A MULTIPOLE FAST ASYMPTOTIC ALGORITHM FOR A CLASS OF EQUATIONS BASED ON THE FLOW FUNCTION METHOD WITH FRACTIONAL ORDER LAPLACE TRANSFORM

by

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Original scientific paper
https://doi.org/10.2298/TSCI2403361D

As a mature and reliable method, this study is based on the flow function method for mathematical modeling and establishes a class of mathematical models that are approximately realistic, flexible, and easy to calculate. According to the characteristics of fractional order calculus, the initial boundary conditions are modified and optimized to reduce the model error of this class of equations. According to the minimum energy principle and linearized integral calculation method, the multi-field multi-parameter non-linear coupling problem in the calculation process is solved, and the rapid calculation of the initial boundary model is realized. The accuracy of the model is tested by numerical simulation and simulation validation of different processes. A reliable theoretical and technical support is provided for the calculation of this type of equations.

Key words: *fractional order, Laplace transform, asymptotic*

Introduction

With fast development of fractional-order partial differential equations, their numerical algorithm has become an urgent problem to reveal the basic properties of the fractional models [1-5]. We know that most PDE do not get analytical solutions, and for fractional order PDE, even if they can get analytical solutions, their analytical solutions also have special functional terms (Gamma function, Beta function, Mitag-Leffler function) [6-8]. Therefore, the study of numerical algorithms of fractional order PDE and their algorithmic analysis is a crucial topic, which is also a hot spot and a difficult point in the development of fractional order calculus in recent years. The difficulties are: firstly, the regularity of the solutions of fractional order PDE has a subtle influence on the stability and convergence of the numerical algorithm, which requires us to conduct a rigorous numerical analysis in the process of studying the numerical algorithm to ensure the effectiveness of the algorithm. Secondly, the fractional order operator is actually a calculus operator, which contains both integration and differentiation, and the discretization of the fractional order operator requires both the complexity of the discretization of the fractional order operator is much greater than that of the PDE of integer order, therefore, it is more important to construct an efficient, stable and convergent numerical algorithm for a specific PDE of fractional order.

The Caputo space fractional order heat equation can be used to describe many complex natural phenomena in physics [1], biology [2, 3] and mechanics [9-12]. For example, the

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fractional-order Allen-Cahn equation [13] can be used to model the formation of mesoscale morphological patterns and interface motions. Fractional-order Gray-Scott [5] can be used to describe the mechanisms of different graph generation. The fractional order Bloch-Torrey model has also been successfully applied to magnetic resonance [14-16].

We discuss the following Caputo fractional order differential equation [6]:

$$ {}_0^c\mathrm{D}_t^\alpha u(\mathrm{x},t) = f[t, v(t)], \quad t \in [0,T] \tag{1} $$

$$ u^{(k)}(0) = u_0^{(k)}, \quad k = 0,1,\cdots,[\alpha] \tag{2} $$

$$ u(x,0) = u(x,T) = 0, \; x \in R^3 \setminus \Omega \tag{3} $$

where $\Omega = (a,b)\times(c,d)\times(m,n)$ is the 3-D region, $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$ is the vectorized 2-D spatial variable. $t \in (0,T]$ is the time variable, $1 < \alpha_s < 2(s=1,2)$ is the order of the derivative, ν is the kinematic viscosity, $\nu > 0, k > 0,$ are the given real numbers. $u = u(x,t)$ is a complex-valued function with respect to time t and space x, and $u_0(x)$ is a given smooth function. The operator ${}_0^c\mathrm{D}_t^\alpha u(x,t)$ denotes the Caputo fractional order derivative operator of order α_s. The $u = u(x,t)$ denotes the velocity field and is a function of space $x \in \Omega$ with respect to time $t \in [0,T]$, $p = p(x,t)$ denotes the pressure field, $f = f(x,t)$ is the external force, and $u_0 = u_0(x)$ is the initial velocity.

Appropriate theory and methodss

First, we introduce some basic inequalities and related conclusions that will be used for the analysis and solution of this equation.

Definition 1. The function $v(\mathbf{x})$ is zero-extended to the whole $\mathbb{R}^d$, and then defined by the Fourier transform $\mathcal{F}[(-\Delta)^\alpha v](\omega) = |\omega|^{2\alpha} \mathcal{F}[v](\omega)$.

