SOLITARY WAVE SOLUTION FOR THE NON-LINEAR BENDING WAVE EQUATION BASED ON HE'S VARIATIONAL METHOD

by

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A beam vibration originating in the beam porous structure or on a non-smooth boundary might make its vibrating energy concentrated on a single wave, leading to a solitary wave. This paper applies the variational approach to analysis of the soliton basic property, and the effect of the fractal dimensions on the solitary wave is elucidated. This paper is to draw attention the beam soliton property beyond its widely known resonance and periodic and chaotic properties.

Key words: *fractal traveling transformation, He's fractal derivative, He's variational method, the deflection vibration equation*

Introduction

In engineering, many vibration equations are inherently non-linear due to various influencing factors [1-4], the deflection vibration a beam is a good example. The previous literature mainly focused on the vibration equation instability [5-10], chaos [11] and bifurcation [12], however, the solitary waves were rarely studied, it is quite different from the resonance, and it might be dangerous if the wave peak is too large. Kim and Hong [13] found some new solitons of the well-known Duffing oscillator, and its dynamical properties are different from the forced vibration system resonance [14].

The solitary wave originally came from the KdV equation [15], and now the fractal solitary theory [16-18] provides a connection between the solitary wave motion dynamics and the fractal dimensions, allowing the fractal geometry to control solitary wave properties. Ji, *et al*. [19] studied a transvers vibration of a porous concrete using a fractal vibration theory, and the fractal solitons were found. Kou, *et al*. [20, 21] showed that the chatter vibration can be effectively treated by the fractional convolutional neural network.

In this paper, we will consider the a porous beam vibration using Hamilton principle [22, 23] to establish a fractal vibration model, and the best-case scenario is elucidated to control the solitons geometrically.

Basic assumptions and equations

Harada and Asakura [24] considered the influence of rotational inertia based on the Rayleigh's theory, Liu, *et al*. [25] considered the geometric non-linearity effect caused by the

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maximum deflection of the beam and derived the non-linear deflection vibration equation using Hamilton's variational principle. This article is to extend the theory in [25] into a fractal space.

According to [25], for a continuous beam, we assume that:

$$u_x = -z\frac{\partial W}{\partial x}, \quad u_y = 0, \quad u_z = W \tag{1}$$

where u_x, u_y, u_z represent the displacement in the directions of x, y, z, respectively. The W is the deflection in the direction of z.

When considering the large deflection of the beam, the axial strain can be expressed:

$$\varepsilon_x = -z\frac{\partial^2 W}{\partial x^2} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 \tag{2}$$

Assuming a linear relationship between axial stress $\sigma_x$and strain ε_x, we can obtain:

$$\sigma_x = E\varepsilon_x = E\left[-z\frac{\partial^2 W}{\partial x^2} + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2\right] \tag{3}$$

where E is the elastic modulus of the material.

When considering large deflection, the strain energy per unit length of the beam can be derived using the following formula:

$$U = \frac{1}{2}\iint_S \sigma_x\varepsilon_x \mathrm{d}y\mathrm{d}z = \frac{1}{2}EI\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \frac{1}{8}ES\left(\frac{\partial W}{\partial x}\right)^4 \tag{4}$$

Subsequently, when considering rotational inertia, the kinetic energy per unit length of the beam under this condition is comprised of two components: lateral kinetic energy and rotational kinetic energy. Its expression is:

$$T = \frac{1}{2}\rho\iint_S \left[\left(\frac{\partial u_x}{\partial t}\right)^2 + \left(\frac{\partial u_z}{\partial t}\right)^2\right]\mathrm{d}y\mathrm{d}z = \frac{1}{2}\rho I\left(\frac{\partial^2 W}{\partial x\partial t}\right)^2 + \frac{1}{2}\rho S\left(\frac{\partial W}{\partial t}\right)^2 \tag{5}$$

where ρ is density of beams, $I = \iint_S z^2\mathrm{d}y\mathrm{d}z$ – the rotational inertia, and $S = \iint_S \mathrm{d}y\mathrm{d}z$ – the cross-sectional area of the beam.

