STABILITY OF INITIAL RESPONSE OF EXPONENTIALLY DAMPED OSCILLATORS

by

Guozhong XIU[a,b], Hongmei LIU[c*], and Jianjing MAO[b]

[a] Sino-German Intelligent Manufacturing School, Shenzhen City Polytechnic, Shenzhen, China
[b] School of Software, Zhengzhou University of Industrial Technology, Xinzheng, Henan, China
[c] Journal Center of Donghua University, Shanghai, China

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A damping system always results in energy consumption. This paper studies an exponentially damped oscillator with historical memory for a viscoelastic damper structure, its stability under an initial response is analyzed analytically and verified numerically.

Key words: *exponentially damped oscillators, initial response, stability, convolutional nonviscously damped model*

Introduction

With the extremely fast development of materials science, more and more new materials, including bionic materials, polymer materials, polymer viscoelastic materials, nanomaterials, piezoelectric materials, 3-D printed materials and metamaterials [1-5], have been extensively applied to vibration-related fields, especially for vibrations absorption and noise control [6], improvement of safety and reliability of aircraft, ship control panels, gyro instruments and other mechanical equipment [7], and energy harvesting devices [8].

The traditional approach to the viscous damping model is assumed that the damping force is proportional to the relative velocity of the moving subject. This mathematical description method cannot reflect the actual structural characteristics and complex damping energy dissipation characteristics of materials, except for the convenience of calculation, so a new mathematical description is much needed.

Liang and Wang [9] suggested a fractal viscoelastic model, where the two-scale fractal theory [10, 11] was adopted. In recent years, more and more attention was paid to the integral constitutive model, that is, the convolutional non-viscously damped model [12-14]. This model has many advantages in the memory characteristics and the time-delay property. Compared with the viscous damping model, it can better reflect the rheological characteristics of materials, and can also explain the characteristics of structural damping mechanism in a physical sense. However, because the damping force contains an integral term, it changes the original linear property and brings difficulties to the analysis and calculation.

The research on the convolutional non-viscously damped oscillators mainly focuses on two types: one is fractional oscillators [15], in which the viscoelastic kernel function is expressed by the power law function or Mittag-Leffler function. The other is an exponen-

* Corresponding author, e-mails: xiuguozhong2013@163.com, hongmeiliu@dhu.edu.cn

tially damped oscillator whose damping force is represented by an exponentially decayed memory kernel [14, 16]. They can all be expressed as convolution integrals. The equations of motion established by an integral constitutive model are a group of coupled second-order integral differential equations. The existence of an integral term makes the vibration analysis and control design more complicated than that of the classical design.

Adhikari and Woodhouse [13] systematically studied the exponentially damped model, including the dynamics of exponentially damped single degree of freedom and multi degree of freedom systems. Li *et al.* [16] studied the sensitivity of dynamic analysis of linear non-viscously damped systems. Shen *et al.* [17] considered the exponentially damped system and used Taylor expansion formula to eliminate the integral term, so as to conduct time history analysis of the system. In [18], the exponentially damped system was studied, and the method to determine the parameters of the specific material exponentially damped model was given, which was verified by experiments. Lazaro and Perez-Aparicio [19] proposed a new method to calculate the eigenvalue of linear viscoelastic vibrator. The exponentially damped model, fractional derivative damping model and viscous model were taken as examples to verify the method. Guedria and Smaoui [20] proposed a new method to simultaneously calculate the eigenvalue derivatives of left and right eigenvectors of exponentially damped systems and their related derivatives. A large number of scholars have conducted extensive research on the exponentially damped system [21-23]. However, the research on the initial response of the exponentially damped vibrator is very rare and preliminary.

The stability of initial response of exponentially damped oscillators is discussed in this paper. Unlike classical viscous damped oscillators, the equation of motion of this class of oscillators is a set of coupled second-order Volterra integral differential equations. This paper deals with the initial value problem of integral differential equations. The determination of the initial displacement and initial velocity of the motion equation of exponentially damped oscillators with a history is not sufficient to understand its dynamic behavior. The initial condition should contain the time history of the velocity of vibration motion [24]. In this way, the initial response of the exponentially damped oscillators can be obtained accurately. In this paper, it is proved theoretically that the initial response has no effect on the stability of the exponentially damped oscillators, and through numerical simulation, the effect of the initial response on the exponentially damped oscillators gradually disappears.