That is, $(-\Delta)^\alpha v(x) = \mathcal{F}^{-1}[|\omega|^{2\alpha} \mathcal{F}[v](\omega)](x)$.

If the function extends to the whole space $\mathbb{R}^d$, then there is the equivalent expression is:

$$ (-\Delta)^\alpha v(x) = C_{d,\alpha} \text{ p.v.} \int_{\mathbb{R}^d} \frac{v(x) - v(\tau)}{|x-\tau|^{n+2\alpha}} \mathrm{d}\tau \tag{4} $$

where p.v. is the Coasean principal value and $C_{d,s}$ is a positive constant.

Definition 2. Definition of the spectral decomposition by the Laplace operator $-\Delta$:

$$ (-\Delta)^\alpha v = \sum_{n=1}^{\infty} \hat{v}_n \lambda_n^\alpha \psi_m, \quad \hat{v}_n = \int_\Omega v\varphi_n \tag{5} $$

where $\{\varphi_n\}_{n=1}^\infty$ is an orthogonal basis function in $L^2(\Omega)$ space of norm one and for any positive integer $n \ge 1$. $-\Delta\varphi_n = \lambda_n\varphi_n, \quad \varphi_n\big|_{\partial\Omega} = 0.$

We write $\{\lambda_n, \varphi_n\}$ for the eigenvalues of the previous problem with unit orthogonal eigenfunctions.

Definition 3. Discrete fractional order Sobolev parametrization, defining the function spac $S_N = \mathrm{span}\{g_j(x), j = 0,1,\cdots, N-1\}$, and the trigonometric polynomial $g_j(x)$:

$$g_j(x) = \frac{1}{N}\sum_{k=-N/2}^{N/2}\frac{1}{c_k}\mathrm{e}^{\mathrm{i}k\mu\left(x-x_j\right)},\ \text{where}\ c_k = \begin{cases} 1, |k| < \dfrac{N}{2} \\ 2, |k| = \dfrac{N}{2} \end{cases} \quad \mu = \frac{2\pi}{L}$$

Subsequently, define the interpolation operator:

$$I_N : L^2(\Omega) \to S_N\text{, and } I_N u(x) = \sum_{j=0}^{N-1} u_j g_j(x) = \sum_{k=-N/2}^{N/2}\hat{u}_k \mathrm{e}^{\mathrm{i}k\mu(x-\mathrm{a})}$$

where

$$\hat{u}_k = \frac{1}{Nc_k}\sum_{j=0}^{N-1} u_j \mathrm{e}^{-\mathrm{i}k\mu\left(x_j - a\right)}, \quad -\frac{N}{2} \le k \le \frac{N}{2}, \quad \hat{u}_{-N/2} = \hat{u}_{N/2}\,.$$

Inversion is $u_j = (I_N u)(x_j) = \sum\limits_{k=-N/2}^{N/2-1}\hat{u}_k \mathrm{e}^{\mathrm{i}k\mu\left(x_j - a\right)}$.

For any $u \in l_h^2 := \{u \mid u \in \mathcal{V}_h, \| u \|_h^2 < \infty\}$, we have:

$$\hat{u} \in l^2 := \{x = \hat{u} \in l^2 = \left\{ x = \{x_k\} \mid \sum_{k=-\infty}^{\infty} x_k^2 < \infty \right\}$$

according to Parseval's theorem we get

$$(u, v)_h = \sum_{k=-N/2}^{N/2-1}\hat{u}_k \overline{v}_k$$

Given the constant $\sigma \in [0,1]$, we define the discrete fractional order Sobolev parametrization $\|\cdot\|_{H^\sigma}$ and the semi-parametrization $|\cdot|_{H_n^\sigma}$ to be.

Obviously, $\| u \|_{H^\sigma}^2 = \| u \|_h^2 + | u |_{H^\sigma}^2, | u |_{H_n^0}^2 = \| u \|_h^2$. It is easy to prove that the discrete Sobolev space defined above constitutes a fugitive linear space, and then we introduce the following lemma, which is important for the analysis of the unconditional convergence of the conservation Fourier fitting method.