The total energy density per unit length is:

$$L = T - U = \frac{1}{2}\rho I\left(\frac{\partial^2 W}{\partial x\partial t}\right)^2 + \frac{1}{2}\rho S\left(\frac{\partial W}{\partial t}\right)^2 - \frac{1}{2}EI\left(\frac{\partial^2 W}{\partial x}\right)^2 - \frac{1}{8}ES\left(\frac{\partial W}{\partial x}\right)^4 \tag{6}$$

The Euler equation reads:

$$\frac{\partial L}{\partial W} - \frac{\partial}{\partial t}\frac{\partial L}{\partial W_t} - \frac{\partial}{\partial x}\frac{\partial L}{\partial W_x} + \frac{\partial^2}{\partial x^2}\frac{\partial L}{\partial W_{xx}} + \frac{\partial^2}{\partial t^2}\frac{\partial L}{\partial W_{tt}} + \frac{\partial^2}{\partial x\partial t}\frac{\partial L}{\partial W_{xt}} - \frac{\partial^3}{\partial x^3}\frac{\partial L}{\partial W_{xxx}} ... = 0 \tag{7}$$

where

$$W_t = \frac{\partial W}{\partial t}, \quad W_x = \frac{\partial W}{\partial x}$$

Then, by substituting eq. (6) into eq. (7), we can obtain:

$$\frac{\partial^2 W}{\partial t^2}-\frac{c_0^2}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial W}{\partial x}\right)^3\right]-r_1^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 W}{\partial t^2}-c_0^2\frac{\partial^2 W}{\partial x^2}\right)=0 \tag{8}$$

Making $\omega=\partial W/\partial x$, the equation can be written:

$$\frac{\partial^2 \omega}{\partial t^2}-\frac{c_0^2}{2}\frac{\partial^2(\omega^3)}{\partial x^2}-r_1^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 \omega}{\partial t^2}-c_0^2\frac{\partial^2 \omega}{\partial x^2}\right)=0 \tag{9}$$

where

$$c_0=\sqrt{\frac{E}{\rho}}$$

represents the longitudinal elastic wave velocity and $r_1=\sqrt{I/S}$ – the radius of rotation of the cross-section relative to the neutral axis.

The article focuses on solitary wave solutions, so the following transformation is considered:

$$\omega=\omega(\xi),\ \ \xi=k(x-ct) \tag{10}$$

where k is the wave number, $k = 2\pi/\lambda$, c – the wave velocity, $c = \lambda f$, f – the frequency.

Then, by substituting eq. (10) into eq. (9), we can obtain:

$$c^2\frac{\mathrm{d}^2\omega}{\mathrm{d}\xi^2}-\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{c_0^2}{2}\omega^3\right)-r_1^2\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[(c^2-c_0^2)k^2\frac{\mathrm{d}^2\omega}{\mathrm{d}\xi^2}\right]=0 \tag{11}$$

Integrating eq. (11) twice, and setting the integral constants to be zero, we obtain:

$$c^2\omega-\frac{c_0^2}{2}\omega^3-r_1^2\left[(c^2-c_0^2)k^2\frac{\mathrm{d}^2\omega}{\mathrm{d}\xi^2}\right]=0 \tag{12}$$

Re-write eq. (12) in the form:

$$k^2\frac{\mathrm{d}^2\omega}{\mathrm{d}\xi^2}+\delta_1\omega+\delta_3\omega^3=0 \tag{13}$$

where

$$\delta_1=-\frac{c^2}{r_1^2(c^2-c_0^2)},\quad \delta_3=\frac{c_0^2}{2r_1^2(c^2-c_0^2)}$$

Equation (13) is the Duffing oscillator, it has been widely studied, see for examples [26-30]. The aforementioned derivation is for a continuum beam, for a porous beam or a beam moving along a non-smooth boundary, a fractal modification is needed.

He, *et al*. [31] unlocked that the porous concrete beam vibration can be modelled by the fractal vibration theory, and its low frequency property was found, which has made a profound impact on architectural engineering and civil engineering. It was found that the fractal dimensions of a porous beam affected greatly the vibration properties [32-34].

In this article, the porous beam is considered like that for a porous concrete. It was reported that the fractal dimensions affect the porous concrete mechanical properties [35-38] and the thermal property [39]. Hinted by above literature, we extend eq. (13) to its fractal partner in the form:

$$k^2 \frac{\mathrm{D}^2\omega}{{}^H\mathrm{D}\xi^{2\mu}} + \delta_1\omega + \delta_3\omega^3 = 0 \tag{14}$$

where μ represents the two-scale fractal dimensions [40-42] and $\mathrm{D}^2\omega/{}^H\mathrm{D}\xi^{2\mu}$ – the two-scale fractal derivative, $\xi^\mu = k(x^\mu - ct^\mu)$. The definition and the basic properties of the fractal derivative are discussed in details in [40-42], and it was widely applied to deal with complex problems, for examples, nanoscale circuits [43], nano/micro devices [44], fractal diffusion [45], fractal economics [46], fractal population dynamics [47], fractal boundary layer theory [48], fractal solitary theory [49] and electrospinning process [50]. It can also model Euler-Bernoulli beams in a microgravity space [51], and viscoelastic polymer materials vibration [52].

