+a_{12})-a_{10}\sin(a_{10}y+a_{12})]$$

$$\zeta_6=\left[k_2e^{a_5(-x)-a_8}+k_3\cos(a_{10}y+a_{12})\right]^2$$

Case 7:

$$\delta(t)=\delta(t),\quad \varepsilon(t)=\varepsilon(t),\quad a_1=0,\quad a_3(t)=a_3(t),\quad a_{11}(t)=a_{11}(t),\quad a_{13}=0,\quad k_1=0$$
$$k_2=k_2,\quad k_3=0,\quad a_5=-\frac{2a_6[\varepsilon(t)-1]}{\varepsilon(t)},\quad a_7(t)=a_7(t)$$

and a_2, a_6, a_8, a_9, $a_{12,}$ k_2 are arbitrary constants. Incorporating the determinations of the coefficients into transformation eq. (2) provides the interaction solution to eq. (4):

$$u_7(x,y,t)=\frac{\chi_7}{\zeta_7} \tag{13}$$

where

$$\chi_7=2a_5(a_2+a_5)k_1k_2e^{a_3(t)-a_7(t)-a_5x+a_2y+a_4-a_8}$$

$$\zeta_7=[k_2e^{-a_7(t)+a_5(-x)-a_8}+k_1e^{a_3(t)+a_2y+a_4}]^2$$

Case 8:

$$\delta(t)=0,\quad \varepsilon(t)=2,\quad a_1=0,\quad a_3(t)=-4a_2a_9^2t,\quad a_5=0,\quad a_{10}=-a_9$$

$$a_{11}(t)=\frac{19a_9^3t}{9}-2a_2^2a_9t,\quad a_{13}=0,\quad k_1=k_1,\quad k_2=k_2,\quad k_3=0,\quad a_6=0,\quad a_7(t)=4a_2a_9^2t$$

and a_2, a_6, a_8, a_9, $a_{12,}$ $k_{1,}$ k_2 are arbitrary constants. Substituting the value of the previous coefficients into transformation eq. (2) gives the interaction solution of eq. (4):

$$u_8(x,y,t)=\frac{\chi_8}{\zeta_8} \tag{14}$$

where

$$\chi_8=2a_2a_9k_1k_3\sin\left[\frac{1}{9}a_9(18a_2^2t-9x+9y)-\frac{19}{9}a_9^3t-a_{12}\right]e^{a_2(y-4a_9^2t)+a_4}$$

$$\zeta_8=\left\{k_1e^{a_2(y-4a_9^2t)+a_4}+k_2e^{-4a_2a_9^2t-a_8}+k_3\cos\left[\frac{1}{9}a_9(18a_2^2t-9x+9y)-\frac{19}{9}a_9^3t-a_{12}\right]\right\}^2$$

Case 9:

$$\delta(t)=\delta(t,\varepsilon(t)=0,\quad a_1=0,\quad a_3(t)=a_3(t),\quad a_7(t)=a_7(t),\quad a_{10}=0,\quad a_{11}(t)=0,\quad a_{13}=0$$

$$k_1=0,\quad k_2=0,\quad k_3=0$$

and a_2, a_5, a_6, a_8, a_9, a_{12} are arbitrary constants. Plugging the coefficients from earlier into transformation eq. (2) renders the interaction solution of eq. (4):

$$u_9(x,y,t)=-2a_9^2\tan^2(a_9x+a_{12})-2a_9^2 \tag{15}$$

Case 10:

$$\delta(t)=\frac{a_9^2}{9},\quad \varepsilon(t)=0,\quad a_1=0,\quad a_3(t)=0,\quad a_7(t)=a_7(t),\quad a_{10}=0$$

$$a_{11}(t)=0,\quad a_{13}=0,\quad k_2=0$$

and a_2, a_5, a_6, a_8, a_9, $a_{12,}$ $k_{1,}$ k_3 are arbitrary constants. Inserting the previously mentioned into eq. (2) gives the interaction solution for eq. (4):

