ANALYSIS OF THE STATIONARY PROBABILITY DENSITY OF A GENERALIZED AND BISTABLE VAN DER POL SYSTEM EXCITED BY COLORED NOISE

by

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The stochastic P-bifurcation behavior of bi-stability in a generalized van der Pol oscillator with the fractional damping under colored noise and thermal excitation is investigated. Firstly, using the principle of minimal mean square error and linearization method, the non-linear stiffness terms can be equivalent to a linear stiffness which is a function of the system amplitude, and the original system is simplified to an equivalent integer order van der Pol system. Secondly, the system amplitude stationary probability density function is obtained by the stochastic averaging, and then based on the singularity theory, the critical parametric conditions for the system amplitude stochastic P-bifurcation are found. Finally, the types of the stationary probability density function of the system amplitude are qualitatively analyzed in each area divided by the transition set curves. The consistency between the analytical results and the numerical results acquired from Monte-Carlo simulation also testifies the theoretical analysis in this paper and the method used in this paper can directly guide the design of the fractional order controller to adjust the response of the system.

Key words: *stochastic P-bifurcation, Gaussian colored noise, fractional damping, critical parametric conditions, Monte-Carlo simulation*

Introduction

Fractional calculus [1-5] is a generalization of the traditional integer-order calculus. It is well-known that the integer-order derivative can not express the memory characteristics of the viscoelastic substances, while the fractional derivative contains convolution, which can

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express the memory effect and shows a cumulative effect over time. Therefore, the fractional derivative is a more suitable mathematical tool to describing memory characteristics [6-9] and has become a powerful mathematical tool for the study in the research fields such as anomalous diffusion, non-Newtonian fluid mechanics, viscoelastic mechanics and soft matter physics. Comparing with the integer-order calculus, the fractional calculus can describe various reaction processes more accurately [10-12], thus, it is necessary and significant to study the mechanical characteristics and the fractional order parametric influences on such systems.

Recently, many scholars have studied the dynamic behavior of non-linear multi-stable systems under different noise excitations and achieved fruitful results. Liu *et al*. [13] studied the response of a strongly non-linear vibro-impact system with Coulomb friction excited by real noise, and analyzed the P-bifurcation by a qualitative change of the friction amplitude and the restitution coefficient on the stationary probability distribution. Some researchers [14-16] studied the van der Pol-Duffing oscillators under Levy noise, color noise, combined harmonic, and random noise, respectively, the stochastic P-bifurcation behaviours of the noise oscillators were discussed by analyzing changes in the system stationary probability density function (PDF), and the analytical results of the bimodal stationary PDF were obtained, showing that the system parameters and noise intensity can each induce stochastic P-bifurcation of the systems. Wu and Hao [17-19] investigated the tri-stable stochastic P-bifurcation in a generalized Duffing-van der Pol oscillator under additive Gaussian white noise, multiplicative colored noise, combined additive and multiplicative Gaussian white noise, respectively, they obtained an analytical expression of the system stationary PDF of amplitude and analyzed the influences of noise intensity and system parameters on the system stochastic P-bifurcation. Chen and Zhu [20] studied the response of the Duffing system with fractional damping under the combined white noise and harmonic excitations, and showed that variation of the fractional derivative order can arouse the system stochastic P-bifurcation. Huang and Jin [21] discussed the response and the stationary PDF of a single-degree-of-freedom strongly non-linear system under Gaussian white noise excitation. Li *et al*. [22] studied the bi-stable stochastic P-bifurcation behavior of the van der Pol- Duffing system with the fractional derivative under additive and multiplicative colored noise excitations and found that changes in the linear damping coefficient, the fractional derivative order and the noise intensity can each lead to stochastic P-bifurcation in the system. Liu *et al*. [23] investigated the Duffing oscillator system with fractional damping under combined harmonic and Poisson white noise parametric excitation, and then analyzed the asymptotic Lyapunov stability with probability of the original system based on the largest Lyapunov exponent. Chen *et al*. [24] studied the primary resonance response of the van der Pol system under fractional-order delayed negative feedback and forced excitation, and obtained the approximate analytical solution based on the averaging method. Chen *et al*. [25] proposed a stochastic averaging technique which can be used to study the randomly excited strongly non-linear system with delayed feedback fractional-order proportional-derivative controller, and obtained the stationary PDF of the system.