The fractional-order Laplacian $(-\Delta)^{2/2}$ acts on the interpolation function to obtain the discrete fractional-order Laplacian operator:

$$(-\Delta)^{\frac{\mathrm{a}}{2}} I_N u(x) = \frac{1}{N}\sum_{j=0}^{N-1} u_j \sum_{p=-N/2}^{N/2}\frac{1}{c_p} |\mu p|^\alpha\ \mathrm{e}^{\mathrm{i}p\mu\left(x-x_j\right)}$$

Therefore:

$$(-\Delta)^{\frac{\Omega}{2}} I_N u(x_k) = \sum_{p=-N/2}^{N/2-1} d_p \left(\frac{1}{N}\sum_{j=0}^{N-1} u_j \mathrm{e}^{-\frac{2\pi ij;j}{N}}\right)\mathrm{e}^{\frac{2\pi ijpk}{N}} \quad \text{where} \quad d_p = |\mu p|^\alpha, \quad -\frac{N}{2} \le p \le \frac{N}{2} - 1$$

For $U \in \mathcal{V}_h$, define the discrete fractional order Laplacian $(-\Delta)_d^{a/2}$ as:

$$\left[(-\Delta)_d^{\frac{\mathrm{a}}{2}} U\right]_k = \sum_{p=-N/2}^{N/2-1} d_p \left(\frac{1}{N}\sum_{j=0}^{N-1} U_j \mathrm{e}^{-\frac{2\pi\mathrm{iniv}}{N}}\right)\mathrm{e}^{\frac{2\mathrm{inNeN}}{N}} \tag{6}$$

According to the Caputo fractional order derivative, we can get a important result:

$$\frac{\partial^\alpha u(x,y,t)}{\partial |x|^\alpha} = -\frac{1}{2\cos\frac{\alpha\pi}{2}}\left[{}_a\mathrm{D}_x^\alpha u(x,y,t) + {}_x\mathrm{D}_b^\alpha u(x,y,t)\right] \tag{7}$$

$${}_a\mathrm{D}_x^\alpha u(x,y,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_a^x \frac{u(\xi,y,t)}{(x-\xi)^{\alpha-1}}\mathrm{d}\xi$$

$${}_x\mathrm{D}_b^\alpha u(x,y,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_x^b \frac{u(\xi,y,t)}{(\xi-x)^{\alpha-1}}\mathrm{d}\xi$$

where ${}_a\mathrm{D}_x^\alpha u(x,y,t)$ and ${}_x\mathrm{D}_b^\alpha u(x,y,t)$ are the left and right Riemann-Liouville fractional order derivatives, respectively.

Solution of the equation

Using Holder's inequality and Young's inequality together [6, 7], the previous equation is equivalent to the following Volterra integral equation through the fractional order Laplace transform.

$$x(t) = \sum_{k=0}^{[\alpha]-1}\frac{t_{n+1}^k}{k!}x_0^{(k)} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f[\tau, x(\tau)]\mathrm{d}\tau \tag{8}$$

where $h = T/N, N \in \mathbb{Z}, t_n = nh (n = 0,1,\cdots,N)$

Let $x_h(t) \approx x(t)$ be the approximate solutions, using the stream function method, we get:

$$x_h(t_{n+1}) = \sum_{k=0}^{[\alpha]-1}\frac{t_{n+1}^k}{k!}x_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f[t_{n+1}, x_h^p(t_{n+1})] + \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{j,n+1} f[t_j, x_h(t_j)] \tag{9}$$

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha+1}, \text{ if } j = 0 \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} - 2(n-j+1)^{\alpha+1}, \text{ if } 1 \le j \le n \\ 1, \text{ if } j = n+1 \end{cases} \tag{10}$$

$$x_h^p(t_{n+1}) = \sum_{k=0}^{[\alpha]-1}\frac{t_{n+1}^k}{k!}x_0^{(k)} + \frac{1}{\Gamma(\alpha)} b_{j,n+1} f[t_j, x_h(t_j)] \tag{11}$$

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}[(n+1-j)^\alpha - (n-j)^\alpha], \quad 0 \le j \le n \tag{12}$$

Using x denotes the horizontal coordinate in the direction of the tangent line at any point. By parabolizing, the geometric equations are:

$$y_1(x) = R + h + \frac{x^2}{2R}$$
$$y_2(x) = R - \frac{x^2}{2R} \tag{13}$$
$$y_3(x) = \left(y - R - \frac{h}{2}\right)\left[y - y_2(x)\right]\left[y - y_1(x)\right]$$