Initialization of convolutional non-viscously damped oscillators

The integral constitutive relation of viscoelastic materials is expressed by Volterra type integro-differential equation:

$$\sigma(t)=\int_{-\infty}^{t} G(t-\tau)\dot{\varepsilon}(\tau)\mathrm{d}\tau \tag{1}$$

where $\sigma(t)$ is the stress, $\varepsilon(t)$ – the strain, and $G(t)$ – the stress relaxation function. The lower end of the integral is $-\infty$, because the stress of viscoelastic materials depends on all the time histories of strain.

Volterra’s integro-differential equation can model many practical problems, for example, population dynamics [25], and the homotopy perturbation method has been proved to be effective to solve such problems [26].

Figure 1 shows a single DoF oscillator with a convolutional non-viscously damper. The vibration equation of the exponentially damped oscillator is:

$$m\ddot{x}(t)+c\int_{-\infty}^{t}G(t-\tau)\dot{x}(\tau)\mathrm{d}\tau+kx(t)=f(t) \tag{2}$$

where *m* is the mass, *k* – the stiffness, *c* – the damping coefficient, and *f*(*t*) – the external force acting on the system.

The integral term in eq. (2) makes the dynamic model different from the classical model. It contains not only the information of vibration displacement and velocity, but also the time history of velocity. This means that, unlike viscous damping systems, instantaneous displacements and velocities are not sufficient to predict the dynamic behavior of the system. The motion time history should be added to the initial condition to completely determine the dynamics of the convolutional non-viscously damped oscillators. Therefore, the dynamic equation with past history is described:

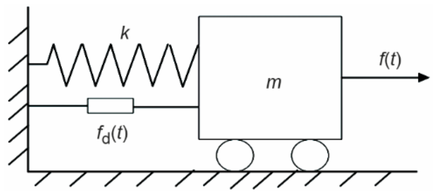

Figure 1. Equivalent mechanical model of convolutional non-viscously damped oscillator

$$\begin{gathered}m\ddot{x}(t)+c\int_{t=-\infty}^{t}G(t-\tau)\dot{x}(\tau)\mathrm{d}\tau+kx(t)=f(t),\quad t>0\\ x(0)=x_0,\quad \dot{x}(0)=v_0\\ \dot{x}(t)=v(t),\quad -\infty<t<0\end{gathered} \tag{3}$$

where $t = 0$ is the initial time, and the lower end of the integral $t = -\infty$ is the beginning time of the vibration. In fact, it is more reasonable to set the start time to $t = -a$, which means that the system is in a static state before $t = -a$.

In order to better represent the historical memory of vibrator dynamics, the integral term in eq. (3) can be divided into two parts:

$$\int_{-a}^{t}G(t-\tau)\dot{x}(\tau)\mathrm{d}\tau=\int_{-a}^{0}G(t-\tau)\dot{x}(\tau)\mathrm{d}\tau+\int_{0}^{t}G(t-\tau)\dot{x}(\tau)\mathrm{d}\tau \tag{4}$$

The first part on the right of the above equation describes the historical effect of historical motion on system dynamics, expressed as $\psi(t)$, then:

$$\psi(t)=\int_{-a}^{0}G(t-\tau)\dot{x}(\tau)\mathrm{d}\tau=\int_{-a}^{0}G(t-\tau)v(\tau)\mathrm{d}\tau \tag{5}$$

where $\psi(t)$ describes the historical effect of vibration, which affects the behavior of the dynamic system after the initial time $t = 0$. Equation (3) can be rewritten:

$$\begin{gathered}m\ddot{x}(t)+c\int_{0}^{t}G(t-\tau)\dot{x}(\tau)\mathrm{d}\tau+kx(t)=f(t)-c\psi(t),\quad t>0\\ x(0)=x_0,\quad \dot{x}(0)=v_0\end{gathered} \tag{6}$$

