A SPLIT ITERATIVE ASYMPTOTIC METHOD FOR THE NUMERICAL SOLUTION OF A CLASS OF FRACTIONAL HEAT TRANSFER EQUATIONS

by

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In this paper, a new split iterative compact difference scheme for a class of system is constructed. Then, the conservation properties of the scheme are discussed, and the convergence of the split iterative difference scheme is analyzed by using the discrete energy method on the basis of the prior estimation. Finally, numerical experiments verify these properties of the new scheme. In addition, the numerical results also show the influence of fractional derivative on the variation of the transport equation.

Key words: *splitting, asymptotic, transport*

Introduction

Fractional calculus is arbitrary-order differentiation and integral, it is the promotion of integer-order calculus, the study of fractional calculus began at the end of the 17^{th} century, in the nearly three centuries, through the unremitting efforts of many mathematicians, finally formed a variety of fractional calculus theory including Riemann-Liouville, Growald-Letnikov, Caputo, and Riesz. Due to the inability to give suitable physical and geometric explanations, the study of fractional calculus remained purely mathematical for a long time. However, in recent decades, with the research and development of various subject areas, the memory-preserving properties of fractional differentiation can be more accurate than that of integer-order differentiation. As a result, fractional differential equations have been successfully used to study problems in thermophysics, chaos, complex viscoelastic materials, fluid power systems and other fields [1-7].

The analytical solutions of differential equations of fractional order usually contain some special functions, for example, Mittag-Leffler function, Fox function, Wright function, *etc.* These functions are obtained from infinite series and are very difficult to calculate numerically. Especially for some non-linear differential equations, their analytical solutions are difficult to obtain. Therefore, it is of great theoretical importance and practical value to construct numerical methods to solve fractional order differential equations. So far, there are still a lot of challenging problems in the numerical computation of fractional order differential equations, such as the computation of long time histories and large spatial regions, and most of the research algorithms are focused on finite difference methods and finite element methods. Therefore, it is still an urgent and important research topic to find the numerical solutions of fractional order differential equations quickly and to further improve the numerical methods.

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Numerous scholars have constructed different numerical formats for spatial fractional order non-linear equations. For example, the Sine fitting spectral method for the splitting of the spatial fractional-order non-linear Schrodinger equation. The spatial fourth-order tight ADI format method [8, 9]. The spectral Galerkin method for the splitting of 2-D spatial fractional-order non-linear equations. Split Fourier spectral methods [10, 11]. Local extrapolation methods for splitting formats of second-order exponential operators, and structure-preserving numerical methods. In long-time numerical simulations, structure-preserving numerical methods exhibit better simulation results than traditional numerical methods because they can inherit the inherent geometric properties of a given thermal system [12-16]. In the past decades, there have been many structure-preserving numerical methods for solving classical non-linear equations, among which the mass-conserving Fourier spectrum method for non-linear equations of spatial fractional order is an effective numerical method, and the efficiency of numerical simulations with this method can be verified by numerical experiments.

Dynamic model of heat transport and its analytical expression

We consider the following heat transport [6]:

$$\begin{aligned} &{}_{RL}\mathrm{D}_t^{\alpha}(u-v-tb)-\Delta u(x,t)=0, \quad 0<t\le T, \quad x\in\Omega \\ &u(x,0)=v(x), \quad u_t(x,0)=f(x), \quad x\in\Omega \\ &u(x,t)=0, \qquad\qquad\qquad\qquad x\in\partial\Omega, \quad 0<t\le T \end{aligned} \tag{1}$$

where $\alpha\in(1,2)$, ${}_{RL}\mathrm{D}_t^{\alpha}$ is the Bernhard Riemann-Joseph Liouville fractional derivative, $\Delta u(x, t)$ is the Laplace operator defined on some polygonal region $\Omega\subset\mathbb{R}^d\,(d=1,2,3)$, v and b are given functions, and $u,v,f\in H_0^1(\Omega)\cap H^2(\Omega)$.

