APPROXIMATE ANALYTICAL SOLUTION OF GENERALIZED FRACTAL EQUAL-WIDTH WAVE EQUATION

by

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In this paper, a generalized equal width wave equation involving space fractal derivatives and time Caputo fractional derivatives is studied and its approximate analytical solution is presented by the Adomian decomposition method. An example shows that the method is efficient to solve fractal non-linear partial differential equations.

Key word: *generalized fractal equal width wave equation, caputo derivative, space fractal derivative, adomian decomposition method*

Introduction

Fractional calculus has been received a skyrocketing attention from mathematics, physics to engineering, it can model effectively a seemingly stochastic diffusion [1, 2] or seemingly irregular Brownian motion [3], it can be also used for noise detection [4], and thermal displacement prediction [5]. The machine learning or deep learning [6-10] will become more powerful to deal with innumerable data for neural networks and imagine processing, and now a fractional model makes a complex and anomalous problem accessible.

A wave near the coast is affected by various random factors including the unsmooth boundary and its memory property, the traditional models can not model these factors, and a fractal-fractional model has to be considered. In this study, we consider the following generalized fractal-fractional equal width wave equation:

$$\mathrm{D}_t^\alpha u(x,t) + a u^k \frac{\partial u}{\partial x^\beta} - \mathrm{D}_t^\alpha \left(b \frac{\partial u}{\partial x^{2\beta}} \right) = 0, \quad 0 < \alpha \le 1, \quad 0 < \beta \le 1 \tag{1}$$

subject to the initial condition:

$$u(x,0) = \phi \left[\frac{x^\beta}{\Gamma(1+\beta)} \right] \tag{2}$$

where D_t^α is the Caputo time fractional derivative of order α [11, 12], $\partial/\partial x^\beta$ – the He's space fractal derivative of order β [13-16], ϕ – the known function, and a, k, and b – the positive parameters.

When $\alpha = \beta = 1$, eq. (1) was first introduced by Morrison, *et al.* [17] as the model equation to describe non-linear dispersive waves, and it has a simple relation with the Benja-

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min-Bona-Mahony equation [18-20]. Other similar equations were widely appeared in literature, and much achievement has been obtained. In this paper, we consider a modification with the Caputo time-fractional derivative and He's space fractal derivative [13-16]. This adaptation considers that the Caputo fractional derivative has the memory property [21], and He's fractal derivative can effectively model phenomena in a porous medium [22-27].

Though eq. (1) can model exactly a memorial wave travelling along an unsmooth boundary, it is difficult to be solved analytically, some famous analytical methods, *e.g.*, the homotopy perturbation method [28-32], are not appliable to eq. (1) directly. This paper adopts the Adomian decomposition method (ADM) [33-36] for this purpose.

Basic definitions

In this section, we recall the basic definitions of fractional calculus and fractal calculus which shall be used in this paper. For more details see [37-41].

Definition 1. A real function $f(x), x>0$ is said to be in the space $C_\lambda, \lambda \in R$ if there exists a real number $p>\lambda,$ such that $f(x)=x^p f_1(x)$ where $f_1(x)\in C[0,\infty)$ and it is said to be in the space C_n if and only if $f^{(n)} \in C_\lambda,\ n\in N.$

Definition 2. The Riemann-Liouville fractional integral operator of order $\alpha>0$ of a function $f(x)\in C_\lambda, \lambda \geq -1$ is defined:

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)\mathrm{d}s \tag{3}$$

while the definition of Riemann-Liouville fractional derivative is:

$${}^{RL}\mathrm{D}_t^q u(x,t)=\frac{\partial^n}{\partial t^n}[J_t^{n-q}u(x,t)] \tag{4}$$

where $n-1\leq q<n$ and n is an integer.

Properties of the operator J^α can be found in [37] and we mention only the following: For $\alpha,\beta\geq 0, x>0,$ and $\lambda>-1$:

$$J^\alpha J^\beta f(x)=J^{\alpha+\beta} f(x) \tag{5}$$

$$J^\alpha J^\beta f(x)=J^\beta J^\alpha f(x) \tag{6}$$

$$J^\alpha (x^\lambda)=\frac{\Gamma(\lambda+1)}{\Gamma(1+\alpha+\lambda)}x^{\lambda+\alpha} \tag{7}$$

