SOLITARY WAVE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS BY HE'S VARIATIONAL METHOD

by

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Existence of variational principles for Navier-Stokes equations has been discussing for hundreds of years, but it has not yet been solved. In this study, a new perspective is proposed, which uses a traveling wave transform, so that a variational formulation can be established. Furthermore, the solitary wave solutions are solved by He's variational method.

Key word: *Navier-Stokes millennium-prize problem, traveling wave transform, solitary wave solutions, He's variational methods, He-Weierstrass function, variational principle*

Introduction

Any a motion should follow a nature law, the most famous one is the Hamilton principle [1-3], which is a minimum variational principle. Navier-Stokes equations describe the motion of a fluid, which can be expressed:

$$\begin{gathered}\frac{\partial \xi}{\partial s}+(\xi\cdot\nabla)\xi=-\frac{1}{\rho}\nabla p+\frac{\mu}{\rho}\nabla^2\xi\\ \nabla\xi=0\end{gathered} \tag{1}$$

where p is the pressure, ρ – the density, μ – the kinematic viscosity, and s – the time. In 3-D space, velocity vector is $\xi=\xi(\xi_1,\xi_2,\xi_3)$, the components of Navier-Stokes equations in *i*-, *j*-, and *k*-directions are given by the following equations:

$$\begin{gathered}\frac{\partial \xi_1}{\partial s}+\xi_1\frac{\partial \xi_1}{\partial i}+\xi_2\frac{\partial \xi_1}{\partial j}+\xi_3\frac{\partial \xi_1}{\partial k}=\frac{\mu}{\rho}\left(\frac{\partial^2\xi_1}{\partial i^2}+\frac{\partial^2\xi_1}{\partial j^2}+\frac{\partial^2\xi_1}{\partial k^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial i}\\ \frac{\partial \xi_2}{\partial s}+\xi_1\frac{\partial \xi_2}{\partial i}+\xi_2\frac{\partial \xi_2}{\partial j}+\xi_3\frac{\partial \xi_2}{\partial k}=\frac{\mu}{\rho}\left(\frac{\partial^2\xi_2}{\partial i^2}+\frac{\partial^2\xi_2}{\partial j^2}+\frac{\partial^2\xi_2}{\partial k^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial j}\\ \frac{\partial \xi_3}{\partial s}+\xi_1\frac{\partial \xi_1}{\partial i}+\xi_2\frac{\partial \xi_3}{\partial j}+\xi_3\frac{\partial \xi_3}{\partial k}=\frac{\mu}{\rho}\left(\frac{\partial^2\xi_3}{\partial i^2}+\frac{\partial^2\xi_3}{\partial j^2}+\frac{\partial^2\xi_3}{\partial k^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial k}\end{gathered} \tag{2}$$

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where

$$\frac{\partial \xi_1}{\partial i}+\frac{\partial \xi_2}{\partial j}+\frac{\partial \xi_3}{\partial k}=0$$

Navier-Stokes equations should also follow a variational principle, though much effort has been paid, its existence is still a big problem. Scientists only find some variational formulations for simple fluids [4-6]. A variational principle can give profound, original, and challenging insights of a fluid problem, especially the travelling waves.

The well-known KdV equations [7-9] is the approximate case of the Navier-Stokes equations, there are various variational principles for KdV equations [10-12], and the modern soliton theory is originally developed from the KdV equation. This paper aims at searching for solitary waves directly from the Navier-Stokes equations by establishment of a suitable variational principle.

Variational principle

The variational formulation for eq. (1) is extremely difficult to be obtained. This paper is to search for solitary waves from Navier-Stokes equations, so we focus ourselves on a constrained variational formulation by the following transformations [13-16]:

$$\begin{aligned}&\xi_1(i,j,k,s)=\Xi(\varepsilon)\\&\xi_2(i,j,k,s)=\mathrm{A}(\varepsilon)\\&\xi_3(i,j,k,s)=\mathrm{B}(\varepsilon)\\&p(i,j,k,s)=P(\varepsilon)\end{aligned}\tag{3}$$

$$\varepsilon=\lambda_1 i+\lambda_2 j+\lambda_3 k-cs+\varepsilon_0\tag{4}$$

$$\begin{aligned}&\alpha\xi_2=\xi_1\\&\beta\xi_3=\xi_1\\&\vartheta p=\xi_1\end{aligned}\tag{5}$$

where α, β, and ϑ are all non-zero functions.