In Lagrange coordinate system, using fractional-order Fourier transform and Darboux transform [7], the equation can be converted into a non-linear rigid Caputo-type fractional-ODE initial value problem:

$$y'(t)=f\left[t,y(t),{}_{t_0}^{C}\mathrm{D}_t^{\gamma}y(t)\right], \quad \gamma\in(0,1), t\in\left(t_0,T\right]; \quad y(t_0)=y_0$$

where ${}_{t_0}^{C}\mathrm{D}_t^{\gamma}y(t)$ denotes the γ-order Caputo derivative with respect to the solution $y(t)$, $f,\Psi:\left(t_0,T\right]\times\mathbb{C}^d\times\mathbb{C}^d\to C^d$ is given by a sufficiently smooth function satisfying the following one-sided and classical Lipschitz conditions with constants L_1, $L_2\ge 0$.

$$\Psi\langle y-\hat{y}, f(t,y,z)-f(t,\hat{y},z)\rangle\le L_1\left\|y-\hat{y}\right\|^2, \quad t\in\left(t_0,T\right], y,\hat{y},z\in\mathbb{C}^d$$
$$\left\|f(t,y,z)-f(t,y,\hat{z})\right\|\le L_2\left\|z-\hat{z}\right\|, \quad t\in\left(t_0,T\right], y,z,\hat{z}\in\mathbb{C}^d$$

where $\langle\cdot,\cdot\rangle$ is the inner product on the m-dimensional complex space $\mathbb{C}^d$, and $\|\cdot\|$ is the parametrization induced by this inner product, $\Psi:\left(t_0,T\right]\times\mathbb{C}^d\times\mathbb{C}^d\to C^d$, and L_1, $L_2\ge 0$ are the Lipschitz constant.

Let N be an even positive integer, and define the space lattice $\Omega_h=\{x_j=a+jh, j=0,1,\ldots,N-1\}$, where $h = L/N$ is the space step. For any positive integer N_t, define the time step $\tau = T/N$, then the division of time and space is defined as $\Omega_{h\tau}=\Omega_h\times\Omega_\tau$, where $\Omega_\tau=\{t_n=n\tau, n=0,1, ldots, N_t\}$. For the lattice function:

$$u=\left\{u_j^n\,\middle|\,(x_j,t_n)\in\Omega_{h\tau}\right\}$$

introduce the following notation:

$$\delta_x^+ u_i^n = \frac{u_{i+1}^n - u_i^n}{h}, \quad \sigma_t^+ u_i^{n+k} = \frac{u_{i+k}^{n+1} + k u_i^n}{k\lambda}, \quad \delta_t^+ u_i^n = \frac{u_{i+1}^{n+1} - \lambda u_i^n}{\lambda \tau}$$

Let $\mathcal{V}_h = \{u | u = (u_j), x_j \in \Omega_h\}$ denote the space of nodal functions defined on Ω_h. For any lattice function $u, v \in \mathcal{V}_h$, define the discrete inner product and the corresponding L^2 parametrization:

$$(u,v)_h = \frac{1}{N}\sum_{j=0}^{N-1} u_j \bar{v}_j, \quad \|u\|_h^2 = (u,u)_h$$

Furthermore, define the discrete L^p parametrization:

$$\|u\|_{t_h^p}^p = \frac{1}{N}\sum_{j=0}^{N-1} |u_j|^p, \quad 1 \le p < +\infty$$

and the discrete L^∞ parametrization:

$$\|u\|_{l_h^\infty} = \max_{0 \le j \le N-1} |u_j|$$

Using the discrete Gronwall inequality [5], let h, ψ, x_l, a_l, b_l, r_l be some nonnegative numbers such that:

$$x_n + \vartheta \sum_{i=0}^{n} a_i \le \delta \sum_{i=0}^{n} r_i x_i + \mu \sum_{i=0}^{n} b_i + \tau, \quad n \ge 0 \tag{2}$$

Suppose for any i with $hr_i < 1$, then when $\sigma_i = (1 - hr_i)^{-1}$, we have:

$$x_n \le \left(\sigma \sum_{i=0}^{n} b_i + \tau \sum_{i=0}^{n} a_i\right) \exp\left(\kappa \sum_{l=0}^{n} \sigma_i r_i\right), \quad n \ge 0 \tag{3}$$