Multipole fast representations ϕ_I, ϕ_II and ϕ_III are expressed:

$$\phi_1 = \varphi \frac{y - y_1(x)}{H}\left(y - R - \frac{h}{2}\right) \tag{14}$$

$$\phi_\mathrm{II} = \varphi \frac{y - y_2(x)}{y_1(x) - y_2(x)} + \varphi(ax^2 + b) \tag{15}$$

$$\phi_\mathrm{III} = \varphi \frac{y - y_3(x)}{h}\left[y_2(x) - y_1(x)\right] \tag{16}$$

where $\varphi = v_0 H = v_1 h$ denotes the flow; $\varphi(ax^2 + b)[y - y_1(x)]$ $[y - y_2(x)]$ denotes the additional flow function field, $(ax^2 + b)$ $[y_2(x) - y_1(x)$ $]$ \$ denotes the shape correction function of the additional stream function field, $[y - y_1(x)][y - y_2(x)]$ denotes the boundary mixing control function, and $(y - R - h/2)$ is the neutral layer control function. This leads to the geometric relationship of the heat transfer line Γ.

– According to the characteristics of each boundary region, in order to facilitate the calculation and optimize the zone boundary conditions, so that the boundary conditions are closer to the actual situation and provide an important reference for strain modeling, we unify the front-slip zone and back-slip zone with the division method of thermal conductivity to avoid complex stress analysis calculation. The flow function method is used to establish the kinematic allowable velocity field closer to the real one, and the Gaussian integral model is used to solve the difficult problem of solving the non-linear model under multi-field and multi-variable coupling, which can greatly shorten the calculation time and simplify the calculation.

Notice that:

$$\lim_{x \to l}\left(\frac{x^2}{2R}\right) = \Delta h, \quad l = \sqrt{2R\Delta h}$$

we can further derive the velocity discontinuity Γ expressions, So when given the parameters, one can get

$$h = (1-\varepsilon)H, \quad \Delta h = \varepsilon H, \quad R = \frac{l^2}{2\varepsilon H}, \quad \bar{x} = \frac{x}{l}, \quad \bar{y} = \frac{y - y_2(x)}{H}, \tag{17}$$

$$y_1(\bar{x}) = 1 + \frac{1}{\varepsilon} + \frac{\varepsilon \bar{x}^2}{2} - \varepsilon, \quad y_2(\bar{x}) = \frac{1}{\varepsilon} - \frac{\varepsilon \bar{x}^2}{2}, \quad y_3 = \frac{\bar{x}}{1-\varepsilon}, \tag{18}$$

$$\phi_\mathrm{I} = \varphi \bar{y}\left[\bar{y} - y_1(\bar{x})\right], \quad \phi_2 = \left[\bar{y} - y_2(\bar{x})\right]\frac{\varphi \bar{y}}{1-\varepsilon}, \quad \phi_3 = \frac{\bar{y} - \dfrac{1}{\varepsilon}}{1 - \varepsilon \bar{x}^2} + \varphi(a\bar{x}^2 + b)\left(\bar{y} - \frac{1}{\varepsilon}\right) \tag{19}$$

$$\bar{y} = 1.25(1+\varepsilon)^2 - \frac{\dfrac{1}{1-\varepsilon} - \dfrac{1}{1+\varepsilon}}{a\bar{x}^2 + b} \tag{20}$$

– The flow function method belongs to the class of energy methods, in which the thermally deformed body is considered as an incompressible object and the thermoplastic deformation problem is considered as a sourceless field with zero dispersion of the velocity vector. In each individual deformation region, the velocity field and its derivative (strain rate field) are continuous. Based on the nature of the flow function, the thermoplastic deformation region is solved for the partial derivatives of the flow line ϕ_{II}, and the thermoplastic motion-permitted velocity field and strain-rate field are obtained:

$$v_{x_1} = \frac{\partial \phi_1}{\partial \bar{x}_1} = \frac{\varphi}{1-\varepsilon+\varepsilon\bar{x}^2} - \varphi(a\bar{x}^2+b)\left(\bar{y} - \frac{1}{\varepsilon} + \frac{\varepsilon+1}{2}\right)\left(\bar{y} - \frac{1}{\varepsilon} + \frac{\varepsilon\bar{x}^2}{2}\right) \tag{21}$$