Definition 3. The time fractional derivative of $u(x, t)$ in Caputo sense is defined:

$$\mathrm{D}_t^\alpha u(x,t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}u_s^{(m)}(x,s)\mathrm{d}s \tag{8}$$

for $m-1<\alpha\leq m, m\in N^+, x>0,$ and $u(x,t)\in C_{-1}^m.$

Definition 4. The space fractal derivative of $u(x, t)$ in the He's sense is defined as [41]:

$$\frac{\partial u(x,t)}{\partial x^\beta}=\Gamma(1+\beta)\lim_{\substack{x_1-x\to\Delta x\\ \Delta x\neq 0}}\frac{u(x_1,t)-u(x,t)}{(x_1-x)^\beta}, \quad 0<\beta\leq 1 \tag{9}$$

and

$$\frac{\partial u(x,t)}{\partial x^{2\beta}} = \frac{\partial}{\partial^{\beta}}\left[\frac{\partial u(x,t)}{\partial x^{\beta}}\right] \tag{10}$$

The fractal derivative is also called as the two-scale fractal derivative, and it can be widely applied to various unsmooth boundary value problems and porous medium problems, [42-47].

Adomian decomposition method

The ADM [33-36] is a technique for solving nonlinear equations in the form:

$$u(x,t) = \tau + \Theta(u) \tag{11}$$

where $\Theta : H \to H$ is a non-linear mapping from a Banach space H into itself and $\tau \in H$ is known.

The Adomian decomposition method assumes that the solution u can be expanded as an infinite series:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \tag{12}$$

and the non-linear term $\Theta(u)$ can be decomposed:

$$\Theta\left(\sum_{n=0}^{\infty} u_n\right) = \sum_{n=0}^{\infty} A_n(u) \tag{13}$$

where the polynomials $A_n(u)$ are given by:

$$A_n(u_0, u_1, \cdots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial q^n}\left[\Theta\left(\sum_{k=0}^{n} q^k u_k\right)\right]_{q=0}, \qquad n = 0, 1, 2, \cdots \tag{14}$$

Substituting eqs. (12) and (13) into (11) gives:

$$\sum_{n=0}^{\infty} u_n = \tau + \sum_{n=0}^{\infty} A_n \tag{15}$$

which is satisfied formally if we set:

$$u_0(x,t) = \tau \tag{16}$$

$$u_1 = A_0 \tag{17}$$

$$\vdots$$

$$u_{m+1} = A_m \tag{18}$$

Then k-term approximate solution of eq. (11) is given by $u = u_0 + u_1 + \cdots + u_{k-1}$.

Approximate solutions

In this section, we derive the main algorithm for solving the problem (1)-(2). Now, eq. (1) can be written:

$$L_t u(x,t) = {}^{RL}\mathrm{D}_t^{1-\alpha}\left[a u^k \frac{\partial u}{\partial x^\beta} + \mathrm{D}_t^\alpha\left(b\frac{\partial u}{\partial x^{2\beta}}\right)\right] \tag{19}$$

where

$$L_t \equiv \frac{\partial}{\partial t}$$

which is an easily invertible linear operator.

Taking the operator L_t^{-1} on both sides of eq. (19), we obtain:

$$u(x,t) = u(x,0) + L_t^{-1}\left[{}^{RL}\mathrm{D}_t^{1-\alpha}\left(a u^k \frac{\partial u}{\partial x^\beta}\right)\right] + b\frac{\partial u}{\partial x^{2\beta}} \tag{20}$$

Suppose that the solutions take the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \tag{21}$$

and the non-linear term can be decomposed:

$$u^k \frac{\partial u}{\partial x^\beta} = \sum_{n=0}^{\infty} A_n \tag{22}$$

where the polynomials A_n can be expressed:

$$A_0 = u_0^k \frac{\partial}{\partial x^\beta} u_0 \tag{23}$$

$$A_1 = k u_0^{k-1} u_1 \frac{\partial}{\partial x^\beta} u_0 + u_0^k \frac{\partial}{\partial x^\beta} u_1 \tag{24}$$

$$A_2 = \frac{k(k-1)}{2} u_0^{k-2} u_1^2 \frac{\partial u_0}{\partial x^\beta} + k u_0 u_2 \frac{\partial}{\partial x^\beta} + \frac{k}{2} u_0^{k-1} u_1^2 + u_0^k \frac{\partial u}{\partial x^\beta} \tag{25}$$

and so on.

Therefore, by using the ADM, we have:

$$u_0(x,t) = u(x,0) \tag{26}$$

$$u_1(x,t) = a L_t^{-1}\left({}^{RL}\mathrm{D}_t^{1-\alpha} A_0\right) + b\frac{\partial u_0}{\partial x^{2\beta}} \tag{27}$$

$$u_2(x,t) = aL_t^{-1}\left({}^{RL}\mathrm{D}_t^{1-\alpha} A_1 \right) + b\frac{\partial u_1}{\partial x^{2\beta}} \tag{28}$$

$$u_3(x,t) = aL_t^{-1}\left({}^{RL}\mathrm{D}_t^{1-\alpha} A_2 \right) + b\frac{\partial u_2}{\partial x^{2\beta}} \tag{29}$$

and so on.