Define the function space $S_N = \text{span}\{g_j(x), j\ 0\ 0, 1,\ldots, N-1\}$, and the trigonometric polynomial $g_j(x)$:

$$g_i(x) = \frac{1}{k}\sum_{k=-\sigma/2}^{M/2} \frac{1}{c_k} \mathrm{e}^{\mathrm{i}kt(x-x_i)}, \quad \text{in which} \quad c_k = \begin{cases} 1, |m| < \dfrac{\lambda}{3}, \\ 2, |m| = \dfrac{\lambda}{3}, \end{cases} \quad t = \frac{2\pi}{L}$$

Define the interpolation operator I_k: $L^2(\Omega) \to S_k$ as:

$$I_k u(x) = \sum_{i=0}^{m-1} u_i(x) g_i(x) = \sum_{k=-m/2}^{m/2} \mathrm{e}^{\mathrm{i}k\mu(x-a)}, \quad \hat{u}_k(x) = \frac{c_k}{m} u_0(x) \sum_{i=0}^{m-1} \mathrm{e}^{-\mathrm{i}k\mu(x_j - a)}, \quad \frac{t}{3} \le k \le 2t \tag{4}$$

$$u_1^{(n)} = x[\sigma(E)t_n] - \sum_{i=0}^{k-1}\left(\beta_i - \frac{\beta_k}{\alpha_k}\alpha_i\right) x_{n+i} - \frac{\beta_k}{\alpha_k} hy'[\sigma(E)x_n] \tag{5}$$

$$u_2^{(n)} = x\tau(E)x_n - \theta y'[\delta(E)t_n] \tag{6}$$

Furthermore, if the first summand on the right-hand side of eq. (4) expands only to $n-1$ terms, then eqs. (4) and (6) hold for all $\sigma > 0$ and $\sigma_l \equiv 1$.

From eqs. (4)-(6) we know that there exists a constant $c_2 > 0$ such that:

$$u_1^{(n)} \le \kappa\varepsilon^3, \quad u_2^{(n)} \le \kappa\varepsilon^{K+1}$$

where κ, $\varepsilon > 2$ such that $1/\kappa + 1/\varepsilon = 1/2$. In particular, taking $\kappa = 2/\alpha$, $\varepsilon = 2/(1-\alpha)$, using the Sobolev embedding theorem and Taylor's expansion [6], we get:

$$x[\sigma(E)t_n] = x(t_n) + h\sigma'(1)x'(t_n) + \frac{[\varepsilon\sigma'(1)]^3}{3!}x''''(t_n) + o(\varepsilon^3) \tag{7}$$

$$x'[\sigma(E)t_n] = x''(t_n) + \varepsilon\sigma'(1)x'''(t_n) + o(\xi^2) \tag{8}$$

$$\hat{x}_{n+k} = x(t_n) + \varepsilon x'(t_n) + \frac{(\phi i)^2}{2!}x''(t_n) + \varepsilon^{1/2}\sigma^2 x''(t_n) + o(\varepsilon^3) \tag{9}$$

According to Parseval's theorem [6], we get:

$$(u,v)_\varepsilon = \sum_{k=0}^{m/2}\hat{u}_k\bar{\hat{v}}_k \le \{x_k\}\Big|\sum_{k=-\infty}^{\infty}x_k^2 < \infty, \quad 0<\alpha\le 1, \quad \tau>0 \tag{10}$$

$$x(t_{n+1}) = \varphi(0) + \frac{1}{\Gamma(\alpha)}\int_0^{t_{n+1}}(t_{n+1}-s)^{\alpha-1}f[s,x(s),x(s-\tau)]\mathrm{d}s \tag{11}$$

$$\eta = \frac{T}{K}, \quad k = \frac{\tau}{\lambda}, \quad k, \quad K\in\mathbb{Z}, \quad t_n = n\lambda, \quad \lambda =, \quad 0,1,\cdots,N$$

$$x_\varepsilon(t_{n+1}) = \varphi(0) + \frac{\varepsilon^\alpha}{\Gamma(\alpha+2)}f[x_\varepsilon^{(n)}(t_{n+1})] + \frac{\varepsilon^{\alpha+1}}{\Gamma(\alpha+2)}\sum_{j=0}^{n}a_{i,\lambda+1}x_\varepsilon(t_{i+k}) \tag{12}$$

$$x_\varepsilon^{(k)}(t_{n+1}) = x(0) + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n}b_{i,k+1}f[x_\varepsilon(t_i),x_\varepsilon(t_{i+k})] \tag{13}$$