$$u_{10}(x,y,t)=-\frac{\chi_{10}}{\zeta_{10}} \tag{16}$$

where

$$\chi_{10}=2a_9k_3(k_1e^{a_2y+a_4}[a_9\cos(a_9x+a_{12})-a_2\sin(a_9x+a_{12})]+a_9k_3$$

$$\zeta_{10}=[k_3\cos(a_9x+a_{12})+k_1e^{a_2y+a_4}]^2$$

Case 11:

$$\delta(t)=\delta(t),\quad \varepsilon(t)=0,\quad a_1=0,\quad a_2=0,\quad a_3(t)=0,\quad a_7(t)=a_7(t),\quad a_{10}=0$$

$$a_{11}(t)=0,\quad a_{13}=0,\quad k_2=0$$

and a_5, a_6, a_8, a_9, a_{12}, k_1, k_3 are arbitrary constants. The previous coefficients are substituted into transformation eq. (2) to obtain the interaction solution of eq. (4):

$$u_{11}(x,y,t) = -\frac{\chi_{11}}{\zeta_{11}} \tag{17}$$

where

$$\chi_{11} = 2a_9^2 k_3 \{k_1 e^{a_3(t)+a_4} \cos[a_{11}(t) + a_9 x + a_{12}] + k_3\}$$

$$\zeta_{11} = \{k_3 \cos[a_{11}(t) + a_9 x + a_{12}] + k_1 e^{a_3(t)+a_4}\}^2$$

Case 12:

$$\delta(t) = \delta(t), \quad \varepsilon(t) = 0, \quad a_1 = 0, \quad a_3(t) = 0, \quad a_5 = -a_6, \quad a_7(t) = a_7(t), \quad a_{10} = 0$$
$$a_{11}(t) = a_{11}(t), \quad a_{13} = 0, \quad k_1 = 0$$

and a_2, a_4, a_6, a_8, a_9, a_{12}, k_2, k_3 are arbitrary constants. The interaction solution for eq. (4) is calculated by incorporating the previous coefficients into transformation eq. (2):

$$u_{12}(x,y,t) = -\frac{\chi_{12}}{\zeta_{12}} \tag{18}$$

where

$$\chi_{12} = 2k_3(k_2 e^{-a_7(t)+a_5(-x)-a_8}\{(a_5^2 - a_9^2)\cos[a_{11}(t) + a_9 x + a_{12}] -$$
$$-2a_5 a_9 \sin[a_{11}(t) + a_9 x + a_{12})] - a_9^2 k_3\}$$

$$\zeta_{12} = \{k_2 e^{-a_7(t)+a_5(-x)-a_8} + k_3 \cos[a_{11}(t) + a_9 x + a_{12}]\}^2$$

Case 13:

$$\delta(t) = \delta(t), \quad \varepsilon(t) = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3(t) = 0, \quad a_5 = 0, \quad a_6 = 0, \quad a_7(t) = 0, \quad a_{10} = 0$$

$$a_{11}(t) = 0, \quad a_{13} = 0, \quad k_1 = k_1, \quad k_2 = k_2, \quad k_3 = k_3$$

and a_4, a_8, a_9, a_{10}, a_{12}, k_i $(1 \leq i \leq 3)$ are arbitrary constants. To acquire the interaction solution for eq. (4), the previous coefficients are put into transformation eq. (2):

$$u_{13}(x,y,t) = -\frac{\chi_{13}}{\zeta_{13}} \tag{19}$$

where

$$\chi_{13} = 2a_9^2 k_3[(e^{a_4} k_1 + e^{-a_8} k_2)\cos(a_9 x + a_{12}) + k_3]$$

$$\zeta_{13} = [k_3 \cos(a_9 x + a_{12}) + e^{a_4} k_1 + e^{-a_8} k_2]^2$$

Case 14:

$$\delta(t) = \frac{a_9^2}{9}, \quad \varepsilon(t) = 0, \quad a_1 = 0, \quad a_3(t) = 0, \quad a_5 = 0, \quad a_7(t) = 0$$
$$a_{10} = 0, \quad a_{11}(t) = 0, \quad a_{13} = 0$$

