GENERALIZED VARIATIONAL PRINCIPLES FOR THE MODIFIED BENJAMIN-BONA-MAHONY EQUATION IN THE FRACTAL SPACE

by

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Because variational principles are very important for some methods to get the numerical or exact solutions, it is very important to seek explicit variational formulations for the non-linear PDE. At first, this paper describes the modified Benjamin-Bona-Mahony equation in fractal porous media or with irregular boundaries. Then, by designing skillfully the trial-Lagrange functional, variational principles are successfully established for the modified Benjamin-Bona-Mahony equation in the fractal space, respectively. Furthermore, the obtained variational principles are proved correct by minimizing the functionals with the calculus of variations.

Key words: *variational principle, modified Benjamin-Bona-Mahony equation, semi-inverse method, fractal dimension*

Introduction

It is always an attractive and hot topic in different scientific fields to solve a PDE with integer or fractional orders [1-7], because of its exorbitant existence for modeling various phenomena, for examples, a mathematical model can be used for prediction the course of pandemics [8, 9]. A famous female Ukrainian mathematician, said that mathematics is an unknown land [10]. A whole new land is now opening for mathematicians to explore the approximate and exact solutions of a variety of physical and mathematical models, and variational-based methods have been very useful and effective, such as Ritz technique [11-15], the variational iteration method [16-20], and Hamiltonian-based method [21].

When contrasted with other methods, *e. g.*, the Taylor series method [22-24], variational-based methods show obvious advantages in effectiveness and simplicity and reliability [25-28]. The variational principles are so important in numerical simulation and analytical analysis that it becomes a branch of mathematics to seek an explicit variational formulation from non-linear PDE, and there are many schools worldwide, among which the semi-inverse school is an unprecedented monolith. The semi-inverse method [29-38] was firstly proposed in 1997 by Dr. Ji-Huan He, who is a famous Chinese mathematician. The semi-inverse method has been widely used to establish variational principles from the governing equations di-

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rectly, and has become a significant and effective tool in the variational theory far beyond the well-known Lagrange multiplier method, and variational variational principles are established for Korteweg-de Vries equation [39], Boussinesq equation systems [40], fractional dispersive long wave equations [41], Broer-Kaup equations [42], fractal high-order long water-wave equation [43], Burger equation [44], 1-D compressible flow [45], dynamic economics [46], Telegraph Equation [47], Chen-Lee-Liu Equation [48], and fractal solitary waves [49], furthermore, the Lagrange crisis [50] frequently encountered in constructing variational principles can be avoided effectively by the semi-inverse method.

In this paper, variational principles are established by the semi-inverse method for the modified Benjamin-Bona-Mahoni (BBM) equation [51-53] with spatio-temporal fractal derivatives, Wang and He [20] revealed that when time is fractal, the space must be also fractal, and vice versa, this is the spatio-temporal relation in physics. Although the modified BBM equation has been extensively studied for a long time by some scientists [51-53], but, up to now, its variational principle in a fractal space and fractal time has not been dealt with. Therefore, finding its variational principle is of great value, and might find lots of applications in numerical modelling and scientific researches.

The fractional partners

Usually, we can view physical motions and phenomena using two distinctly different scales [54-57]. One is the large scale, where Newton's calculus is approximately valid and the traditional mechanics can be roughly applied. The other scale is a much smaller one, a scale of molecule size. Under such a small scale, the media becomes discontinuous, and the fractal calculus [58, 59] has to be adopted. Usually, the smooth space (X, T) should be replaced by a fractal space (X^β, T^α), where β and α are fractal dimensions in space and time. In the fractal space, the modified BBM equation [51-53] can be transformed into the following form:

$$\frac{\partial u}{\partial T^{\alpha}}+\frac{\partial u}{\partial X^{\beta}}+au^2\frac{\partial u}{\partial X^{\beta}}+b\frac{\partial^3 u}{\partial X^{2\beta}\partial T^{\alpha}}=0 \tag{1}$$

where the He's fractal derivatives are defined [60]:

$$\frac{\partial u}{\partial T^{\alpha}}(T_0,X)=\Gamma(1+\alpha)\lim_{\substack{T-T_0\to\Delta T\\ \Delta T\neq 0}}\frac{u(T,X)-u(T_0,X)}{(T-T_0)^{\alpha}} \tag{2}$$

$$\frac{\partial u}{\partial X^{\beta}}(T,X_0)=\Gamma(1+\beta)\lim_{\substack{X-X_0\to\Delta X\\ \Delta X\neq 0}}\frac{u(T,X)-u(T,X_0)}{(X-X_0)^{\beta}} \tag{3}$$