Given the constant $\sigma\in[0, 1]$, we define the discrete fractional order Sobolev parametrization $\|\cdot\|_{H_\varepsilon^\sigma}$ and the semi-parametrization $\|\cdot\|_{H_\varepsilon^\sigma}$ as:

$$|u|_{H_\tau^\sigma}^2 = \sum_{k=-1}^{m+1}|\mu k|^{2\sigma}|\hat{u}_k|^2, \quad \|u\|_{H_{h\tau}^\delta}^2 = \sum_{k=1}^{m+1}\left(1+|\mu k|^{2\delta}\right)|\hat{u}_k|^2 \tag{14}$$

Let m and N be two specified positive integers:

$$y(t)\in C^{(m+1)}([t_0,T]), \quad t_n = t_0 + nh(n=0,1,\ldots,N), \quad h = \frac{T-t_0}{N}$$

and consider a numerical approximation to the t_n at numerical approximation of the Caputo derivative.

– When $t\in[t_{j-m}, t_j](m<j\le n,\ m<n\le N)$, we can use m times Lagrange interpolation:

$$L_{m,j}(t) = \sum_{i=0}^{m}y(t_{j-i})\prod_{l=0,l\ne i}^{m}\frac{t-t_{j-l}}{t_{j-i}-t_{j-l}} \tag{15}$$

to approximate $y(t)$, then obviously, $\|u\|_{H_h^\sigma}^2 = \|u\|_h^2 + |u|_{H_h^\sigma}^2, |u|_{H_h^0}^2 = \|u\|_h^2$. It is easy to prove that the discrete Sobolev spaces defined above constitute fugitive linear spaces. Thus, it is shown that the unconditional convergence analysis for the conservation Fourier virtual spectrum method is very accurate:

$$\frac{1}{\Gamma(1-\gamma)}\int_{t_{j-m}}^{t_j}\frac{y'(v)}{(t_n-v)^\gamma}\,\mathrm{d}v \approx \frac{1}{\Gamma(1-\gamma)}\int_{t_{j-m}}^{t_j}\frac{L'_{m,j}(v)}{(t_n-v)^\gamma}\,\mathrm{d}v \approx \frac{\varepsilon^\gamma}{\Gamma(1-\gamma)}\sum_{i=0}^{m}\omega_{i,j,n}^m y(t_{j-i}) \tag{16}$$

– When $\alpha = 2$, the system (1) simplifies to the classical KGS [7] system. From the literature [14], it is known that this classical system has the following analytic solution:

$$\psi(x,t,x_0,u) = \frac{3\sqrt{2}}{4\sqrt{1-u^2}}\left(\frac{\partial^\alpha u}{\partial|x|^\alpha}+\frac{\partial^\beta u}{\partial|t|^\beta}\right)\mathrm{e}^{-t} - (1-u\mathrm{e}^{-t})\exp\left[\mathrm{i}\left(ux+\frac{1-u^2+u^4}{2(1-u^2)}t\right)\right] \tag{17}$$

$$\phi(x,t,x_0,u) = u\mathrm{e}^t - \left(\frac{\partial^\alpha u}{\partial|x|^\alpha}+\frac{\partial^\beta u}{\partial|y|^\beta}\right)\mathrm{e}^t - \mathrm{e}^{u\mathrm{e}^t} \tag{18}$$

It is easy to observe that $\psi_0(x)$ and $\phi_0(x)$ decay rapidly to zero with $|x| \to \infty$, so the fluctuation function is negligible outside of $x \in (a,b),(a \ll 0, b \gg 0)$.