Due to complexity of the fractional derivative, the parametric vibration characteristics of the fractional system can only be analyzed qualitatively, while the critical conditions of the parametric influences can not be obtained. In practice, the critical conditions of the parametric influences play a vital role for the analysis and design of the fractional order systems. Additionally, the stochastic P-bifurcation of bi-stability for the generalized van der Pol system with the fractional damping has not been reported in the open literature. In this paper, taking a generalized van der Pol system with a fractional damping excited by multiplicative Gaussian white noise from thermal excitation as an example, non-linear vibration of this kind of fractional order systems is studied through the fractional derivative. The transition set curves and critical parameter conditions for the system stochastic P-bifurcation are obtained by the singularity method. The types of the system stationary PDF curves in each area of the parameter plane are analyzed. We also compare the numerical results from Monte-Carlo simulation with analytical solutions obtained by the stochastic averaging. The comparison shows that the numerical results are in good agreement with the analytical solutions, verifying our theoretical analysis.

Derivation of the equivalent system

The initial condition of the Riemann-Liouville derivative has no physical meaning, while the initial condition of the system described by the Caputo derivative has not only clear physical meaning but also forms the same initial condition with the integer-order differential equation. Therefore, in this paper we adopt the Caputo fractional derivative:

$$\begin{gathered} {}_a^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)}\int_a^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}}\mathrm{d}u, \quad m < p < m+1 \\ {}_a^C\mathrm{D}^p[x(t)] = x^{(m)}(t), \quad p = m \end{gathered} \tag{1}$$

where $m \in N$, $t \in [a,b]$, $x^{(m)}(t)$ is the m-order derivative of $x(t)$ and $\Gamma(m)$ is the Gamma function, which satisfies $\Gamma(m + 1) = m\Gamma(m)$.

For a given physical system, the initial moment of oscillators is $t = 0$ and the Caputo derivative is usually expressed:

$${}_0^C\mathrm{D}^p[x(t)] = \frac{1}{\Gamma(m-p)}\int_0^t \frac{x^{(m)}(u)}{(t-u)^{1+p-m}}\mathrm{d}u \tag{2}$$

where $m-1 < p \le m, m \in N$.

In this paper, we study the generalized van der Pol system with the fractional damping excited by additive Gaussian colored noise excitation:

$$\ddot{x}(t) - [-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t)]\,{}_0^C D^p[x(t)] + w^2 x(t) = \zeta(t) \tag{3}$$

where $x(t)$ is the displacement of the system, ε – the linear damping coefficient, α_1 and α_2 – the non-linear damping coefficients of the system, w – the system natural frequency, ${}_0^C\mathrm{D}^p[x(t)]$ – the $p(0 \le p \le 1)$ order Caputo derivative of $x(t)$, which is defined by eq. (2), and $\zeta(t)$ – the Gaussian colored noise from thermal excitation with zero mean and auto-correlation function, which satisfies:

$$E[\zeta(t_1)\zeta(t_2)] = \frac{D}{\tau}\exp\left[-\frac{|t_1 - t_2|}{\tau}\right] \tag{4}$$

where τ and D denote the correlation time and the intensity of the colored noise, respectively.

Meanwhile, $\zeta(t)$ can be obtained through a first-order low pass filter by passing the Gaussian white noise $\zeta(t)$:

$$\dot{\zeta}(t) = -\frac{1}{\tau}\zeta(t) + \frac{1}{\tau}\xi(t) \tag{5}$$

where $E[\xi(t)] = 0$ and $E[\xi(t_1)\xi(t_2)] = 2D\delta(t_1 - t_2)$.

The fractional derivative has the contributions of damping force and restoring force [26], hence, we introduce the equivalent system:

$$\ddot{x}(t)-[-\varepsilon+\alpha_1 x^2(t)-\alpha_2 x^4(t)][C(p)\dot{x}(t)+K(p)x(t)]+w^2 x(t)=\zeta(t) \tag{6}$$

where $C(p)$ and $K(p)$ are the coefficients of the equivalent damping and equivalent restoring forces of the fractional derivative ${}_0^C\mathrm{D}^p[x(t)]$, respectively.