Based on the previous transformation, we can convert the Navier-Stokes equations into the following ordinary differential equation:

$$-\mathrm{c}\Xi'+\lambda_1\Xi\cdot\Xi'+\frac{\lambda_2}{\alpha}\Xi\cdot\Xi'+\frac{\lambda_3}{\beta}\Xi\cdot\Xi'=\frac{\mu\lambda_1^{\,2}}{\rho}\Xi''+\frac{\mu\lambda_2^{\,2}}{\rho}\Xi''+\frac{\mu\lambda_3^{\,2}}{\rho}\Xi''-\frac{\lambda_1}{\rho\vartheta}\Xi'\tag{6}$$

Through the previous equation, we have:

$$-\frac{\mu}{\rho}(\lambda_1^{\,2}+\lambda_2^{\,2}+\lambda_3^{\,2})\Xi''+\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\Xi\cdot\Xi'+\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\Xi'=0\tag{7}$$

Integrating (7), we have:

$$-\frac{\mu}{\rho}(\lambda_1^{\,2}+\lambda_2^{\,2}+\lambda_3^{\,2})\Xi'+\frac{1}{2}\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\Xi^2+\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\Xi=H\tag{8}$$

According to (7) and (8), we have:

$$\Xi' = \frac{\rho}{\mu(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)}\left[\frac{1}{2}\left(\lambda_1 + \frac{\lambda_2}{\alpha} + \frac{\lambda_3}{\beta}\right)\Xi^2 + \left(\frac{\lambda_1}{\rho\vartheta} - c\right)\Xi - H\right] \tag{9}$$

Hence, eq. (7) can be represented:

$$\frac{\frac{2\mu^2}{\rho}(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)^2\Xi'' - \rho\left(\lambda_1 + \frac{\lambda_2}{\alpha} + \frac{\lambda_3}{\beta}\right)^2\Xi^3 - 3\rho\left(\lambda_1 + \frac{\lambda_2}{\alpha} + \frac{\lambda_3}{\beta}\right)\left(\frac{\lambda_1}{\rho\vartheta} - c\right)\Xi^2}{2\mu(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)} +$$

$$+\frac{-2\rho\left(\frac{\lambda_1}{\rho\vartheta} - c\right)^2\Xi + 2\rho H\left(\lambda_1 + \frac{\lambda_2}{\alpha} + \frac{\lambda_3}{\beta}\right)\Xi + 2\rho H\left(\frac{\lambda_1}{\rho\vartheta} - c\right)}{2\mu(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)} = 0 \tag{10}$$

So the variational formulation of (10) can be established by the semi-inverse method [17], which is:

$$J(\Xi) = \int\left[-\frac{\mu}{2\rho}(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)(\Xi')^2 - \frac{\rho\left(\lambda_1 + \frac{\lambda_2}{\alpha} + \frac{\lambda_3}{\beta}\right)^2}{12\mu(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)}\Xi^4 - \right.$$

$$\left. - \frac{\rho\left(\lambda_1 + \frac{\lambda_2}{\alpha} + \frac{\lambda_3}{\beta}\right)\left(\frac{\lambda_1}{\rho\vartheta} - c\right)}{2\mu(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)}\Xi^3\right]\mathrm{d}\varepsilon +$$

$$+\int\left[-\rho\frac{\left(\frac{\lambda_1}{\rho\vartheta} - c\right)^2 - H\left(\lambda_1 + \frac{\lambda_2}{\alpha} + \frac{\lambda_3}{\beta}\right)}{2\mu(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)}\Xi^2 + \frac{\rho\left(\frac{\lambda_1}{\rho\vartheta} - c\right)H}{\mu(\lambda_1{}^2 + \lambda_2{}^2 + \lambda_3{}^2)}\Xi\right]\mathrm{d}\varepsilon \tag{11}$$

The semi-inverse method [17] has been widely used to establish a suitable variational formulation from a governing equation, for examples, the variational principle for singular waves [18], water waves [19], nano/microelectromechanical systems [20], two-point boundary value problems [20], KdV-Burgers-Kuramoto equation [21], 3D unsteady fluids [22], thin films [23], solitary waves [24], long water waves [25], Schrodinger equation [26].

From eq. (11), He-Weierstrass function [27] can be obtained:

$$E(\varepsilon, \Xi, \Xi', \omega) = \frac{1}{2}\omega^2 - \frac{1}{2}(\Xi')^2 - (\omega - \Xi')\Xi' \tag{12}$$

where $\omega = \partial\Xi/\partial\varepsilon$.

From eq. (12), It is evident that:

$$E(\varepsilon, \Xi, \Xi', \omega) = 0, \quad \frac{\partial^2 E}{\partial\omega^2} > 0 \tag{13}$$

Equation (13) shows that eq. (12) is a minimal variational principle.

Solitary wave solutions

The objective of this section is to identify solitary wave solutions for Navier-Stokes equations by the obtained variational principle. The idea goes back to [28] variational approach to solitons, and it has been showing its validity for various wave equations, for examples, Boussinesq equation [29] and various the wave equations [30-36].