– When $t \in [t_0, t_j](1 \le j \le m, j \le n \le N)$, $s = 2 - 2\alpha$, at this point we can approximate $y(t)$ by $L_{m,m}(t)$, so:

$$\frac{1}{\Gamma(1-\gamma)}\int_{t_0}^{t_j}\frac{y'(v)}{(t_n-v)^\gamma}\mathrm{d}v \approx \frac{h^{-\gamma}}{\Gamma(1-\gamma)}\sum_{i=0}^{m}\varpi_{i,j,n}^m y(t_{m-i}) \tag{19}$$

– When $s \in [2(1-\alpha), 2-\alpha]$. At this point, using the Gagliardo-Nirenberg inequality [6]:

$$\left\|\Lambda^{2-\alpha}\theta\right\|_{L^2} \leqslant C\left\|\Lambda^{s+\alpha}\theta\right\|_{L^2}^{\beta}\left\|\Lambda^s\theta\right\|_{L^2}^{1-\beta} \tag{20}$$

where $\beta = (2-\alpha-s)/\alpha] \in [0, 1]$. Using the Young's inequality [7], we know that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\xi^k\theta\right\|_{L^2}^2 + \kappa\left\|\xi^{s+\alpha}\theta\right\|_{L^2}^2 \leqslant C\left\|\xi^{s+\alpha}\theta\right\|_{L^2}^{1+\beta}\left\|\xi^s\theta\right\|_{L^2}^{2-\beta} \leqslant \left\|\xi^s\theta_0\right\|_{L^2}, \quad \forall t>0 \tag{21}$$

– If the exact solution of the initial margin value problem is sufficiently smooth, and $1 < \beta \le \alpha \le 2$. Then the numerical solution U^n of the difference format (18) converges to the exact solution of the initial-edge-value problem with $O(\tau^2 + h^4)$, depending on l_h^∞-paradigm, and Φ^n converges to the exact solution of the initial-edge-value problem, depending on P_h^2-paradigm.

$$\varepsilon^n = u^n - U^n, \quad \eta^n = \phi^n - \Phi^n, \quad \vartheta^n = v^n - V^n(\vartheta^0 = 0), \quad G\delta_x^{(\beta)}\vartheta^n = \delta_t\eta^n \tag{22}$$

The following iterative asymptotic splitting algorithm is obtained:

$$\mathrm{i}\delta_t\varepsilon^n + D\delta_x^{(\alpha)}A_t\varepsilon^n - \Theta^n = r^n, \quad n = 0,1,\cdots,N-1$$

$$\delta_t^2\eta^n - G\delta_x^{(\beta)}A_f\eta^n - \Xi^n = s^n, \quad n = 1,2,\cdots,N-1$$

$$\eta^1 = s^0, \quad \varepsilon^0 = 0, \quad \eta^0 = 0$$

$$\varepsilon_0^n = \varepsilon_J^n = 0, \quad \eta_0^n = \eta_J^n = 0, \quad n = 0,1,\cdots,N$$

$$\varepsilon^n = (\varepsilon_1^n, \varepsilon_2^n, \ldots, \varepsilon_{J-1}^n)^{\mathrm{T}}, \quad \eta^n = (\eta_1^n, \eta_2^n, \ldots, \eta_{J-1}^n)^{\mathrm{T}}$$

$$u(t) = \omega(t)\cos\varphi(t), \quad \frac{\partial}{\partial t}u(t) = -\omega(t)\omega_0 ch\sin\varphi(t), \quad \phi(t) = \omega_0 t + \varphi(t), \tag{23}$$

where ω(t), φ(t), and ψ(t) are random processes. Substituting equation into equation (23), we can obtain the equations for the amplitude ω(t) and the phase angle ϕ(t):

$$\frac{\mathrm{d}A}{\mathrm{d}t} = M_{11} + M_{12} + M_{13} + G_{11}\xi(t)$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = M_{21} + M_{22} + M_{23} + G_{22}\xi(t) \tag{10}$$

Numerical simulation and analysis

Consider the following two-dimensional non-linear fractional-order thermal transport system differential equation initial value problem ($0 < \gamma < 1$):

$$\begin{gathered} {}_{RL}\mathrm{D}_t^{\gamma} u(t) = u'(t) - \Delta u(t) + v(t) + u^2(t) sh[v(t)] \\ {}_{RL}\mathrm{D}_t^{\gamma} v(t) = v'(t) - \Delta v(t) + u(t) + v^2(t) ch[u(t)] \\ u(0,t) = 0, \quad v(0,t) = 0; \quad u'(1,t) = 1, \quad v'(1,t) = 1 \end{gathered} \tag{11}$$

where u and v are the propagation velocity of the isolated wave, and x_0 denotes the initial phase. For the initial conditions of the studied fractional order system, the value of the initial moment is given, *i.e.*