$$v_{x_2} = \frac{\partial \phi_2}{\partial \bar{x}_2} = \varphi\frac{2\varepsilon\bar{x}\left(\bar{y} - \dfrac{1}{\varepsilon}\right)}{(1-\varepsilon+\varepsilon\bar{x}^2)^2} + \varepsilon\bar{x}(a\bar{x}^2-b)\left(\bar{y} - \frac{\varepsilon-1}{2}\right)\left(\bar{y} - \frac{1}{\varepsilon} - \frac{\varepsilon\bar{x}^2}{2}\right) \tag{22}$$

$$v_{x_3} = \frac{\partial \phi_3}{\partial \bar{x}} = \varphi(a\bar{x}^2+b)\left(\bar{y} - \frac{1}{\varepsilon} - \frac{\varepsilon\bar{x}^2}{2}\right) \tag{23}$$

$$\dot{\varepsilon}_x = \frac{\partial v}{\partial \bar{x}} = \varphi\frac{2\varepsilon\bar{x}}{(1-\varepsilon+\varepsilon\bar{x}^2)^2} + \varphi\varepsilon\bar{x}(a\bar{x}^2+b)\left(\bar{y} - \frac{1}{\varepsilon} + \frac{\varepsilon-1}{2}\right) \tag{24}$$

The sum of the strain rate components in previous equation is zero, and the flow function model satisfies the property of zero velocity dispersion in the passive field.

– In order to implement numerical simulations, verify the accuracy of the thermal strain model, estimate the minimum relative error and the maximum relative error between the theoretical computational model and the experiment, we should also estimate the error of the fast asymptotic algorithm in order to improve the strain computational model to be reliable and accurate and used for thermal strain prediction:

$$W_V = \frac{2\sqrt{3}}{3\sigma_s}\left[\int_{y_2(\bar{x})}^{y_1(\bar{x})}\int_0^1 (\dot{\varepsilon}_{\max} - \dot{\varepsilon}_{\min})\,\mathrm{d}\bar{x}\mathrm{d}\bar{y} - \int_0^1\int_{x_3(\bar{y})}^1 \dot{\varepsilon}_{\max} - \dot{\varepsilon}_{\min}\,\mathrm{d}\bar{x}\mathrm{d}\bar{y}\right] \tag{25}$$

$$W_T = kH\int_0^1 \sqrt{(v_x - v_0)^2 + v_y^2}\sqrt{\mathrm{d}\bar{x}^2 + \mathrm{d}y^2} \tag{26}$$

$$W_{x_1} = mk\int_{x_1}^{x_2} \sqrt{v_x^2 + v_y^2}\,\mathrm{d}y_2(\bar{x}) \tag{27}$$

$$W_{x_2} = mk\int_0^{\gamma_2} \sqrt{v_x^2 + v_y^2}\sqrt{1 + \left(\frac{\varepsilon^2\bar{x}H}{4}\right)^2}\mathrm{d}\bar{x} - \int_{\gamma_2}^{\theta} \sqrt{v_x^2 + v_y^2}\sqrt{1 + \left(\frac{\varepsilon^2\bar{x}H}{4}\right)^2}\mathrm{d}\bar{x} \tag{28}$$

$$W_{x_3} = mk\int_0^{\theta_1}\sqrt{v_x^2+v_y^2}\sqrt{1+\left(\frac{\varepsilon^2\overline{x}H}{4l}\right)^2}\mathrm{d}\overline{x}-\sqrt{1+\left(\frac{\Delta h}{2l}\right)^2}\,\mathrm{d}_1{}^2(\overline{x})\mathrm{d}\overline{x} \tag{29}$$