For the fractal derivatives, we have the following chain rules:

$$\frac{\partial^2}{\partial X^{2\beta}}=\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}} \tag{4}$$

$$\frac{\partial^3}{\partial X^{2\beta}\partial T^{\alpha}}=\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial X^{\beta}}\frac{\partial}{\partial T^{\alpha}} \tag{5}$$

In the definitions given in eqs. (2) and (3), ΔX and ΔT are the smallest spatial scale for discontinuous boundary and the smallest temporal scale for watching the physical phenomena, respectively. When the spatial scale is larger than ΔX, the boundary is considered as a smooth one, and traditional continuum mechanics works. However, on the scale of ΔX, the boundary is discontinuous, and it is considered a fractal curve. When we watch the wave motions on a scale larger than ΔT, a smooth wave morphology is predicted. However, when we observe the wave on the scale of ΔX, discontinuous wave morphology can be found [54-60]. In the fractal space, all variables depend upon the scales used for observation and the fractal dimensions of the discontinuous boundary. The fractal derivatives are widely used in applications for discontinuous media or discontinuous boundaries [61, 62].

Variational principles for modified BBM equation in the fractal space

According to the basic properties of previously given fractal calculus, we have the following time and space scale transforms [54-60]:

$$t = T^{\alpha} \tag{6}$$

$$x = X^{\beta} \tag{7}$$

The modified BBM eq. (1) in fractal space becomes:

$$u_t + u_x + au^2 u_x + bu_{xxt} = 0 \tag{8}$$

The modified BBM equation has the ability to simulate the propagation of shallow water waves. The traveling wave and soliton solutions in particular have been studied extensively [51-53]. In eq. (8), a and b are parameters of constant values, which indicate the effects of high-order non-linearity and dissipation, respectively. Most of the problems in science and engineering are complex in nature due to the presence of non-linear terms and higher order derivatives in their governing equation.

In order to find its variational principles, the modified BBM equation can be transformed into the following form:

$$(u + bu_{xx})_t + \left(u + \frac{a}{3}u^3 \right)_x = 0 \tag{9}$$

It is obvious that finding Lagrangian representations for the modified BBM equation is a non-trivial problem. Additionally, it is necessary to replace the physical field $u(x, t)$ by its derivatives of potential fields. According to eq. (9), a potential function Π can be introduced:

$$\begin{aligned} \Pi_x &= u + bu_{xx} \\ \Pi_t &= -\left(u + \frac{a}{3}u^3 \right) \end{aligned} \tag{10}$$

Thus, eq. (9) will be automatically satisfied. We hope to construct different variational principles, according to eq. (9) and the field eq. (10).

For establishing the variational principles, whose Euler-Lagrange equations will be equivalent to the modified BBM equation, we can firstly set a trial-functional in the form:

$$J(u, \Pi) = \iint L(u, u_t, u_x, u_{xx}, u_{xxt}, \Pi, \Pi_t, \Pi_x)\mathrm{d}x\mathrm{d}t \tag{11}$$

where L is the trial-Lagrange functional. In view of eqs. (9) and (10), we design by the semi-inverse method [22-24], that L can be written:

$$L = (u + bu_{xx})\Pi_t + \left(u + \frac{a}{3}u^3\right)\Pi_x + F(u) \tag{12}$$

in which F is an unknown function of only variable u and its derivatives to be determined later. There are many alternative methods for constructing the trial-functional [22-24]. The great merit of the above trial-Lagrange functional (12) is whose stationary condition with respect to Π leads to the following Euler-Lagrange equation:

$$\frac{\partial L}{\partial \Pi} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \Pi_x} - \frac{\partial}{\partial t}\frac{\partial L}{\partial \Pi_t} = 0 \tag{13}$$