According to He's variational theory [28], we assume that the solitary solution of eq. (11) take the following form:

$$\Xi(\varepsilon) = \kappa \operatorname{sech}^2(\upsilon\varepsilon) \tag{14}$$

where $\kappa \neq 0,\ \upsilon \neq 0,\ \kappa$ and υ are unknown constants to be determined later.

Upon simultaneous solution of eq. (11) and eq. (14), the resulting expression is shown [28]:

$$J(\kappa,\upsilon) = \int_0^\infty \left\{ -\frac{2\mu}{\rho}(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)\kappa^2\upsilon^2[\operatorname{sec}h^2(\upsilon\varepsilon)tg(\upsilon\varepsilon)]^2 - \right.$$
$$\left. -\frac{\rho\kappa^4}{12\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)}\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)^2[\operatorname{sec}h^2(\upsilon\varepsilon)]^4 \right\}\mathrm{d}\varepsilon -$$
$$-\int_0^\infty \left\{ \frac{\rho\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\kappa^3}{2\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)}[\operatorname{sec}h^2(\upsilon\varepsilon)]^3 + \right.$$
$$\left. +\frac{\left[\left(\frac{\lambda_1}{\rho\vartheta}-c\right)^2-\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)H\right]\rho\kappa^2}{2\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)}[\operatorname{sec}h^2(\upsilon\varepsilon)]^2 \right\}\mathrm{d}\varepsilon +$$
$$+\int_0^\infty \left\{ \frac{\rho H\kappa}{\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)}\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\operatorname{sec}h^2(\upsilon\varepsilon) \right\}\mathrm{d}\varepsilon \tag{15}$$

The following are the results obtained:

$$J(\kappa,\upsilon) = -\frac{\rho\kappa 28\dfrac{\mu^2}{\rho^2}(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2\kappa\upsilon^2}{105\upsilon\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)} -$$
$$-\frac{\rho\kappa\left[4\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)^2\kappa^3+28\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\left(\frac{\lambda_1}{\rho\vartheta}-c\right)\kappa^2\right]}{105\upsilon\mu\left(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2\right)} +$$
$$+\frac{\rho\kappa\left[-35\left(\frac{\lambda_1}{\rho\vartheta}-c\right)^2\kappa+35H\left(\lambda_1+\frac{\lambda_2}{\alpha}+\frac{\lambda_3}{\beta}\right)\kappa+105\left(\frac{\lambda_1}{\rho\vartheta}-c\right)H\right]}{105\upsilon\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)} \tag{16}$$

Considering He's variational method [28], it gives:

$$\frac{\partial J}{\partial \kappa}=0 \tag{17}$$

$$\frac{\partial J}{\partial \upsilon}=0 \tag{18}$$

When

$$\left(\frac{\lambda_1}{\rho\vartheta}-c\right)$$

tends to zero, the eq. (16) can bring the following results:

$$\frac{-56\dfrac{\mu^2}{\rho}(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2\kappa\upsilon^2-16\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)^2\rho\kappa^3+70\mathrm{H}\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)\rho\kappa}{105\upsilon\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)}=0 \tag{19}$$

$$\frac{-28\dfrac{\mu^2}{\rho}(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2\kappa^2\upsilon^2+4\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)^2\rho\kappa^4-35H\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)\rho\kappa^2}{105\upsilon^2\mu(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)}=0 \tag{20}$$

Solving eqs. (19) and (20) we can determine κ and υ:

$$\kappa=\sqrt{\frac{35H}{6\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}} \tag{21}$$

$$\upsilon=\rho\sqrt{\frac{5\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}{12H\mu^2(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2}} \tag{22}$$

So the eq. (14) can be replaced by:

$$\Xi(\varepsilon)=\sqrt{\frac{35H}{6\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}}\,\mathrm{sech}^2\left[\rho\sqrt{\frac{5\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}{12H\mu^2(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2}}(\lambda_1 i+\lambda_2 j+\lambda_3 k-cs+\varepsilon_0)\right] \tag{23}$$

With eq. (11), the solitary wave solution of eq. (1) can be approximated:

$$\Xi(i,j,k,s)=\sqrt{\frac{35H}{6\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}}\,\mathrm{sech}^2\cdot$$

$$\cdot\left[\rho\sqrt{\frac{5\left(\lambda_1+\dfrac{\lambda_2}{\alpha}+\dfrac{\lambda_3}{\beta}\right)}{12H\mu^2(\lambda_1{}^2+\lambda_2{}^2+\lambda_3{}^2)^2}}(\lambda_1 i+\lambda_2 j+\lambda_3 k-cs+\varepsilon_0)\right] \tag{24}$$

Two examples

This section gives two examples to verify the accuracy and effectiveness of He's variational method [28].

Example 1. Consider variables in eq. (24), let $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\alpha = 2$, $\beta = 2$, $\rho = 1$, $\vartheta = 1$, $c = 3$, $\mu = -1$, and $H = 1$.