– Considering that the fractional-order Laplace operator is equivalent in space to the Caputo fractional-order derivative, which is defined by the left and right Riemann-Liouville fractional-order derivatives, we construct the discrete Riemann-Liouville fractional-order derivatives of the fourth-order tight-difference method and use eqs. (6) and (7) to discrete fractional-order Laplace operator. Its basic idea is to use weighted averaging to eliminate the error in the Grunwald formula for displacement, the lower order terms in the asymptotic expansion, and to combine the tight technique to improve the approximation accuracy. Thus, we can further obtain the following result:

$$u_0(x,y,t)=\sin(\pi x)\cos(\pi y) \tag{30}$$

$$u_1(x,y,t)=\frac{ht^\alpha}{\Gamma(1+\alpha)}(2\pi^\beta-1)u_0 \tag{31}$$

$$u_2(x,y,t)=\left[\frac{ht^{2\alpha}}{\Gamma(1+2\alpha)}+(2\pi^\beta-1)^2\right]u_0 \tag{32}$$

$$u_3(x,y,t)=\left[\frac{h^3t^{3\alpha}}{\Gamma(1+3\alpha)}(2\pi^\beta-1)^3+(h+1)^2t^\alpha\Gamma(1+\alpha)\right]u_0 \tag{33}$$

Numerical experiments

Consider the following fractional order heat transport model initial edge value problem $(0<\gamma<1)$ [15, 16]:

$$\begin{gathered}{}_0^C\mathrm{D}_t^\gamma u(x,t)=\frac{\partial^2 f(x,t)}{\partial x^2}+g(x,t),\quad (x,t)\in\left(0,\frac{1}{2}\right)\times(0,10]\\ u(x,0)=\sin(\pi x),\quad x\in\left[0,\frac{1}{2}\right]\\ u(0,t)=u(1,t)=0,\quad t\in[0,10]\end{gathered} \tag{34}$$

where $f(u)=u(1-u^2), g(x,t)$ is a specified function such that the above problem has an exact solution $u(x,t)=\mathrm{e}^t\sin(\pi x)$. Next, we use a linearization method to discretize the spatial variables x using a tight difference algorithm. Let $\mathcal{W}:=\{v_i:0\le i\le M\}$ be the lattice function on $\Omega:=\{x_i:0\le i\le M\}$. By discretizing and simplifying, we get:

$$x_1(t)=[u_1(t),u_2(t),\ldots,u_{M-1}(t)]^T \tag{35}$$

$$x_2(t)=\frac{1}{\lambda}[u_1(t)+g(x_0,t),0,\ldots,0,u_{M-1}(t)+g(x_{M-1},t)]^T \tag{36}$$

$$x_3(t)=\{f(u_1(t)]+g(x_1,t), f[u_2(t)]+g(x_2,t),\ldots, f[u_{M-1}(t)]+g(x_{M-1},t)\}^T \tag{37}$$

We can verify the convergence and conservation of the format through numerical experiments and analyze the effect of fractional order on the numerical solution waveform. When $\alpha = 2$, the FKGS system reduces to the classical KGS system and has the following analytic isolated waveform solution. Let:

$$\Lambda^{\frac{1}{2}} u_{i,j} := \left(4\cos\frac{\alpha\pi}{2}\cos\frac{\beta\pi}{2} h_x^{\alpha} h_y^{\beta} \right)^{-1/2} (\mathrm{Lu})_{(j-1)(M_1-1)+i}$$

Therefore:

$$A_{11} == \frac{1}{2\tau}\delta_t \frac{3}{(1-v^2)}\delta_x^{\alpha} \left\| \Lambda_1^{\frac{1}{2}} \mathrm{e}^{k+\frac{1}{2}} \right\|^2 \tag{38}$$

$$A_{12} = -\frac{1}{2\alpha}\mathcal{A}\delta_y^{\beta}\mathrm{e}^{k+1} \left\| \Lambda_2^{\frac{1}{2}} \delta_t \mathrm{e}^{k} \right\|^2 + \mathrm{sech}^2 \frac{x - vt - x_0}{2\sqrt{1-v^2}} \tag{39}$$

