A REMARK ON A STRONG MINIMUM CONDITION OF A FRACTAL VARIATIONAL PRINCIPLE

by

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The fractal variational principle gives a good physical understanding of a discontinuous problem in an energy way, and it is a good tool to revealing the physical phenomenon which cannot be done by the traditional variational theory. A minimum variational principle is very important in ensuring the convergence of artificial intelligence algorithms for numerical simulation and image processing. The strong minimum condition of a fractal variational principle in a fractal space is discussed, and two examples are given to illustrate its simplicity and feasibility.

Key words: *strong minimum condition, fractal variational principle, He's fractal derivative, Toda oscillator, MEMS system*

Introduction

The variational principle [1-3] is an energy functional that can represent energy conservation during artificial intelligence algorithm and numerical simulation, it can also reveal possible solution structures for a complex problem. The variational theory is widely used in physics, mechanics, thermal science, and engineering. The well-known virtual work principle, the minimum potential energy principle, the complementary energy principle and the Hamilton principle have been widely studied and applied [4-7]. Now, the variational principle has become the theoretical basis of numerical methods for the medical image computing [8-14], chaotic systems monitoring [15, 16], and fractal-fractional calculus [17-20], but it is usually difficult to find an exact functional, and the semi-inverse method [21] seems to the best candidate for searching for a suitable variational formulation from the governing equations.

The variational principle is also used for the finite element method [22], and the minimum variational principle is very important in ensuring the convergence of the numerical simulation. The Weierstrass theorem [23] can be used to identify the minimum variational principle, and the most famous minimum principles are the Hamilton principle and the least action principle [24].

The variational theory opens an energy approach to complex physical problems in continuous space and the fractal space as well [25-29]. As an unprecedented application, the fractal variational principle has given birth to a new branch of mathematics, that is the fractal solitary theory [30-34]. In 2021, He *et al.* [35] suggested a simple but effective criterion for judgment of a minimum variational principle by extending the Weierstrass theorem to fractal variational principles in non-continuum mechanics to study the strong minimum. This paper is

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to verify the criterion for the strong minimum condition of a fractal variational principle using the two examples.

A strong minimum condition of a fractal variational principle

Consider the following fractal variational formula [35]:

$$\mathrm{J}(y)=\int L\left(t,y,\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)\mathrm{d}t^{\varepsilon} \tag{1}$$

with the constraint:

$$\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}=\tau \tag{2}$$

where L is the Lagrange function and the Euler-Lagrange equation has the form [35]:

$$\frac{\mathrm{d}L}{\mathrm{d}y}-\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\frac{\mathrm{d}L}{\mathrm{d}y_{t^{\varepsilon}}}=0 \tag{3}$$

where $\mathrm{d}y/\mathrm{d}t^{\varepsilon}$ is the two-scale fractal derivative and defined [36-40]:

$$\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}(t_0^{\varepsilon})=\Gamma(1+\varepsilon)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{y(t^{\varepsilon})-y(t_0{}^{\varepsilon})}{(t-t_0)^{\varepsilon}} \tag{4}$$

where Γ is the gamma function, Δt – the time scale, and ε – the fractal dimensions [41], $\varepsilon\in R$. An approximate continuum can be predicted when $t>\Delta t$, while random properties can be observed when $t<\Delta t$. The fractal calculus can be used to explain the discontinuous phenomena and it is helpful in establishing a governing equation in a fractal space [42-45].

When $\varepsilon=1$, we can easily have:

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\Gamma(2)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{y(t)-y(t_0)}{t-t_0}=y'$$

Similarly, when $\varepsilon=2$:

$$\frac{\mathrm{d}y}{\mathrm{d}t^2}=\Gamma(3)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{y(t^2)-y(t_0^2)}{(t-t_0)^2}=y''$$

It is easy to find that the two-scale fractal derivative agrees with the traditional differential derivative when the value of the fractal dimensions ε is a positive integer [46].

To better understand the fractional derivative, take the function $y=t^{\mu}$ as an example. Using eq. (4), we can obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t^{\varepsilon}}t^{\mu}=\frac{\Gamma(1+\mu)\Gamma(1+\varepsilon-N)}{\Gamma(1+\mu-N)}t^{\mu-\varepsilon} \tag{5}$$

where N is a natural number, $N\le\varepsilon$.