Applying the equivalent method mentioned in [23], we get the ultimate forms of $C(p)$ and $K(p)$:

$$\begin{aligned} K(p) &= w^p \cos\frac{p\pi}{2} \\ C(p) &= -w^{p-1}\sin\frac{p\pi}{2} \end{aligned} \tag{7}$$

Therefore, the equivalent van der Pol oscillator associated with system (6) can be written:

$$\ddot{x}(t)+w_0{}^2 x(t)-\alpha_1 w^p\cos\frac{p\pi}{2}x^3(t)+\alpha_2 w^p\cos\frac{p\pi}{2}x^5(t)-$$

$$-w^{p-1}\sin\frac{p\pi}{2}\gamma\dot{x}(t)=\zeta(t) \tag{8}$$

where

$$\begin{aligned} \gamma &= -\varepsilon+\alpha_1 x^2(t)-\alpha_2 x^4(t) \\ w_0{}^2 &= w^2+\varepsilon w^p\cos\frac{p\pi}{2} \end{aligned} \tag{9}$$

Stationary PDF of the system amplitude

Linearizing the cubic and quintic stiffness terms and taking the undetermined damping and stiffness coefficients as functions of the system amplitude, the vibrational structure of the equivalent system can be rewritten [27]:

$$\ddot{x}(t)+w_0{}^2 x(t)+C(a)\dot{x}(t)+K(a)x-w^{p-1}\sin\frac{p\pi}{2}[-\varepsilon+\alpha_1 x^2(t)-\alpha_2 x^4(t)]\dot{x}(t)=\zeta(t) \tag{10}$$

To determine the coefficients $C(a)$ and $K(a)$ in eq. (10), the error between system (8) and system (10) is defined by:

$$e=-\alpha_1 w^p\cos\frac{p\pi}{2}x^3(t)+\alpha_2 w^p\cos\frac{p\pi}{2}x^5(t)-C(a)\dot{x}(t)-K(a)x(t) \tag{11}$$

Assuming that the system (10) has the solution of the following form:

$$\begin{aligned} x(t) &= a(t)\cos\varphi(t) \\ \varphi(t) &= w_0 t+\theta(t) \end{aligned} \tag{12}$$

where ${w_0}^2 = w^2 + \varepsilon w^p \cos(p\pi/2)$, using the generalized harmonic balance technique and making the error (11) minimized in the mean square sense, the undetermined coefficients $C(a)$ and $K(a)$ can be obtained [27]:

$$C(a) = 0$$
$$K(a) = w^p \cos\frac{p\pi}{2}\frac{(5\alpha_2 a^2 - 6\alpha_1)a^2}{8} \tag{13}$$

Substituting eq. (13) into eq. (10) gives the equivalent system

$$\ddot{x}(t) + \Omega^2 x(t) - w^{p-1}\sin\frac{p\pi}{2}[-\varepsilon + \alpha_1 x^2(t) - \alpha_2 x^4(t)]\dot{x}(t) = \zeta(t) \tag{14}$$

where

$$\Omega^2 = w^2 + \varepsilon w^p \cos\frac{p\pi}{2} + w^p \cos\frac{p\pi}{2}\frac{(5\alpha_2 a^2 - 6\alpha_1)a^2}{8}$$

Assuming that system (14) has the solution of the periodic form, we introduce the following transformation [28]:

$$X = x(t) = a(t)\cos\Phi(t)$$
$$Y = \dot{x}(t) = -a(t)\Omega\sin\Phi(t) \tag{15}$$
$$\Phi(t) = \Omega t + \theta(t)$$

where Ω is natural frequency of the above equivalent system (14), $a(t)$ and $\theta(t)$ represent the amplitude and phase processes of the system response, respectively, and they are both random processes.

Substituting eq. (15) into eq. (14), we obtain:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = F_{11}(a,\theta) + G_{11}(a,\theta)\zeta(t)$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = F_{21}(a,\theta) + G_{21}(a,\theta)\zeta(t) \tag{16}$$

in which:

$$F_{11}(a,\theta) = w^{p-1}\sin\frac{p\pi}{2}a\sin^2\Phi(-\varepsilon + \alpha_1 a^2\cos^2\Phi - \alpha_2 a^4\cos^4\Phi)$$
$$F_{21}(a,\theta) = w^{p-1}\sin\frac{p\pi}{2}\sin\Phi\cos\Phi(-\varepsilon + \alpha_1 a^2\cos^2\Phi - \alpha_2 a^4\cos^4\Phi)$$
$$G_{11} = -\frac{\sin\Phi}{\Omega} \tag{17}$$
$$G_{21} = -\frac{\cos\Phi}{a\,\Omega}$$

