TWO-DIMENSIONAL HEAT TRANSFER WITH MEMORY PROPERTY IN A FRACTAL SPACE

by

Jiang-Jun LIU *

Basic Teaching Department, Huanghe Jiaotong University, Wuzhi, China

Original scientific paper
https://doi.org/10.2298/TSCI2403993L

This paper considers a temperature-dependent thermal conductivity with memory property in a fractal space. The two-scale fractal derivative is adopted to model the temperature field in the spatial dimensions, and Caputo fractional derivative is used to describe its memory property. The variational iteration method is employed to solve the mixed model with great success. This paper offers a new window for studying intractable problems arising in porous media or unsmooth boundaries.

Key word: *two-dimensional heat transfer, Caputo fractional derivative, memory property, He's fractal derivative*

Introduction

It is well-known that the Caputo fractional derivative [1-6] has memory and long-range spatial correlation, and can describe physical phenomena and biochemical reaction processes with memory, heredity and path dependence properties more accurately than integer order, which are often ignored in classical integer order models.

Fractional calculus [7] is an extension of integral calculus, and it is a non-standard operator theory to describe many intractable problems with memory property which can not be treated by the traditional calculus. On the other hand, fractal calculus [8] have attracted much attention due to their wide applications to complex problems, for examples, fractal solitary waves [9, 10], porous concretes [11-13], fractal MEMS systems [14-18], however, it can not deal with the memory property.

It might be promising to couple Caputo fractional derivative and the fractal derivative to deal with phenomena with memory property in a fractal space. For this purpose, this paper considers a 2-D heat transfer equation with memory property in a fractal space, the governing equation can be expressed_

$$\mathrm{D}_t^{\alpha} u(x,y,t) = \left[\frac{\partial}{\partial x^{\beta}}\left(k_x \frac{\partial u}{\partial x^{\beta}}\right) + \frac{\partial}{\partial y^{\gamma}}\left(k_y \frac{\partial u}{\partial y^{\gamma}}\right)\right] \tag{1}$$

where u is the temperature, k_x and k_y – the heat conduction coefficients in x- and y-directions, respectively, α, β, and γ are two-scale fractal dimensions [13] in time, x- and y-directions, respectively, D_t^{α} – the Caputo time fractional derivative, and $\partial/\partial x^{\beta}$ and $\partial/\partial x^{\gamma}$ – the He's space fractal derivatives [19] in x- and y-directions, respectively:

---

* Author's, e-mail: liushuxue2004@163.com

$$\mathrm{D}_t^{\alpha}u(x,y,t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t(t-s)^{m-\alpha-1}u_s^{(m)}(x,y,s)\mathrm{d}s,\quad m-1<\alpha\le m,\quad m\in N^+,\quad x>0 \tag{2}$$

$$\frac{\partial u(x,y,t)}{\partial x^{\beta}}=\Gamma(1+\beta)\lim_{\substack{x_1-x\to\Delta x\\ \Delta x\neq 0}}\frac{u(x_1,y,t)-u(x,y,t)}{(x_1-x)^{\beta}},\quad 0<\beta\le 1 \tag{3}$$

$$\frac{\partial u(x,y,t)}{\partial y^{\gamma}}=\Gamma(1+\gamma)\lim_{\substack{y_1-y\to\Delta y\\ \Delta y\neq 0}}\frac{u(x,y_1,t)-u(x,y,t)}{(y_1-y)^{\gamma}},\quad 0<\gamma\le 1 \tag{4}$$

The mixed model given in eq. (1) involves Caputo time fractional derivative for memory property, and the fractal derivative for temperature field in a fractal space. There were much literature using either a fractional derivative or a fractal derivative, see for examples, [20-23], but the mixed model was rare and it is much promising for a complex problem.