Then k-term approximate solutions of eq. (19) are given by $u = u_0 + u_1 + \ldots + u_{k-1}$. To illustrate the effectiveness of the previous algorithm, we consider a special case of eqs. (1) and (2):

$$\mathrm{D}_t^{\alpha} u(x,t) = -u^2 \frac{\partial u}{\partial x^{\beta}} + \mathrm{D}_t^{\alpha}\left(\frac{\partial u}{\partial x^{2\beta}} \right) \tag{30}$$

with the initial condition:

$$u(x,0) = \frac{2\sqrt{6}E}{E^2+1} \tag{31}$$

where

$$E = \exp\left[\frac{-x^{\beta}}{\Gamma(1+\beta)} \right]$$

By eqs. (26)-(29), and (23)-(25), we obtain:

$$\begin{gathered} u_0 = \frac{2\sqrt{6}E}{E^2+1} \\ u_1(x,t) = \frac{-2\sqrt{6}\,E(E^2-1)t^{\alpha}}{(E^2+1)^2\Gamma(1+\alpha)} \\ u_2(x,t) = \frac{-2\sqrt{6}\,E(6E^2-E^4-1)t^{2\alpha}}{(E^2+1)^3\Gamma(1+2\alpha)} \\ u_3(x,t) = \frac{2\sqrt{6}\,E(E^2-1)(22E^2-E^4-1)t^{3\alpha}}{(E^2+1)^4\Gamma(1+3\alpha)} \\ \vdots \end{gathered} \tag{32}$$

Thus, the 4-term approximate solution of problem (30)-(31) is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) \tag{33}$$

where $u_i\,(i = 0 \sim 3)$ are given in eq. (32).

When $\alpha = \beta = 1$, eqs. (30) and (31) become, respectively:

$$\frac{\partial u}{\partial t} + u^2\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^2 \partial t} = 0 \tag{34}$$

$$u(x,0) = \sqrt{6}\cosh^{-1}(x). \tag{35}$$

The approximate solution is:

$$u(x,t)=\frac{2\sqrt{6}\,\mathrm{e}^{-x}}{1+\mathrm{e}^{-2x}}+\frac{-2\sqrt{6}\mathrm{e}^{-x}(\mathrm{e}^{-2x}-1)t}{(1+\mathrm{e}^{-2x})^2}-\frac{\sqrt{6}\,\mathrm{e}^{-x}(6\mathrm{e}^{-2x}-\mathrm{e}^{-4x}-1)t^2}{(1+\mathrm{e}^{-2x})^3}+$$

$$+\frac{\sqrt{6}\,\mathrm{e}^{-x}(\mathrm{e}^{-2x}-1)(22\mathrm{e}^{-2x}-\mathrm{e}^{-4x}-1)t^3}{3(1+\mathrm{e}^{-2x})^4} \tag{36}$$

which converges to the exact solution if the iteration process continues.

Conclusion

In this paper, the ADM has been successfully applied to obtaining the approximate analytical solution of generalized fractal equal width wave equation. The example is given to illustrate the validity and accuracy of the method. The results show that the method is efficient to handle fractal wave equations with singularity [48] or He's fractional derivative [49, 50].