Solitary waves

This paper focuses itself on the solitons of the porous beam vibration. The variational formula of the non-linear bending wave equation established by the semi-inverse method [53], which is:

$$J(\omega) = \int_0^\infty \left[\frac{1}{2}\left(\frac{\mathrm{D}\omega}{{}^H\mathrm{D}\xi^\mu}\right)^2 - \frac{\delta_1}{2k^2}\omega^2 - \frac{\delta_3}{4k^2}\omega^4 \right] \mathrm{d}\xi^\mu \tag{15}$$

The semi-inverse method is an effective tool to establishment of a variational formulation from a complex differential equation, see for examples [54-58]. Fractal variational formulations were discussed in [59-61].

The Euler-Lagrange equation of eq. (15) can be described:

$$-\frac{\mathrm{D}^2\omega}{{}^H\mathrm{D}\xi^{2\mu}} - \frac{\delta_1}{k^2}\omega - \frac{\delta_3}{k^2}\omega^3 = 0 \tag{16}$$

From eq. (15), we can obtain He's modified Weierstrass function [62]:

$$H = \frac{1}{2}z^2 - \left(\frac{\delta_1}{2k^2}\omega^2 + \frac{\delta_3}{4k^2}\omega^4\right) - \left[\frac{1}{2}\left(\frac{\mathrm{D}\omega}{{}^H\mathrm{D}\xi^\mu}\right)^2 - \left(\frac{\delta_1}{2k^2}\omega^2 + \frac{\delta_3}{4k^2}\omega^4\right)\right] -$$

$$-\left(z - \frac{\mathrm{D}\omega}{{}^H\mathrm{D}\xi^\mu}\right)\frac{\mathrm{D}\omega}{{}^H\mathrm{D}\xi^\mu} = \frac{1}{2}z^2 - \frac{1}{2}\left(\frac{\mathrm{D}\omega}{{}^H\mathrm{D}\xi^\mu}\right)^2 - \left(z - \frac{\mathrm{D}\omega}{{}^H\mathrm{D}\xi^\mu}\right)\frac{\mathrm{D}\omega}{{}^H\mathrm{D}\xi^\mu} \tag{17}$$

where the variable z is defined:

$$z = \frac{\mathrm{D}\omega}{{}^H\mathrm{D}\xi^\mu} \tag{18}$$

From eq. (17), it is obvious that:

$$H(\xi,\omega,\omega^{\mu},z)=0 \tag{19}$$

and

$$\frac{\partial^2 H}{\partial p^2}>0 \tag{20}$$

Equations (20) and (19) indicate that eq. (15) is a minimal variational principle.

Subsequently, the solitary wave solution of eq. (14) can be assumed in the following form [18]:

$$\omega(\xi^{\mu})=p\,\text{sech}(\xi^{\mu}) \tag{21}$$

where p is conversion coefficient.

Substituting eq. (21) into eq. (15), we obtain:

$$J(p)=\int_0^{\infty}\left\{\frac{1}{2}[-p\tanh(\xi^{\mu})\text{sech}(\xi^{\mu})]^2-\frac{\delta_1}{2k^2}\text{sech}^2(\xi^{\mu})-\frac{\delta_3}{4k^2}\text{sech}^4(\xi^{\mu})\right\}\text{d}\xi^{\mu}=$$

$$=\left(\frac{1}{6}-\frac{\delta_1}{2k^2}\right)p^2-\frac{\delta_3}{6k^2}p^4 \tag{22}$$

According to the stability conditions of the previous equation, it can be obtained that:

$$\frac{\text{D}J}{\text{D}p}=\frac{1}{3}p-\frac{\delta_1}{k^2}p-\frac{2\delta_3}{3k^2}p^3=0 \tag{23}$$

By solving eq. (23), the value of p can be obtained:

$$p=\pm\sqrt{\frac{k^2-3\delta_1}{2\delta_3}} \tag{24}$$

So the solution of eq. (14) can be obtained:

$$\omega(\xi^{\mu})=\pm\sqrt{\frac{k^2-3\delta_1}{2\delta_3}}\,\text{sech}(\xi^{\mu}) \tag{25}$$

To sum up, we obtain the solitary wave solutions of the non-linear bending wave equation:

$$\omega(x^{\mu},t^{\mu})=\pm\sqrt{\frac{k^2-3\delta_1}{2\delta_3}}\,\text{sech}[k(x^{\mu}-ct^{\mu})] \tag{26}$$