It was reported that a porous medium has temperature-dependent heat conduction coefficients [20], in this paper we assume that:

$$k_x=k_y=\mu u^n \tag{5}$$

where μ and n are constants. So eq. (1) becomes:

$$\mathrm{D}_t^{\alpha}u(x,y,t)=\mu\left[\frac{\partial}{\partial x^{\beta}}\left(u^n\frac{\partial u}{\partial x^{\beta}}\right)+\frac{\partial}{\partial y^{\gamma}}\left(u^n\frac{\partial u}{\partial y^{\gamma}}\right)\right] \tag{6}$$

When $\alpha=\beta=\gamma=1$ and $\mu=n=1$eq. (6) is called classical Boussinesq equation [24-27]. The main purpose of the present work is to solve eq. (3) by He's fractional variational iteration method [28].

He's variational iteration method

In this section, we briefly describe the variational iteration method, which is a useful mathematical tool to solving various non-linear problems, see for examples [29-32]. To illustrate the basic idea of this method, we consider the following non-linear equation:

$$Lu(x,y,t)+Nu(x,y,t)=g(x,y,t) \tag{7}$$

where L and N are linear and non-linear operators, respectively, and g – the source inhomogeneous term.

The variational iteration method is to construct a correction functional for eq. (7) in the form:

$$u_{k+1}(x,y,t)=u_k(x,y,t)+\int_0^t\lambda\{L[u_k(x,y,s)]+N[\tilde{u}_k(x,y,s)]-g(x,y,s)\}\mathrm{d}s \tag{8}$$

where λ is the general Lagrange multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_k$ is a restricted variation which means $\delta\tilde{u}_k=0$.

The main steps of He's variational iteration method requires first the determination of Lagrange multiplier, λ. Then the successive approximations $u_{k+1}, k\ge 0$, of the solution u

will be obtained by using a given trial solution, u_0. The approximate converges fast to the exact solution, so the exact solution is:

$$u = \lim_{k\to\infty} u_k \tag{9}$$

For fractional differential equation, we consider the following general case:

$$\mathrm{D}_t^{\alpha} u(x,y,t) + \Omega[u(x,y,t)] = 0 \tag{10}$$

where Ω is a general function involving fractal derivatives. The variational iteration algorithm is:

$$u_{k+1}(x,y,t) = u_k(x,y,t) + \mathrm{J}_t^{\alpha}\{\lambda[\mathrm{D}_t^{\alpha} u_k(x,y,s)] + \Omega[\tilde{u}_k(x,y,s)]\} =$$

$$= u_k(x,y,t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(s)\{\mathrm{D}_s^{\alpha} u_k(x,y,s) + \Omega[\tilde{u}_k(x,s)]\}\mathrm{d}s \tag{11}$$

The generalized Lagrange multiplier can be approximately identified:

$$\lambda = -1 \tag{12}$$

and the following iteration formulation is obtained:

$$u_{k+1}(x,y,t) = u_k(x,y,t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{\mathrm{D}_t^{\alpha} u_k(x,y,s) + \Omega[u_k(x,y,s)]\}\mathrm{d}s \tag{13}$$

Numerical example

In this section, we consider the following 2-D heat transfer equation:

$$\mathrm{D}_t^{\alpha} u(x,y,t) = \frac{1}{2}\left[\frac{\partial}{\partial x^{\beta}}\left(u^2\frac{\partial u}{\partial x^{\beta}}\right) + \frac{\partial}{\partial y^{\gamma}}\left(u^2\frac{\partial u}{\partial y^{\gamma}}\right)\right] \tag{14}$$

subject to the initial condition:

$$u(x,y,0) = \left[\frac{x^{\beta}}{\Gamma(1+\beta)} + \frac{y^{\gamma}}{\Gamma(1+\gamma)}\right]^{\frac{1}{2}} \tag{15}$$

By using the variational iteration method, we construct a correction functional for eq. (14):

$$u_{k+1}(x,y,t) = u_k(x,y,t) - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\{\mathrm{D}_t^{\alpha} u_k(x,y,s) + \Omega_1[u_k(x,y,s)]\}\mathrm{d}s \tag{16}$$

where

$$\Omega_1(u) = -\frac{1}{2}\left[\frac{\partial}{\partial x^{\beta}}\left(u^2\frac{\partial u}{\partial x^{\beta}}\right) + \frac{\partial}{\partial y^{\gamma}}\left(u^2\frac{\partial u}{\partial y^{\gamma}}\right)\right] \tag{17}$$