Initial response of exponentially damped oscillators

Now we are ready to study the initial response of an exponentially damped oscillator with historical effect, which describes the historical effect of past vibration motion. For this purpose, we will ignore the external force $f(t)$ and set it to zero. In this case, the equation of motion with initial conditions is:

$$m\ddot{x}(t)+c\int_0^t G(t-\tau)\dot{x}(\tau)\mathrm{d}\tau+kx(t)=-c\psi(t),\quad t>0$$
$$x(0)=x_0,\quad \dot{x}(0)=v_0 \tag{7}$$

where $G(t)=\mu\mathrm{e}^{-\mu t}$, $\mu>0$.

By applying Laplace transformation [27] to both sides of eq. (7), it can be obtained:

$$m[s^2\bar{x}(s)-sx_0-v_0]+c\left[\frac{\mu s}{\mu+s}\bar{x}(s)-\frac{\mu}{\mu+s}x_0\right]+k\bar{x}(s)=-c\bar{\psi}(s) \tag{8}$$

where $\bar{x}(s)$ is the Laplace transform of $x(t)$ and $\bar{\psi}(s)$ – the Laplace transform of $\psi(t)$. Then eq. (8) is transformed into:

$$\left(ms^2+\frac{c\mu s}{\mu+s}+k\right)\bar{x}(s)=-c\bar{\psi}(s)+msx_0+\frac{c\mu}{\mu+s}x_0+mv_0 \tag{9}$$

Let's call it:

$$\bar{d}(s)=ms^2+\frac{c\mu s}{\mu+s}+k\qquad\text{and}\qquad \bar{h}(s)=\frac{1}{\bar{d}(s)}$$

From eq. (9), the solution of $\bar{x}(s)$ can be written:

$$\bar{x}(s)=-c\bar{h}(s)\bar{\psi}(s)+mx_0s\bar{h}(s)+c\mu x_0\frac{\bar{h}(s)}{s+\mu}+mv_0\bar{h}(s) \tag{10}$$

We perform inverse Laplace transform on eq. (10), and we can get:

$$x(t)=-c\int_0^t h(t-\tau)\psi(\tau)\mathrm{d}\tau+mx_0\dot{h}(t)+c\mu x_0\int_0^t h(t-\tau)\mathrm{e}^{-\mu\tau}\mathrm{d}\tau+mv_0h(t) \tag{11}$$

where $h(t)$ is the inverse Laplace transform of $\bar{h}(s)$.

Next, we determine the expression for $h(t)$, then we can get the initialization response function of the oscillator. Since $\bar{d}(s)$ is zero at $s = s_j$, j = 1, 2, 3 [18], we can get:

$$s_1=-\alpha+\beta i,\quad s_2=-\alpha-\beta i,\quad s_3=-\gamma \tag{12}$$

where $\alpha,\beta,\gamma>0$.

In addition, $\bar{h}(s)$ can be expressed in the form of its residue in the form:

$$\bar{h}(s)=\sum_{j=1}^{3}\frac{R_j}{s-s_j}$$

where R_j is the residue, which can be calculated:

$$R_j = \operatorname*{Res}_{s=s_j} \bar{h}(s) = \lim_{s\to s_j}(s-s_j)\bar{h}(s) = \frac{1}{\lim\limits_{s\to s_j}\dfrac{ms^2+\dfrac{c\mu s}{\mu+s}+k}{s-s_j}} = \frac{1}{\lim\limits_{s\to sj}\dfrac{\bar{d}(s)}{s-s_j}} = \frac{1}{\dfrac{\partial\bar{d}(s)}{\partial s}\Big|_{s=s_j}}$$