Considering the boundedness of $\| U^k \|_\infty$ and $\| U^k \|_\infty$, by the mean value theorem, the Cauchy-Schwartz inequality and Cauchy inequality, there exists a constant C satisfying:

$$\left\| f(U^k) - f(u^k) \right\| = \left\| f'(\theta^k)\mathrm{e}^k \right\| \le C_1 \left\| \mathrm{e}^k \right\|, \quad \left\| f'(U^k) - f'(u^k) \right\| = \left\| f''(\rho^k)\mathrm{e}^k \right\| \le C_1 \left\| \mathrm{e}^k \right\|$$

where $|v| < 1$ is the propagation velocity and x_0 denotes the initial phase. For the initial conditions of this fractional order system, *i.e.*, it is easy to observe that its solution decays rapidly to zero as $|x| \to \infty$, and the wave function decays rapidly to zero outside of $x \in (a,b), (a \ll 0$ and $b \gg 0)$. One can get:

$$A_{13} = -\frac{1}{3\sigma} \left\| \delta_t \mathrm{e}^{k+\frac{1}{2}} \right\|^2 \frac{\sqrt{2}}{\sqrt{1-v^2}} + \varepsilon C \left\| f(u^2)\mathrm{e}^k \right\|^3 \tag{40}$$

$$A_{21} = \frac{1}{4\varepsilon}\frac{\tau}{2} \left\| \delta_t f'(u^k)\mathrm{e}^{k+\frac{1}{2}} \right\|^2 \left(vx + \frac{1-v^2+v^4}{2(1-v^2)} t \right) + \varepsilon\tau^2 \left\| \mathrm{e}^k \right\|^2 \tag{41}$$

$$A_{22} = -\frac{\tau^2}{4} \left\| \Lambda^{\frac{1}{2}} \delta_t \mathrm{e}^{k+\frac{1}{2}} \right\|^2 + \mathrm{sech}^2 \frac{x - x_0}{2\sqrt{1-v^2}} \tag{42}$$

$$A_{23} = \frac{\tau}{2} C \left\| \delta_t f'(u^k)\mathrm{e}^{k+\frac{1}{2}} \right\|^2 \exp(\mathrm{i}vx) \tag{43}$$

$$A_{31} = -\frac{\tau^2}{4}\frac{3\sqrt{2}}{\sqrt{1-v^2}}\delta_x^{\alpha}[f'(U^k) - f'(u^k)]\delta_t U^{k+\frac{1}{2}}\mathrm{e}^{k+1} \tag{44}$$

$$A_{32} = \frac{1}{2}\left(\frac{\sqrt{3}C}{\ln 2} + \Lambda_1^{\frac{1}{2}}\delta_t \mathrm{e}^{k+\frac{1}{2}}\right)\left\|\Lambda_1^{\frac{1}{2}} f(\mathrm{e}^{k+1}) + f'(u^k)\right\|^2 + \operatorname{csch}^2 \frac{x - vt - x_0}{2\sqrt{1-v^2}} \tag{45}$$

$$A_{33} = \left[\frac{\sqrt{2}C_3}{8\ln 2} + \frac{C\varepsilon\left\|R^k\right\|^2}{4} + \frac{3(\delta_t + K^2)\epsilon}{\ln 2}\right]\left[\left\|\Lambda_1^{\frac{1}{2}}\mathrm{e}^{k+\frac{1}{2}}\right\|^2 - \left\|\Lambda_1^{\frac{1}{2}}\mathrm{e}^k\right\|^2\right] + \tanh\frac{x - x_0}{2\sqrt{1-v^2}} \tag{46}$$

In which C is a positive constant independent of k, h_x, h_y and τ. Thus, by the Cauchy inequality, a solution matrix with stable eigenvalues and positive definite values can be constructed, and we have:

$$U = \frac{1}{2\cos\dfrac{\alpha\pi}{2} h_x^\alpha}\begin{bmatrix} A_{11} & A_{12} & A_{12} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{47}$$

Conclusion

In this paper, we combine theoretical analysis and numerical computation through algorithm design, in which the special structure and properties of the coefficient matrix are fully utilized to construct an efficient numerical solution by matrix approximation and matrix splitting. In terms of theory, matrix theory, eigenvalue theory, numerical linear algebra and optimization theory and other related knowledge are applied to make full use of convergence and spectral properties of preprocessed matrices in analytical iteration and multipole fast asymptotic approximation, which greatly shorten the operation time and realize the effectiveness of the algorithm.

Acknowledgment

This work was supported by the Key Scientific Research Project of Colleges and Universities in Henan Province (No. 22B110019) and the Soft Science Project of Henan Provincial Department of Science and Technology (No. 232400411122).

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Paper submitted: August 5, 2022
Paper revised: July 2, 2023
Paper accepted: August 9, 2023