Taking the initial value:

$$u_0(x,y,t) = u(x,y,0) = \left[\frac{x^{\beta}}{\Gamma(1+\beta)} + \frac{y^{\gamma}}{\Gamma(1+\gamma)}\right]^{\frac{1}{2}} \tag{18}$$

By the iteration formulation of eq. (16), we have:

$$u_1(x,y,t) = u_0(x,y,t) + \frac{t^{\alpha}}{4F^{\frac{1}{2}}\Gamma(1+\alpha)} \tag{19}$$

$$u_2(x,y,t) = u_1(x,y,t) - \frac{t^{2\alpha}}{16F^{\frac{3}{2}}\Gamma(1+2\alpha)} \tag{20}$$

$$u_3(x,y,t) = u_2(x,y,t) + \frac{3t^{3\alpha}}{64F^{\frac{5}{2}}\Gamma(1+3\alpha)} \tag{21}$$

and so on.

Hence, the 4-term approximate solution is:

$$u(x,y,t) = F + \frac{t^{\alpha}}{4F^{\frac{1}{2}}\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{16F^{\frac{3}{2}}\Gamma(1+2\alpha)} + \frac{3t^{3\alpha}}{64F^{\frac{5}{2}}\Gamma(1+3\alpha)} \tag{22}$$

where

$$F = \frac{x^{\beta}}{\Gamma(1+\beta)} + \frac{y^{\gamma}}{\Gamma(1+\gamma)}$$

which is the initial condition term.

It is obvious that the solution depends upon the initial condition of F, so the fractional derivative can give a good description of the memory property.

Conclusion

In this work, the variational iteration method has been successfully applied to obtain the approximate analytical solution of 2-D heat transfer equation involving both the fractal derivative and the fractional derivative. An example is given to illustrate the validity and accuracy of the method. The results show that the variational iteration method is efficient to handle the mixed model with memory property.

References

[1] Mendes, E. M. A. M., *et al.*, Numerical Solution of Caputo Fractional Differential Equations with Infinity Memory Effect at Initial Condition, *Communications in Non-linear Science and Numerical Simulation*, *69* (2019), Apr., pp. 237-247

[2] Gundogdu, H., Gozukizil, O. F., On the Approximate Numerical Solutions of Fractional Heat Equation with Heat Source and Heat Loss, *Thermal Science*, *26* (2022), 5A, pp. 3773-3786

[3] Yang, X. J., *Advanced Local Fractional Calculus and Its Applications,* World Science Publisher, New York, USA, 2012