Equation (16) can be treated as the Stratonovich stochastic differential equation, and by adding the relevant Wong-Zakai correction term, we transform it into the corresponding Ito stochastic differential equation:

$$\begin{aligned} \mathrm{d}a &= \left[F_{11}(a,\theta)+F_{12}(a,\theta)\right]\mathrm{d}t+\mathrm{G}_{11}(a,\theta)\mathrm{d}B(t) \\ \mathrm{d}\theta &= \left[F_{21}(a,\theta)+F_{22}(a,\theta)\right]\mathrm{d}t+G_{21}(a,\theta)\mathrm{d}B(t) \end{aligned} \tag{18}$$

where $B(t)$ is the normalized Wiener process and:

$$\begin{aligned} F_{12}(a,\theta) &= \frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial G_{11}(a,\theta)}{\partial a}G_{11}(a,\theta,t+h)+\frac{\partial G_{11}(a,\theta)}{\partial\theta}G_{21}(a,\theta,t+h)\right]R(h)\mathrm{d}h = \\ &= \frac{D\cos^2(\Phi)}{a\,\Omega^2(1+\Omega^2\tau^2)} \\ F_{22}(a,\theta) &= \frac{1}{2}\int_{-\infty}^{\infty}\left[\frac{\partial G_{21}(a,\theta)}{\partial a}G_{11}(a,\theta,t+h)+\frac{\partial G_{21}(a,\theta)}{\partial\theta}G_{21}(a,\theta,t+h)\right]R(h)\mathrm{d}h = \\ &= -\frac{D\sin(2\Phi)}{a^2\Omega^2(1+\Omega^2\tau^2)} \end{aligned} \tag{19}$$

By the stochastic averaging [29] of eq. (18) over Φ, we obtain the following averaged Ito equation:

$$\begin{aligned} \mathrm{d}a &= m_1(a)\mathrm{d}t+\sigma_{11}(a)\mathrm{d}B(t) \\ \mathrm{d}\theta &= m_2(a)\mathrm{d}t+\sigma_{21}(a)\mathrm{d}B(t) \end{aligned} \tag{20}$$

where

$$\begin{aligned} m_1(a) &= w^{p-1}\sin\frac{p\pi}{2}\left(-\frac{\varepsilon}{2}a+\frac{1}{8}\alpha_1 a^3-\frac{1}{16}\alpha_2 a^5\right)+\frac{D}{2a\,\Omega^2(1+\Omega^2\tau^2)} \\ \sigma_{11}{}^2(a) &= \frac{D}{\Omega^2(1+\Omega^2\tau^2)} \\ m_2(a) &= 0 \\ \sigma_{21}{}^2(a) &= \frac{D}{a^2\Omega^2(1+\Omega^2\tau^2)} \end{aligned} \tag{21}$$

Equations (20) and (21) show that da does not depend on θ, the averaged Ito equation of $a(t)$ is independent of $\theta(t)$ and that the random process $a(t)$ is a 1-D diffusion process.

Thus, the reduced Fokker-Planck-Kolmogorov (FPK) equation of $a(t)$ can be written:

$$0=-\frac{\partial}{\partial a}[m_1(a)p(a)]+\frac{1}{2}\frac{\partial^2}{\partial a^2}[\sigma_{11}{}^2(a)p(a)] \tag{22}$$

The boundary conditions are:

$$\begin{aligned} &p(a)=c, \quad c\in(-\infty,+\infty) \quad \text{as} \quad a=0 \\ &p(a)\to 0, \quad \frac{\partial\overline{p}}{\partial a}\to 0 \quad \text{as} \quad a\to\infty \end{aligned} \tag{23}$$

Based on the boundary conditions given in eq. (23), the amplitude stationary PDF can be obtained:

$$p(a)=\frac{C}{\sigma_{11}{}^{2}(a)}\exp\left[\int_0^a\frac{2m_1(u)}{\sigma_{11}{}^{2}(u)}\mathrm{d}u\right] \tag{24}$$

where C is the normalized constant that satisfies:

$$C=\left(\int_0^\infty\left\{\frac{1}{\sigma_{11}{}^{2}(a)}\exp\left[\int_0^a\frac{2m_1(u)}{\sigma_{11}{}^{2}(u)}\mathrm{d}u\right]\right\}\mathrm{d}a\right)^{-1} \tag{25}$$