References

[1] Metzler, R., Klafter, J., The Random Walk's Guide to Anomalous Diffusion: A Fractional dynamics Approach, *Physics Reports-Review Section of Physics Letters*, *339* (2000), 1, pp. 1-77
[2] He, J.-H., Qian, M. Y., A Fractal Approach to the Diffusion Process of Red Ink in a Saline Water, *Thermal Science*, *26* (2022), 3, pp. 2447-2451
[3] Mandelbrot, B. B., Vanness, J. W., Fractional Brownian Motions, Fractional Noise and Applications, *SIAM Review, 10* (1968), 4, pp. 422-437
[4] Kuo, P. H., *et al*., Novel Fractional-Order Convolutional Neural Network Based Chatter Diagnosis Approach in Turning Process with Chaos Error Mapping, *Non-linear Dynamics, 111* (2023), 8, pp. 7547-7564
[5] Kuo, P. H., *et al*., A Thermal Displacement Prediction System with an Automatic LRGTVAC-PSO Optimized Branch Structured Bidirectional GRU Neural Network, *IEEE Sensors Journal, 23* (2023), 12, pp. 12574-12586
[6] You, S. R., *et al*., Fine Perceptive Gans for Brain Mr Image Super-Resolution in Wavelet Domain, *IEEE Transactions on Neural Networks and Learning Systems, 34* (2023), 11, pp. 8802-8814
[7] Hu, S. Y., *et al*., Bidirectional Mapping Generative Adversarial Networks for Brain MR to PET Synthesis, *IEEE Transactions on Medical Imaging, 41* (2021), 1, pp. 145-157
[8] Yu, W., *et al*., Morphological Feature Visualization of Alzheimer's Disease via Multidirectional Perception GAN, *IEEE Transactions on Neural Networks and Learning Systems, 34* (2023), 8, pp. 4401-4415
[9] Wang, S. Q., *et al*., Prediction of Myelopathic Level in Cervical Spondylotic Myelopathy Using Diffusion Tensor Imaging, *Journal of Magnetic Resonance Imaging, 41* (2015), 6, pp. 1682-1688
[10] Wu, K., *et al*., 3D Convolutional Neural Network for Regional Precipitation Nowcasting, *Journal of Image and Signal Processing, 7* (2018), 4, pp. 200-212
[11] Almeida, R., A Caputo Fractional Derivative of a Function with Respect to Another Function, *Communications in Nonlinear Science and Numerical Simulation, 44* (2017), Mar., pp. 460-481
[12] He, J.-H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *International Journal of Theoretical Physics, 53* (2014), Nov., pp. 3698-3718
[13] He, J. H., Fractal Calculus and Its Geometrical Explanation, *Results in Physics, 10* (2018), Sept., pp. 272-276
[14] Wang, K. L., *et al*., Physical Insight of Local Fractional Calculus and Its Application to Fractional KdV-Burgers-Kuramoto Equation, *Fractals, 27* (2019), 7, 1950122
[15] Wang, K. L., He, C. H., A Remark on Wang's Fractal Variational Principle, *Fractals*, *27* (2019), 8, 1950134
[16] He, J.-H., *et al*., Solitary Waves Travelling Along an Unsmooth Boundary, *Results in Physics, 24* (2021), 104104