$$X = x(t) = A(t)\cos\Phi(t), \quad Y = \dot{x}(t) = -A(t)\omega_0 \sin\Phi(t), \quad \Phi(t) = \omega_0 t + \Theta$$

$$U(x) = \int_0^x \omega_0^2 u \mathrm{d}u = \frac{1}{2}\omega_0{}^2 x^2$$

$$\phi_0(x) = \frac{3\sqrt{2}}{4\sqrt{1-v^2}} + \mathrm{sech}^2 \frac{x - x_0}{2\sqrt{1-v^2}} - \mathrm{csch}[\exp(iux)]$$

$$\phi_1(x) = \coth\frac{3}{4(1-v^2)} - \mathrm{sech}^2 \frac{x - x_0}{2\sqrt{1-v^2}}$$

$$\phi_2(x) = \cosh\frac{3v}{4(1-v^2)} - \mathrm{sech}^2 \frac{x - x_0}{2\sqrt{1-v^2}} + \tanh\frac{x - x_0}{2\sqrt{1-v^2}}$$

$$B_{11} = a_1\omega_0 \sinh\Psi \,\mathrm{sgn}(A\cos\mathrm{cant}\Psi) D^{\lambda_1}(|A\cosh\Psi|)$$

$$B_{12} = \frac{a_2}{\omega_0}\sec\mathrm{ant}\Psi\,\mathrm{sgn}(A\coth\Psi) D^{\lambda_2}(|A\sinh\Psi|)$$

$$B_{13} = -A\sinh^2\Psi[(\lambda_1 - \lambda_2 M_0^2\cos^2\Psi) + (1-r)\left|K\omega_0\sinh\Psi\right|\delta(A\cos\text{cant}\Psi)]$$

$$B_{21} = -\frac{\sec\text{ant}\Psi}{\omega_0}\text{sgn}(A\,\text{csch}\,\Psi)$$

$$B_{22} = \frac{a_2}{M_1}\omega_0\,\text{csch}\,\Psi\,\text{sgn}(M_1\cosh\Psi)D^{\lambda_2}(|\,M_1\coth\Psi\,|)$$

$$B_{23} = L\,\text{sech}\,\Psi\,\text{csch}\,\Psi[(\lambda_1 - \lambda_2 A^2\csc^2\Psi) + (1-r)\left|L\omega_0\tanh\Psi(t)\right|\delta(L\,\text{csch}\,\Psi)]$$

$$B_{31} = \frac{a_1}{\sigma\omega_0}\tanh\Psi\,\text{sgn}(\sigma\coth\Psi)D^{\lambda_1}(|\,A\cosh\Psi\,|)$$

$$B_{33} = M_2\omega_0\cosh\Psi\,\text{sgn}(M_2\,\text{csch}\,\Psi) - \text{sech}^2\frac{x-x_0}{2\sqrt{1-v^2}}$$

$$B_{33} = M_2\omega_0\cosh\Psi\,\text{sgn}(M_2\,\text{csch}\,\Psi)$$

Next, the propagation of a single wave is simulated for different values of α. Let $\sigma = 0.8$, $x_0 = 0$, $\lambda = 0.01$, and solve the equations on $x \in [0, 20]$, $t \in [0, 10]$, and taking $\tau = 0.1$ and $\lambda = 0.01$. The numerical simulation results of the isolated waves $|X|$ and U at different moments t are given in tabs. 1-3.

Table1. Value of *U* at different time with λ = 0.01 and σ = 0.8

	$\alpha = 1.5$	$\alpha = 1.2$	$\alpha = 1.0$	$\alpha = 0.6$
$t = 0$	0.000993	1.000004	1.000011	1.000037
$t = 1$	1.000026	1.000009	1.0000015	1.000080
$t = 3$	1.000063	1.000071	1.0000019	1.000065
$t = 5$	1.000103	1.000087	1.0000029	1.000067
$t = 7$	1.000129	1.000083	1.0000037	1.000034

Table 2. Value of *U* at different time with λ = 0.05 and σ = 1.0

	$\alpha = 1.5$	$\alpha = 1.2$	$\alpha = 1.0$	$\alpha = 0.6$
$t = 0$	1.543586	1.586378	2.623363	2.641441
$t = 1$	1.544058	1.148977	2.623344	2.641328
$t = 3$	1.544024	1.146375	2.623335	2.641339
$t = 5$	1.544022	1.163741	2.623308	2.641348
$t = 7$	1.544002	1.863728	2.623292	2.641413