Substituting eq. (21) into eq. (24), we get the explicit expression of stationary PDF of the system amplitude a:

$$p(a)=\frac{Ca\Omega^2(1+\Omega^2\tau^2)}{D}\exp\left(-\frac{\Delta_1\Delta_2}{48D}\right) \tag{26}$$

where C is the normalization constant and:

$$\Delta_1=\Omega^2(1+\Omega^2\tau^2)$$
$$\Delta_2=w^{p-1}\sin\frac{p\pi}{2}(24\varepsilon a^2-3\alpha_1a^4+\alpha_2a^6) \tag{27}$$
$$\Omega^2=w^2+\varepsilon w^p\cos\frac{p\pi}{2}+w^p\cos\frac{p\pi}{2}\frac{(5\alpha_2a^2-6\alpha_1)a^2}{8}$$

Stochastic P-bifurcation of the system amplitude

Stochastic P-bifurcation means that the changes in number of the stationary PDF curve peaks. To obtain the critical parametric conditions for stochastic P-bifurcation, we analyze the influences of parameters on the system stochastic P-bifurcation by using the singularity theory in this section.

For the sake of convenience, $p(a)$ is expressed by:

$$p(a)=\frac{C}{D}R(a,D,\varepsilon,w,p,\alpha_1,\alpha_2)\exp[Q(a,D,\varepsilon,w,p,\alpha_1,\alpha_2)] \tag{28}$$

in which:

$$R(a,D,\varepsilon,w,p,\alpha_1,\alpha_2,\alpha_3)=a\,\Omega^2(1+\Omega^2\tau^2)$$
$$Q(a,D,\varepsilon,w,p,\alpha_1,\alpha_2,\alpha_3)=-\frac{\Omega^2(1+\Omega^2\tau^2)w^{p-1}\sin\dfrac{p\pi}{2}(24\varepsilon a^2-3\alpha_1a^4+\alpha_2a^6)}{48D} \tag{29}$$
$$\Omega^2=w^2+\varepsilon w^p\cos\frac{p\pi}{2}+w^p\cos\frac{p\pi}{2}\frac{(5\alpha_2a^2-6\alpha_1)a^2}{8}$$

Based on the singularity theory [30], the stationary PDF of the system amplitude needs to satisfy:

$$\frac{\partial p(a)}{\partial a}=0,\qquad\frac{\partial^2p(a)}{\partial a^2}=0 \tag{30}$$

Substituting eq. (28) into eq. (30), we obtain:

$$H = \left\{ R' + RQ' = 0, R'' + 2R'Q' + RQ'' + RQ'^2 = 0 \right\} \tag{31}$$

where H is the condition for the changes in number of the PDF curve peaks.

In this part, the influences of p and D on the system are investigated, and the parameters are taken as $\alpha_1 = 1.51$, $\alpha_2 = 2.85$, $w = 1$, and $\tau = 0.1$. According to eqs. (29) and (31), we obtain the transition set for the system stochastic P-bifurcation with the unfolding Parameters p and D shown in fig. 1.

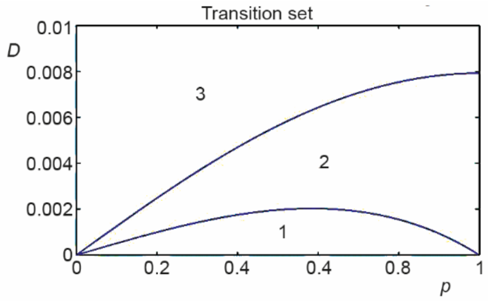


Figure 1. Transition set curves (taking *p* and *D* as unfolding parameters)

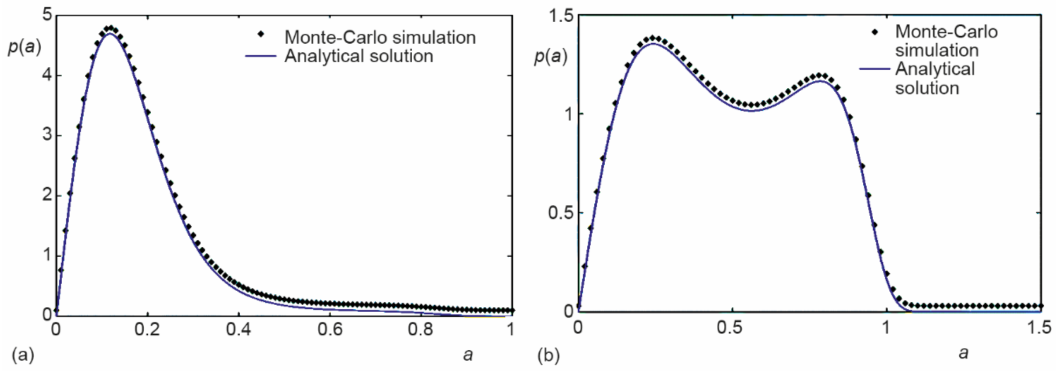
Based on the singularity theory, the topological structures of the stationary PDF curves of different points (p, D) in the same area are qualitatively identical. By taking a point (p, D) in each area, we can obtain all varieties of the system stationary PDF curves that are qualitatively different. The unfolding parameter p–D plane is divided into three sub-areas by the transition set curve. For the sake of convenience, each area in fig. 1 is marked with a number.