[17] Morrison, P. J., *et al.,* Scattering of Regularized-Long-Wave Solitary Waves, *Physica D, 11* (1984), 3, pp. 324-336
[18] Benjamin, T. B., *et al*., Model Equations for Long Waves in Non-linear Dispersive Systems, *Philosophical Transactions of the Royal Society of London, Series A, 272* (1972), 1220, pp. 47-78
[19] Mancas, S. C., *et al*., Weierstrass Traveling Wave Solutions for Dissipative Benjamin, Bona, and Mahony (BBM) Equation, *Journal of Mathematical Physics, 8* (2013), 54, pp. 1-49
[20] Carvajal, X., Panthee, M., On Sharp Global Well-Posedness and Ill-Posedness for a Fifth-Order KdV-BBM Type Equation, *Journal of Mathematical Analysis and Applications, 1* (2019), 479, pp. 688-702
[21] Mendes, E. M. A. M., *et al*., Numerical Solution of Caputo Fractional Differential Equations with Infinity Memory Effect at Initial Condition, *Communications in Nonlinear Science and Numerical Simulation, 69* (2019), Apr., pp. 237-247
[22] Sun, Z. Q., A Variational Approach to a Porous Catalyst, *Thermal Science, 25* (2021), 2, pp. 1437-1442
[23] He, C. H., *et al*., Low Frequency Property of a Fractal Vibration Model for a Concrete Beam, *Fractals, 29* (2021), 5, 2150117
[24] He, C. H., Liu, C., Fractal Approach to the Fluidity of a Cement Mortar, *Non-linear Engineering, 11* (2022), 1, pp. 1-5
[25] He, C. H., *et al*., A Fractal Model for the Internal Temperature Response of a Porous Concrete, *Applied and Computational Mathematics, 21* (2022), 1, pp. 71-77
[26] He, C. H., *et al*., A Novel Bond Stress-Slip Model for 3-D Printed Concretes, *Discrete and Continuous Dynamical Systems-Series S, 15* (2022), 7, pp. 1669-1683
[27] He, C. H., Liu, C., Fractal Dimensions of a Porous Concrete and Its Effect on the Concrete's Strength, *Facta Universitatis Series: Mechanical Engineering, 21* (2023), 1, pp. 137-150
[28] He, C. H., El-Dib, Y. O., A Heuristic Review on the Homotopy Perturbation Method for Non-Conservative Oscillators, *Journal of Low Frequency Noise Vibration and Active Control, 41* (2022), 2, pp. 572-603
[29] He, J.-H., *et al*., Homotopy Perturbation Method for Strongly Non-linear Oscillators, *Mathematics and Computers in Simulation, 204* (2023), Feb., pp. 243-258
[30] He, J.-H., *et al*., Homotopy Perturbation Method for Fractal Duffing Oscillator with Arbitrary Conditions, *Fractals, 30* (2022), 9, 22501651
[31] He, J.-H., *et al*., A Good Initial Guess for Approximating Non-linear Oscillators by the Homotopy Perturbation Method, *Facta Universitatis, Series: Mechanical Engineering, 21* (2023), 1, pp. 21-29
[32] He, J.-H., *et al*., Forced Non-Linear Oscillator in a Fractal Space, *Facta Universitatis, Series: Mechanical Engineering, 20* (2022), 1, pp. 1-20
[33] Wazwaz, A. M., A Comparison Between Adomian Decomposition Method and Taylor Series Method in the Series Solutions, *Applied Mathematics and Computation*, *97* (1998), 1, pp. 37-44
[34] Wazwaz, A. M., Adomian Decomposition Method for a Reliable Treatment of the Bratu-Type Equations, *Applied Mathematics and Computation*, *166* (2005), 3, pp. 652-663
[35] Rach, R., *et al*., A Reliable Modification of the Adomian Decomposition Method for Higher-Order Non-linear Differential Equations, *Kybernetes*, *42* (2013), 2, pp. 282-308
[36] Jafari, H., Daftardar-Gejji, V., Positive Solutions of Non-linear Fractional Boundary Value Problems Using Adomian Decomposition Method, *Applied Mathematics and Computation, 180* (2006), 2, pp. 700-706
[37] Kilbas, A. A., Marzan, S. A., Non-linear Differential Equations with the Caputo Fractional Derivative in the Space of Continuously Differentiable Functions, *Differential Equations*, *41* (2005), 1, pp. 84-89
[38] Mouffak, *et al*., Non-linear Boundary Value Problems for Differential Inclusions with Caputo Fractional Derivative, *Topological Methods in Nonlinear Analysis, 32* (2008), 1, pp. 115-130
[39] Qian, M. Y., He, J. H., Two-Scale Thermal Science for Modern Life: Making the Impossible Possible, *Thermal Science, 26* (2022), 3B, pp. 2409-2412
[40] Anjum, N., *et al*., Two-Scale Fractal Theory for the Population Dynamics, *Fractals*, *29* (2021), 7, 2150182
[41] He, J.-H., El-Dib, Y. O., A Tutorial Introduction to the Two-Scale Fractal Calculus and Its Application to the Fractal Zhiber-Shabat Oscillator, *Fractals, 29* (2021), 8, 2150268
[42] He, J.-H., *et al*., Pull-in Stability of a Fractal System and Its Pull-in Plateau, *Fractals*, *30* (2022), 9, 2250185
[43] Tian, D., *et al*., Fractal N/MEMS: from Pull-in Instability to Pull-in Stability, *Fractals*, *29* (2021), 2, 2150030

[44] Tian, D., He, C. H., A Fractal Micro-Electromechanical System and Its Pull-In Stability, *Journal of Low Frequency Noise Vibration and Active Control, 40* (2021), 3, pp. 1380-1386
[45] He, C. H., A Variational Principle for a Fractal Nano/Microelectromechanical (N/MEMS) System, *International Journal of Numerical Methods for Heat & Fluid Flow, 33* (2023), 1, pp. 351-359
[46] Liu, F. J., *et al*., Thermal Oscillation Arising in a Heat Shock of a Porous Hierarchy and Its Application, *Facta Universitatis Series: Mechanical Engineering*, *20* (2022), 3, pp. 633-645
[47] He, C. H., Liu, C., A Modified Frequency-Amplitude Formulation for Fractal Vibration Systems, *Fractals*, *30* (2022), 3, 2250046
[48] He, C. H., Liu, C., Variational Principle for Singular Waves, *Chaos, Solitons & Fractals, 172* (2023), 113566
[49] Zhang, X. Q., Approximate Analytical Solution of the Kudryashov-Sinelshchikov Equation with He's Fractional Derivative, *Thermal Science*, *27* (2023), 3A, pp. 1795-1802
[50] Lu, J. F., Ma, L., Numerical Analysis of Space-Time Fractional Benjamin-Bona-Mahony Equation, *Thermal Science, 27* (2023), 3A, pp. 1755-1762

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