With the help of Taylor's series, the derivatives of all elementary functions could also be calculated. We can obtain another form of the fractional derivative. For example, the Taylor's series of sin$t$ is

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}t^{2k+1}$$

By eq. (5), we can get:

$$\frac{\mathrm{d}}{\mathrm{d}t^{\varepsilon}}\sin t=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\mathrm{d}}{\mathrm{d}t^{\varepsilon}}t^{2k+1}=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\Gamma(2+2k)\Gamma(1+\varepsilon-N)}{\Gamma(2+2k-N)}t^{2k+1-\varepsilon} \tag{6}$$

After simple calculations, it yields the following results:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sin t=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\Gamma(2+2k)\Gamma(1+1-1)}{\Gamma(2+2k-1)}t^{2k+1-1}=\cos t \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t^{1.5}}\sin t=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{\Gamma(2+2k)\Gamma(1+1.5-1)}{\Gamma(2+2k-1)}t^{2k+1-1.5}=\frac{\sqrt{\pi}}{2}t^{-0.5}\cos t \tag{8}$$

It is obvious that the fractal derivative is useful and convenient to study.

A generalized variational principle was established [35] which is:

$$\mathrm{J}(y,\tau)=\int\left[L(t,y,\tau)-L\left(t,y,\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)-\left(\tau-\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)\frac{\partial L}{\partial\tau}\right]\mathrm{d}t^{\varepsilon} \tag{9}$$

Introducing a modified Weierstrass function [35]:

$$W\left(t,y,\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}},\tau\right)=L(t,y,\tau)-L\left(t,y,\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)-\left(\tau-\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)\frac{\partial L}{\partial\tau} \tag{10}$$

The strong minimum condition for fractal variational principles of eq. (1) is:

$$W\left(t,y,\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}},\tau\right)\geq 0,\quad \frac{\partial^2 W}{\partial\tau^2}>0 \tag{11}$$

The solution processes

The strong minimum condition of a fractal variational principle is so simple that everyone can use it. This paper uses two examples to elucidate the identification process.

Example 1: the well-known Toda oscillator [42, 43]:

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+\mathrm{e}^y-1=0 \tag{12}$$

Consider the oscillator in a fractal space, eq. (12) can be described:

$$\frac{\mathrm{d}}{\mathrm{d}t^{\varepsilon}}\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}+\mathrm{e}^y-1=0 \tag{13}$$

The variational principle of the fractal nonlinear oscillator is obtained according to the semi-inverse method:

$$\mathrm{J}(y)=\int\left\{\frac{1}{2}\left(\frac{\mathrm{d}y}{dt^{\varepsilon}}\right)^{2}-\mathrm{e}^{y}+y\right\}\mathrm{d}t^{\varepsilon} \tag{14}$$

The Lagrange function L is:

$$L=\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)^{2}-\mathrm{e}^{y}+y$$

so the modified Weierstrass function is:

$$W\left(t,y,\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}},\tau\right)=\frac{1}{2}\tau^{2}-\mathrm{e}^{y}+y-\left[\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)^{2}-\mathrm{e}^{y}+y\right]-\left(\tau-\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}=$$

$$=\frac{1}{2}\tau^{2}-\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)^{2}-\left(\tau-\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}} \tag{15}$$

It is obvious that:

$$W\left(t,y,\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}},\tau\right)=0,\quad \frac{\partial^{2}W}{\partial\tau^{2}}=1>0 \tag{16}$$

So eq. (14) is a minimum principle.

Example 2: the MEMS system [47-50]:

$$\frac{\mathrm{d}^{2}y}{\mathrm{d}t^{2}}+y=\frac{\theta}{(1-y)^{2}},\quad \theta\geq 0 \tag{17}$$

Equation (17) can be described in the fractal space:

$$\frac{\mathrm{d}}{\mathrm{d}t^{\varepsilon}}\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}+y-\frac{\theta}{(1-y)^{2}}=0 \tag{18}$$

The Lagrange function L is:

$$L=\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)^{2}-\frac{y^{2}}{2}+\frac{\theta}{1-\mathrm{y}}$$

and the variational principle of the fractal nonlinear oscillator is:

$$\mathrm{J}(y)=\int\left\{\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)^{2}-\frac{y^{2}}{2}+\frac{\theta}{1-\mathrm{y}}\right\}\mathrm{d}t^{\varepsilon} \tag{19}$$

The modified Weierstrass function:

$$W\left(t, y, \frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}, \tau\right) = \frac{1}{2}\tau^2 - \frac{y^2}{2} + \frac{\theta}{1-y} - \left[\frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)^2 - \frac{y^2}{2} + \frac{\theta}{1-y}\right] - \left(\tau - \frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}} =$$

$$= \frac{1}{2}\tau^2 - \frac{1}{2}\left(\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)^2 - \left(\tau - \frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}}\right)\frac{\mathrm{d}y}{\mathrm{d}t^{\varepsilon}} \tag{20}$$

Similar, eq. (19) is a minimum principle of eq. (17).

Discussion and conclusion

This paper shows the criterion to determine a minimum variational principle is extremely simple and reliable, and it gives an opportunity to find many potential applications of the fractal variational principles in future in various fields, and scientists might carve different paths for different technical applications especially in fractal solitary waves and artificial intelligence.

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