When $j = 0$, $k = 0$, and $\varepsilon_0 = 1$, we obtain the solitary wave solution, fig. 1(a) in a single direction:

$$\Xi(i,j,k,s) = \sqrt{\frac{35}{18}}\operatorname{sech}^2\left[\sqrt{\frac{5}{144}}(2i-3s+1)\right] \tag{25}$$

Example 2. In this case, we use $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\alpha = 2$, $\beta = 1$, $\rho = -2$, $\vartheta = 1$, $c = -3$, $\rho = 2$, and $H = 2$.

When $i = 0$, $k = 0$, and $\varepsilon_0 = -2$, the solitary wave solution, fig. 1(b), in another single direction can be obtained:

$$\Xi(i,j,k,s) = \sqrt{\frac{7}{3}}\operatorname{sech}^2\left[\frac{5}{14}\sqrt{\frac{1}{24}}(2j+3s-2)\right] \tag{26}$$

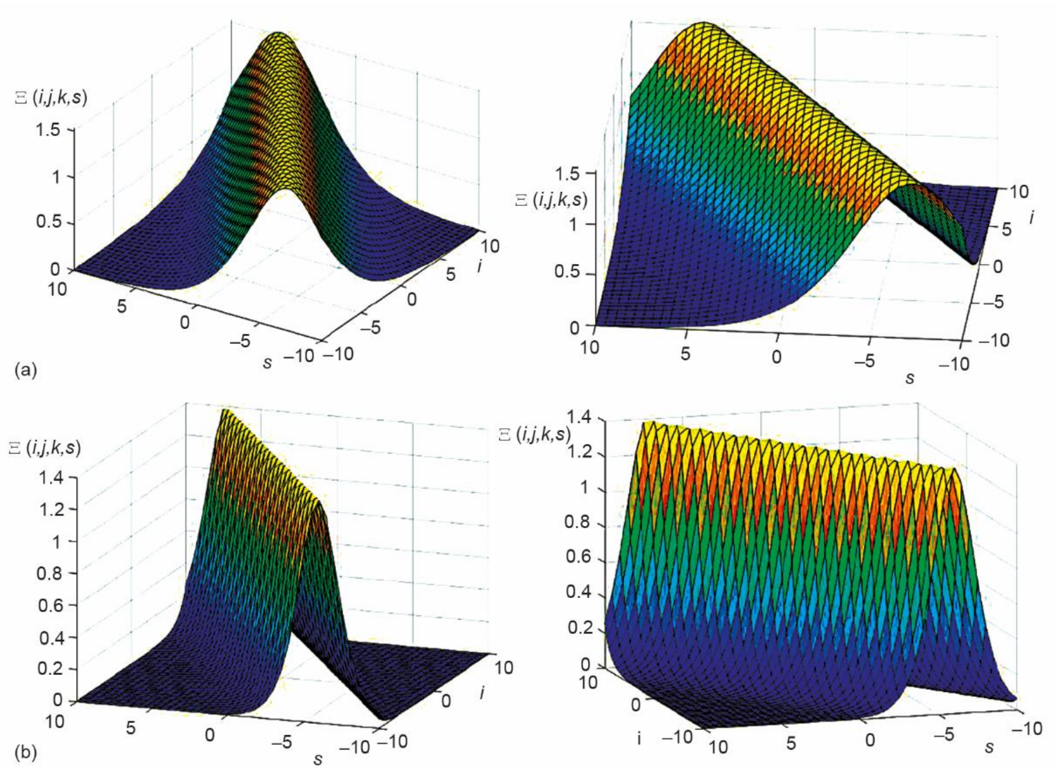

Figure 1. Solitary wave solutions for Navier-Stokes equations; (a) $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 1$, $\alpha = 2$, $\beta = 2$, $\rho = 1$, $\vartheta = 1$, $c = 3$, $\mu = -1$, and $H = 1$, (b) $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, $\alpha = 2$, $\beta = 1$, $\rho = -2$, $\vartheta=1$, $c = -3$, $\mu = 2$, and $H = 2$

Conclusions

Navier-Stokes millennium-prize problem is still an open problem [37], and there might have not exact solution to the Navier-Stokes equations, though the model has been widely used to explain various unsolved problems, *e.g.*, the mountain-river-desert relation [38, 39] and the dynamical properties of a rotating rigid body containing a viscous incom-pressible fluid [40]. This paper gives an alternative approach to the open problem, and exact solutions exist for solitary waves.

In this work, the solitary wave solution of Navier-Stokes equations has been obtained by He's variational method. This paper offers a totally new window for searching for solitary wave solutions directly from Navier-Stokes equations instead of its various approximate forms like KdV equation or Burgers equation, making the soliton theory much accurate to model the solitary waves.

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