and a_2, a_4, a_6, a_8, a_9, a_{12}, k_i $(1 \le i \le 3)$ are arbitrary constants. In order to figure out the interaction solution for eq. (4), the previous coefficients are used in transformation eq. (2):

$$u_{14}(x,y,t) = -\frac{\chi_{14}}{\zeta_{14}} \tag{20}$$

where

$$\chi_{14} = 2a_9 k_3\{k_2 e^{a_6(-y)-a_8}[a_6 \sin(a_9 x + a_{12}) + a_9 \cos(a_9 x + a_{12})] + \\ + a_2 k_1(-e^{a_2 y + a_4})\sin(a_9 x + a_{12}) + a_9 k_1 e^{a_2 y + a_4}\cos(a_9 x + a_{12}) + a_9 k_3\}$$

$$\zeta_{14} = [k_3 \cos(a_9 x + a_{12}) + k_1 e^{a_2 y + a_4} + k_2 e^{a_6(-y)-a_8}]^2$$

Case 15:

$$\delta(t) = [11a_6^2\varepsilon(t)^4 + 9a_{10}^2\varepsilon(t)^4 - 46a_6^2\varepsilon(t)^3 + 54a_6^2\varepsilon(t)^2 - 14a_6^2\varepsilon(t) + 4a_6^2]\{9[\varepsilon(t)-1]\varepsilon(t)^2\}$$

$$\varepsilon(t) = \varepsilon(t), \quad a_1 = 0, \quad a_3(t) = 0, \quad a_5 = -\frac{2a_6[\varepsilon(t)-1]}{\varepsilon(t)}, \quad a_{10} = 0, \quad a_{11}(t) = 0, \quad a_{13} = 0, \quad k_1 = 0$$

$$a_7(t) = \int_a^t \frac{2a_6[2a_6^2\varepsilon(s)^3 + 18a_6^2\varepsilon(s)^2 - 2a_6^2]}{9\varepsilon(s)^2}\mathrm{d}s$$

and a_2, a_4, a_6, a_8, a_9, a_{12}, k_2, k_3 are arbitrary constants. Inserting the previously mentioned into eq. (2) gives the interaction solution for eq. (4):

$$u_{15}(x,y,t) = \frac{\chi_{15}}{\zeta_{15}} \tag{21}$$

where

$$\chi_{15} = 4a_6 k_2 k_3 e^{\frac{-2a_6\left\{\int_a^t \frac{2a_6[2a_6^2\varepsilon(s)^3 + 18a_6^2\varepsilon(s)^2 - 2a_6^2]}{9\varepsilon(s)^2}\mathrm{d}s\right\}}{9\varepsilon(t)}}\{a_{10}\varepsilon(t)\sin(a_{10}y + a_{12}) + a_6[\varepsilon(t)-2][\varepsilon(t)-1]\cdot \\ \cdot\cos(a_{10}y + a_{12})\}\left(k_3\cos(a_{10}y + a_{12}) + k_2 e^{\frac{-2a_6\left\{\int_a^t \frac{2a_6[2a_6^2\varepsilon(s)^3 + 18a_6^2\varepsilon(s)^2 - 2a_6^2]}{9\varepsilon(s)^2}\mathrm{d}s\right\}}{9\varepsilon(t)}}\right)^2$$

$$\zeta_{15} = a_{10}\varepsilon(t)\sin(a_{10}y + a_{12}) + a_6[\varepsilon(t)-2]\cos(a_{10}y + a_{12})$$

Conclusion

In this paper, the (2+1)-D ZK equation with variable coefficients is transformed into a bilinear equation by Hirota bilinear method through variables $u(x, y, t)$, and then the test functions are constructed, which are then substituted into the transformed bilinear equation to obtain the interaction solutions. This paper reveals that the Hirota bilinear method gives much opportunities to find abundant solutions with physical understandings. It can not only catalyse the growth of science and technology, but also promotes the advancement of mathematics, and it can be extended to fractal/fractional ZK equation [36-39] and the thermal displacement prediction models [40, 41], and this paper could offer a starting point for future research into non-linear problems.

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