After introducing eq. (12), eq. (13) is identical to the modified BBM eq. (8). Subsequently, by calculating the stationary conditions of eq. (12) with respect to u, we obtain:

$$\frac{\delta L}{\delta u}: \quad \frac{\partial L}{\partial u} + \frac{\partial}{\partial x^2}\frac{\partial L}{\partial u_{xx}} + \frac{\delta F}{\delta u} = 0 \tag{14}$$

where $\partial F/\partial u$ is called the variational derivative [22-24]. By using eq. (12), eq. (14) can be rewritten:

$$\Pi_t + b\Pi_{xxt} + (1 + au^2)\Pi_x + \frac{\delta F}{\delta u} = 0 \tag{15}$$

We hope to find such an F, so that eq. (15) turns out to be the field eq. (10). Accordingly, after substituting the eq. (10) into eq. (15), we get:

$$\frac{\delta F}{\delta u} = 2abuu_x^2 - \frac{2}{3}u^3 \tag{16}$$

From eq. (16), unfortunately, we cannot identify F through the calculus of variations, because of existing the term $2\alpha\beta uu_x^2$. So, we have to modify the trial-Lagrange function L into a new form [33]:

$$L = A(u + bu_{xx})\Pi_t + B\Pi_x\Pi_t + \left(u + \frac{a}{3}u^3\right)\Pi_x + F(u) \tag{17}$$

Again, by calculating the variational derivatives of L with respect to Π and u, respectively, the new Euler-Lagrange equations can be obtained:

$$\frac{\delta L}{\delta \Pi}: \quad -(A + 2B)(u + bu_{xx})_t - \left(u + \frac{a}{3}u^3\right)_x + \frac{\delta F}{\delta \Pi} = 0 \tag{18}$$

$$\frac{\delta L}{\delta u}: \quad A\Pi_t + Ab\Pi_{xxt} + (1 + au^2)\Pi_x + \frac{\delta F}{\delta u} = 0 \tag{19}$$

In view of eq. (10) and $\partial F/\partial \Pi = 0$, eq. (18) becomes:

$$(A + 2B)(u + bu_{xx})_t + \left(u + \frac{a}{3}u^3\right)_x = 0 \tag{20}$$

Because eq. (20) should be identical to eq. (9), we must set the coefficient of u_t to one. That is:

$$A + 2B = 1 \tag{21}$$

After substituting eq. (10) into eq. (19), we obtain:

$$(1-A)u + (1-A)bu_{xx} + \left(1-\frac{A}{3}\right)au^3 + (1-A)abu^2u_{xx} - 2Aabuu_x^2 + \frac{\delta F}{\delta u} = 0 \tag{22}$$

In order to determine the unknown function *F* successfully, it is necessary to eliminate the term u^2u_{xx} and uu_x^2 simultaneously. Fortunately, it can be proved the following variational identity is correct:

$$\delta(u^2u_x^2) = -(2uu_x^2 + 2u^2u_{xx})\delta u \tag{23}$$

In order to apply the eq. (23) to eq. (22) properly, the coefficients of u^2u_{xx} and uu_x^2 must be equal. So we get:

$$1 - A = -2A \tag{24}$$

From eq. (24), we obtain $A = -1$. Furthermore:

$$\frac{\delta F}{\delta u} = -2u - 2bu_{xx} - \frac{4}{3}au^3 - 2ab(uu_x^2 + u^2u_{xx}) \tag{25}$$

From eq. (25), *F* can be identified easily:

$$F = -u^2 - \frac{a}{3}u^4 + bu_x^2 + abu^2u_x^2 \tag{26}$$

or

$$F = -u^2 - \frac{a}{3}u^4 - buu_{xx} + abu^2u_x^2 \tag{27}$$

Finally, we obtain the variational formulations for the modified BBM eq. (8), which read:

$$J(u,\Pi) = \iint\left[\Pi_x\Pi_t - (u+bu_{xx})\Pi_t + \left(u + \frac{a}{3}u^3\right)\Pi - u^2 - \frac{\alpha}{3}u^4 + bu_x^2 + abu^2u_x^2\right]_x \mathrm{d}x\mathrm{d}t \tag{28}$$