[4] Yang, X. J., *et al*., *Local Fractional Integral Transforms and their Applications*, Academic Press, Pittsburgh, Penn., USA, 2015
[5] Wang, K. L., *et al*., Physical Insight of Local Fractional Calculus and Its Application to Fractional KdV-Burgers-Kuramoto Equation, *Fractals, 27* (2019), 7, 1950122
[6] Kuo, P. H., *et al*., Novel Fractional-Order Convolutional Neural Network Based Chatter Diagnosis Approach in Turning Process with Chaos Error Mapping, *Non-linear Dynamics, 111* (2023), 8, pp. 7547-7564
[7] He, J.-H., A Tutorial Review on Fractal Space-Time and Fractional Calculus, *International Journal of Theoretical Physics*, *53* (2014), 11, pp. 3698-3718
[8] He, J.-H., Fractal Calculus and Its Geometrical Explanation, *Results in Physics, 10* (2018), Sept., pp. 272-276
[9] Wang, K. L., He, C. H., A Remark on Wang's Fractal Variational Principle, *Fractals, 27* (2019), 8, 1950134
[10] He, J.-H., *et al*., Solitary Waves Travelling Along an Unsmooth Boundary, *Results in Physics, 24* (2021), 104104
[11] He, C. H., Liu, C., Fractal Approach to the Fluidity of a Cement Mortar, *Non-linear Engineering, 11* (2022), 1, pp. 1-5
[12] He, C. H., *et al*., A Fractal Model for the Internal Temperature Response of a Porous Concrete, *Applied and Computational Mathematics, 21* (2022), 1, pp. 71-77
[13] He, C. H., Liu, C., Fractal Dimensions of a Porous Concrete and Its Effect on the Concrete's Strength, *Facta Universitatis Series: Mechanical Engineering, 21* (2023), 1, pp. 137-150
[14] He, J.-H., *et al*., Pull-in Stability of a Fractal System and Its Pull-in Plateau, *Fractals, 30* (2022), 9, 2250185
[15] He, J.-H.; *et al*. Periodic Property and Instability of a Rotating Pendulum System. *Axioms, 10* (2021), 3, 191
[16] Tian, D., *et al*., Fractal N/MEMS: from Pull-in Instability to Pull-in Stability, *Fractals, 29* (2021), 2, 2150030
[17] Tian, D., He, C. H., A Fractal Micro-Electromechanical System and Its Pull-in Stability, *Journal of Low Frequency Noise Vibration and Active Control, 40* (2021), 3, pp. 1380-1386
[18] He, C. H., A Variational Principle for a Fractal Nano/Microelectromechanical (N/MEMS) System, *International Journal of Numerical Methods for Heat & Fluid Flow, 33* (2023), 1, pp. 351-359
[19] He, J.-H., El-Dib, Y. O., A Tutorial Introduction to the Two-Scale Fractal Calculus and Its Application to the Fractal Zhiber-Shabat Oscillator, *Fractals, 29* (2021), 8, 2150268
[20] Kochubei, A. N., *et al*., On Fractional Heat Equation, *Fractional Calculus and Applied Analysis, 24* (2021), 1, pp. 73-87
[21] Povstenko, Y. Z., Fractional Heat Conduction Equation and Associated Thermal Stress, *Journal of Thermal Stresses, 28* (2005), 1, pp. 83-102
[22] Liu, F. J., *et al*., Thermal Oscillation Arising in a Heat Shock of a Porous Hierarchy and Its Application, *Facta Universitatis Series: Mechanical Engineering, 20* (2022), 3, pp. 633-645
[23] He, C. H., *et al*., Taylor Series Solution for Fractal Bratu-Type Equation Arising in Electrospinning Process, *Fractals*, *28* (2020), 1, 2050011
[24] Kaur, P., Singh, S., Convective-Radiative Moving Porous fin with Temperature-Dependent Thermal Conductivity, Heat Transfer Coefficient and Wavelength-Dependent Surface Emissivity, *Multidiscipline Modeling in Materials and Structures, 19* (2023), 2, pp. 176-196
[25] Verhoest, N., Troch, P. A., Some Analytical Solutions of the Linearized Boussinesq Equation with Recharge for a Sloping Aquifer, *Water Resources Research*, *36* (2000), 3, pp.793-800
[26] Abdou, M. A., *et al*., New Application of Exp-Function Method for Improved Boussinesq Equation, *Physics Letters A, 369* (2007), 5, pp. 469-475
[27] Abassy, T. A., *et al*., Modified Variational Iteration Method for Boussinesq Equation, *Computers and Mathematics with Applications, 54* (2007), 7, pp. 955-965
[28] He, J.-H., A Short Remark on Fractional Variational Iteration Method, *Physics Letters A, 375* (2011), 38, pp. 3362-3364
[29] Wang, S. Q., He, J. H., Variational Iteration Method for Solving Integro-Differential Equations, *Physics letters A, 367* (2007), 3, pp. 188-191
[30] Deng, S. X., Ge, X. X., The Variational Iteration Method for Whitham-Broer-Kaup System with Local Fractional Derivatives, *Thermal Science*, *26* (2022), 3B, pp. 2419-2426

[31] Wang, S. Q., A Variational Approach to Non-linear Two-Point Boundary Value Problems, *Computers & Mathematics with Applications*, *58* (2009), 11, pp. 2452-2455
[32] Sun, J. S., Approximate Analytic Solution of the Fractal Fisher's Equation *via* Local Fractional Variational Iteration Method, *Thermal Science*, *26* (2022), 3B, pp. 2699-2705

Paper submitted: January 22, 2022
Paper revised: May 21, 2023
Paper accepted: May 21, 2023