When $\mu\to 1$, eq. (26) can be transformed into eq. (27):

$$\lim_{\mu\to 1}\omega(x^{\mu},t^{\mu})=\pm\sqrt{\frac{k^2-3\delta_1}{2\delta_3}}\,\text{sech}(\xi)=\pm\sqrt{\frac{k^2-3\delta_1}{2\delta_3}}\,\text{sech}[k(x-ct)] \tag{27}$$

Soliton property

Without losing the generality, we only consider the following situations:

$$\omega(x^{\mu},t^{\mu})=\sqrt{\frac{k^2-3\delta_1}{2\delta_3}}\,\text{sech}[k(x^{\mu}-ct^{\mu})] \tag{28}$$

Based on the assumptions made earlier, we set λ = 3000 Hz, c = 3000 m/s, $k = 2\pi$, ρ = 7850 kg/m^3, $E = 2.1 \times 10^5$ Mpa, c_0 = 5188.738 m/s, I = 490.9 mm^4, S = 75.540 mm^2, r_1 = 2.549 mm, δ_1 = –772.844/m^2, δ_3 = –1155.961/m^2. Then, we change the value of fractal dimension to draw the function image of deflection ω, fig. 1, under different fractal dimensions.

Through the analysis of fig. 1, the results indicate that the peak value of the non-linear bending wave equation remains unaffected by the fractal dimensions. However, the shape of the non-linear bending wave equation is altered by changes in the fractal dimensions.

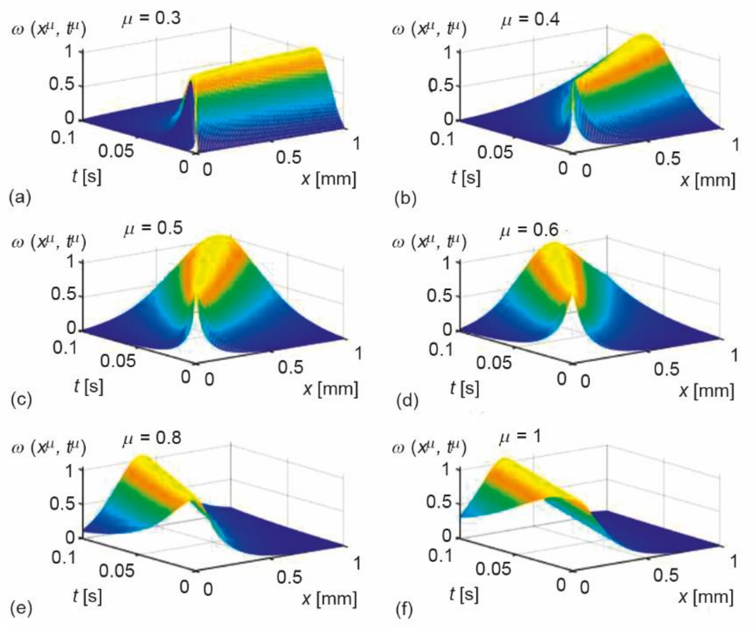

Figure 1. Traveling wave solutions of the non-linear bending wave equation with different fractal dimension

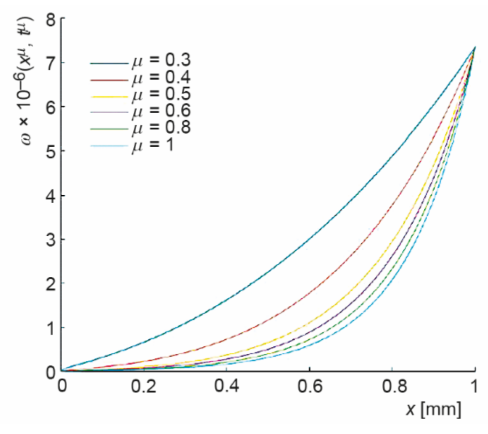

Figure 2. The non-linear bending wave equation under different fractal dimension when t = 1

Figure 2 shows that, when t = 1, the value of the fractal dimensions indirectly impacts the bending change rate by influencing the distribution of influence and wave number within the system. A larger value of the fractal dimensions results in a greater change rate of bend.

Conclusion

On the basis of considering the influence of the moment of inertia and the geometric non-linear effect caused by the large deflection of the beam, this paper derives the non-linear bending wave equation of the beam. Then, He's fractal derivatives are used to obtain the fractal form of the non-linear bending wave equation, and the corresponding solitary wave solution is obtained based on He's variational method. Finally, the images of the solitary wave solutions of the non-linear bending wave equation are illustrated for different fractal dimensions, and the conclusions are drawn: The fractal dimension can change the waveform of solitary wave solution of the non-linear bending wave equation and indirectly impacts the bending change rate by influencing the distribution of influence and wave number within the system.

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