We first analyze the stationary PDF of amplitude $p(a)$ for a point (p, D) in each of the three sub-areas of fig. 1, and then compare the analytical solutions with the numerical data obtained by Monte-Carlo simulation from the original system (3) using the numerical method for fractional derivative [25]. The corresponding results are shown in fig. 2.

From fig. 1 we can see that, the parameter area where the PDF occurs bimodal is surrounded by two curves. And when the parameter (p, D) is taken as $p = 0.5$, $D = 0.001$ in area 1, fig. 2(a), the PDF $p(a)$ has a stable equilibrium. When the parameter (p, D) is taken as $p = 0.6$, $D = 0.004$ in area 2, fig. 2(b), the PDF $p(a)$ has a stable limit cycle far away from the origin and the probability is not zero near the origin, there are both the limit cycle and equilibrium in the system simultaneously. When the parameter (p, D) is taken as $p = 0.3$, $D = 0.008$ in area 3, the PDF $p(a)$ appears in the form of a stable limit cycle at the moment.

Apparently, the stationary PDF $p(a)$ in any two adjacent areas in fig. 1 are very qualitatively different. Regardless of the exact values of the unfolding parameters, if they cross any line in this figure, the system will demonstrate stochastic P-bifurcation behavior. Therefore, the transition set curves are just the critical parametric conditions of the system stochastic P-bifurcation. The analytic results shown in fig. 2 are well consistent with those numerical results obtained by Monte-Carlo simulation from the original system (3), further verifying the theoretical analysis and showing that it is feasible to use the methods in this paper to analyze the stochastic P-bifurcation behavior of fractional order systems.

Compared with the integral-order controllers [31-37], the fractional-order controllers have the better dynamic performances and robustness [25]. In the past several years, various fractional-order controllers have been developed [38-42]. In the previous analyzis we obtained the areas where the stochastic P-bifurcation occurs in system (3), which can make the system switch between mono-stable and bi-stable states by selecting the corresponding unfolding parameters. This could provide theoretical guidance for the analysis and design of the fractional order controllers.



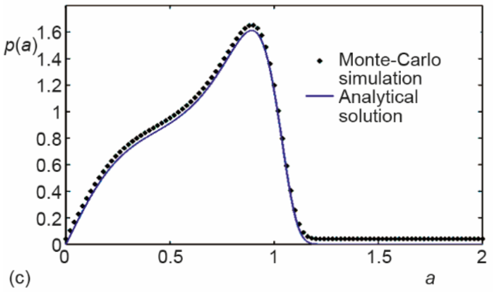


Figure 2. The PDF of $p(a)$ (taking p and D as the unfolding parameters); (a) parameter (p, D) in area 1 of fig. 1, p = 0.5, D = 0.001, (b) parameter (p, D) in area 2 of fig. 1, p = 0.6, D = 0.004, and (c) parameter (p, D) in area 3 of fig. 1, p = 0.3, D = 0.008

Conclusion

In this paper, the stochastic P-bifurcation of a modified fractional and bistable Van der pol system subjected to additive colored noise excitation is investigated. Based on the equivalent principle to make the mean square error minimum, the original system can be transformed into an equivalent integer-order system, and we obtained the system amplitude's stationary PDF by utilizing the stochastic averaging method. Further, the critical parametric conditions for the system's stochastic P-bifurcation behavior are obtained using the singularity theory, which can provide the theoretical guidance for system design. The consistency between the numerical results obtained by Monte-Carlo simulation and the analytical results can also verify the theoretical analysis above. It shows that the fractional order p and noise intensity D can both arise the stochastic P-bifurcation of the system, and the number of peaks of the system's stationary PDF curves $p(a)$ can vary from one to two by selecting the appropriate unfolding parameters.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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