The effect of the fractional order α_1 on the stochastic response of the fractional order system is discussed. Table 1 shows the numerical and analytical results of the steady-state probability density functions $p(A)$, $p(x)$, and $p(y)$ for the amplitude, displacement, and velocity as the fractional order α_1 is varied, with other parameters $\alpha_2 = 1.2$, $\lambda_1 = 0.2$, $\lambda_2 = 0.1$, $\sigma_1 = \sigma_2 = 0.03$, $\omega = 1.0$, $D = 0.4$. Table 2 also shows the numerical and analytical results of the steady-state probability density functions $p(A)$, $p(x)$, and $p(y)$for the amplitude, displacement, and

Table 3. Global error and convergence order of the solution σ = 0.25, 0.5, 0.75

δ	τ	Error	Order	Error	Order	Error	Order	Error	Order
0.25	1/2	$1.3511 \cdot 10^{-3}$	1.8129	$1.7594 \cdot 10^{-4}$	2.7323	$2.6727 \cdot 10^{-5}$	3.6859	$4.2371 \cdot 10^{-6}$	4.8742
	1/3	$1.1415 \cdot 10^{-4}$	1.7421	$2.5183 \cdot 10^{-6}$	3.1295	$1.2015 \cdot 10^{-5}$	3.8017	$1.1298 \cdot 10^{-8}$	3.7981
0.50	1/2	$1.6436e \cdot 10^{-3}$	1.4731	$2.5894 \cdot 10^{-3}$	2.9085	$4.7846 \cdot 10^{-4}$	3.4986	$4.4013 \cdot 10^{-4}$	4.9121
	1/4	$1.0401 \cdot 10^{-3}$	1.4503	$2.5758 \cdot 10^{-5}$	3.0811	$4.5316 \cdot 10^{-6}$	3.4814	$2.1037 \cdot 10^{-5}$	5.1104
0.75	1/3	$4.1523 \cdot 10^{-3}$	1.3025	$3.5341 \cdot 10^{-4}$	2.6583	$2.1315 \cdot 10^{-4}$	3.1989	$2.6012 \cdot 10^{-5}$	4.7451
	1/5	$2.6941 \cdot 10^{-3}$	1.2013	$3.3152 \cdot 10^{-5}$	2.7986	$1.2032 \cdot 10^{-5}$	3.2847	$4.5011 \cdot 10^{-5}$	4.7963

velocity as the fractional order α_1 is varied, with other parameters α_2 = 1.5, λ_1 = 0.2, λ_2 = 0.3, $\sigma_1 = \sigma_2$ = 0.06, ω = 1.2, D = 0.5. From tabs. 1-3 it can be seen that the numerical and analytical results agree well, verifying the validity of this asymptotic approximation method. It is also clear from the table that the fractional order α_1 has an important effect on the stochastic response, and the smaller the order, the higher the probability of the system having larger magnitudes, displacements and velocities.

Conclusions

As α changes, its energy is in a constant state of decay, and the evolution images of the heat field $|\psi|$ and the medium field Φ when taking different α are given in the table.

- For any $1 < \alpha < 2/3$ two symmetric radiation waves can always generate a larger amplitude radiation wave after collision. At the same time, some symmetrically distributed larger radiation waves are also generated. The larger radiation wave will be strengthened, while the smaller one will be weakened.
- When α becomes larger, the waveform of radiation will change slightly, and the waveform will become closer to the waveform when $\alpha < 2$.
- As α decreases, the moment of collision will be delayed and more radiation ripples will be generated after the collision occurs.

As previously mentioned, we propose a new linear implicit conservation format for solving the fractional order heat transport equation. We adopt a new non-linear implicit exponential difference method in time and a Fourier fitting spectral method in space to discretize the heat transport equation, and give energy conservation properties and optimal approximation results. Numerical experiments show that this format not only preserves the mass conservation but also preserves the discrete parametrization of the numerical solution with a bound. The method has significant efficiency in the county compared to some existing structure-preserving formats with the same order in time and space.

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