and

$$J(u,\Pi) = \iint\left[\Pi_x\Pi_t - (u+bu_{xx})\Pi_t + \left(u + \frac{a}{3}u^3\right)\Pi_x - u^2 - \frac{a}{3}u^4 - buu_{xx} + abu^2u_x^2\right]\mathrm{d}x\mathrm{d}t \tag{29}$$

both of which are subject to the constraint equation $\Pi_x = u + bu_{xx}$. The established variational principles are firstly discovered for the modified BBM equation by the semi-inverse method [22-24], and may find lots of applications in numerical simulations and scientific researches. In the following, we will prove the obtained variational principles correct. By making anyone of the previous functionals, eqs. (28) or (29), stationary with respect to independent functions *u* and Π severally, we can obtain two different Euler-Lagrange equations:

$$\delta \Pi: \quad (u+bu_{xx})_t-\left(u+\frac{a}{3}u^3\right)_x-2\Pi_{xt}=0 \tag{30}$$

$$\delta u: \quad -(\Pi_t+b\Pi_{txx})+\left(1+au^2\right)\Pi_x-2u-\frac{4a}{3}u^3-2bu_{xx}-2ab\left(uu_x^2+u^2u_{xx}\right)=0 \tag{31}$$

in which $\delta\Pi$ and δu are the first-order variation for Π and u, respectively. Substituting $\Pi_x = u + bu_{xx}$ into eq. (30) leads to the modified BBM equation, obviously. After substituting $\Pi_x = u + bu_{xx}$ into eq. (31), we can get that:

$$-\left(\Pi_t+u+\frac{a}{3}u^3\right)-b\left(\Pi_t+u+\frac{a}{3}u^3\right)_{xx}=0$$

Because b is a nonzero constant, we can reach that $\Pi_t = -[u + (a/3)u^3]$, which is identical to the second one of eq. (10). Hence, successfully, we proved the obtained variational principles (28) and (29) are correct. In the fractal space (X^β, T^α), the variational formulations can be written into new forms:

$$J(u,\Pi)=\iint\left[\frac{\partial\Pi}{\partial X^\beta}\frac{\partial\Pi}{\partial T^\alpha}-\left(u+b\frac{\partial^2 u}{\partial X^{2\beta}}\right)\frac{\partial\Pi}{\partial T^\alpha}+\left(u+\frac{a}{3}u^3\right)\frac{\partial\Pi}{\partial X^\beta}-\right.$$
$$\left.-u^2-\frac{a}{3}u^4+b\left(\frac{\partial u}{\partial X^\beta}\right)^2+abu^2\left(\frac{\partial u}{\partial X^\beta}\right)^2\right]\mathrm{d}X^\beta\mathrm{d}T^\alpha \tag{32}$$

and

$$J(u,\Pi)=\iint\left[\frac{\partial\Pi}{\partial X^\beta}\frac{\partial\Pi}{\partial T^\alpha}-\left(u+b\frac{\partial^2 u}{\partial X^{2\beta}}\right)\frac{\partial\Pi}{\partial T^\alpha}+\left(u+\frac{a}{3}u^3\right)\frac{\partial\Pi}{\partial X^\beta}-\right.$$
$$\left.-u^2-\frac{a}{3}u^4-bu\frac{\partial^2 u}{\partial X^{2\beta}}+abu^2\left(\frac{\partial u}{\partial X^\beta}\right)^2\right]\mathrm{d}X^\beta\mathrm{d}T^\alpha \tag{33}$$

Conclusion

In this paper, variational principles have been successfully constructed for the modified BBM equation in the fractal space, respectively, by the semi-inverse method [22-24] and designing skillfully trial-Lagrange functionals. Then, the obtained variational principles have proved correct by minimizing the corresponding functionals. From the results of analysis, it is concluded that the variational principle for the modified BBM equation studied in this paper have two different integral formulations, from which the same control equations can be derived. The procedure also reveals that the semi-inverse method [22-24] is effective and powerful. According to the obtained variational principles, on the one hand, we can study possible solution structures for solitary water waves. On the other hand, they also provide hints for numerical algorithms, so eq. (1) can be solved numerically by the variational-based methods. In the numerical simulations and ocean engineering, it is of great importance to choose an appropriate variational principle according to practical applications. Our work in the future will focus on the dynamics of soliton in the modified BBM equation, by the variational approximation method using the established variational principles in this paper.

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