Therefore, by applying the Laplace transformation of $\bar{h}(s)$, $h(t)$ can be obtained:

$$h(t) = L^{-1}[\bar{h}(s)] = \sum_{j=1}^{3} R_j \mathrm{e}^{s_j t} \tag{13}$$

By substituting eq. (13) into eq. (11), the initial response of the exponentially damped oscillator is obtained. It represents the historical effect of motion from $t = -a$:

$$x(t) = -c\int_0^t \left[R_1\mathrm{e}^{s_1(t-\tau)} + R_2\mathrm{e}^{s_2(t-\tau)} + R_3\mathrm{e}^{s_3(t-\tau)}\right]\psi(\tau)\mathrm{d}\tau + mx_0(R_1s_1\mathrm{e}^{s_1t} + R_2s_2\mathrm{e}^{s_2t} + R_3s_3\mathrm{e}^{s_3t}) +$$
$$+c\mu x_0\int_0^t \left[R_1\mathrm{e}^{s_1(t-\tau)} + R_2\mathrm{e}^{s_2(t-\tau)} + R_3\mathrm{e}^{s_3(t-\tau)}\right]\mathrm{e}^{-\mu\tau}\mathrm{d}\tau + mv_0(R_1\mathrm{e}^{s_1t} + R_2\mathrm{e}^{s_2t} + R_3\mathrm{e}^{s_3t}) \tag{14}$$

where $s_1 = -\alpha + \beta i,\ s_2 = -\alpha - \beta i,\ s_3 = -\gamma,\ \alpha, \beta, \gamma > 0.$

Stability of initialization response

Previous section illustrates the influence of past motion history on the initial condition and dynamic response. In this section, we continue to discuss the influence of historical effects on the stability of the initial response of the exponentially damped oscillator. Stability or instability plays an important role in vibration engineering, especially the pull-in instability of micro-electromechanical system [28-32]. Here we will show that the past motion history does not affect the stability of the initial response of the oscillator over time.

Theoretical stability analysis

As can be seen from eq. (14), the initial response consists of four parts:

$$x(t) = -c\int_0^t \left[R_1\mathrm{e}^{s_1(t-\tau)} + R_2\mathrm{e}^{s_2(t-\tau)} + R_3\mathrm{e}^{s_3(t-\tau)}\right]\psi(\tau)\mathrm{d}\tau + mx_0(R_1s_1\mathrm{e}^{s_1t} + R_2s_2\mathrm{e}^{s_2t} + R_3s_3\mathrm{e}^{s_3t}) +$$
$$+c\mu x_0\int_0^t \left[R_1\mathrm{e}^{s_1(t-\tau)} + R_2\mathrm{e}^{s_2(t-\tau)} + R_3\mathrm{e}^{s_3(t-\tau)}\right]\mathrm{e}^{-\mu\tau}\mathrm{d}\tau + mv_0(R_1\mathrm{e}^{s_1t} + R_2\mathrm{e}^{s_2t} + R_3\mathrm{e}^{s_3t}) \tag{15}$$

As can be seen from eq. (15), when $t\to\infty$, we can get $\mathrm{e}^{s_1t}\to 0$, $\mathrm{e}^{s_2t}\to 0$, $\mathrm{e}^{s_3t}\to 0$, $\mathrm{e}^{-\mu t}\to 0$, so the last three parts on the right of eq. (15) gradually approach to zero with the increase of time. Now let's prove that the same is true of the first part on the right of eq. (15).

By substituting eq. (5) into the first term of eq. (15), we can get:

$$-c\int_0^t h(t-\tau)\psi(\tau)\mathrm{d}\tau = -c\int_0^t R_1\mathrm{e}^{s_1(t-\tau)}\psi(\tau)\mathrm{d}\tau - c\int_0^t R_2\mathrm{e}^{s_2(t-\tau)}\psi(\tau)\mathrm{d}\tau - c\int_0^t R_3\mathrm{e}^{s_3(t-\tau)}\psi(\tau)\mathrm{d}\tau \tag{16}$$

By substituting eq. (12) into eq. (16), we can get:

$$-c\int_0^t h(t-\tau)\psi(\tau)\mathrm{d}\tau =$$

$$= -c\int_0^t R_1 \mathrm{e}^{-(\alpha-\beta \mathrm{i})(t-\tau)}\psi(\tau)\mathrm{d}\tau - c\int_0^t R_2 \mathrm{e}^{-(\alpha-\beta \mathrm{i})(t-\tau)}\psi(\tau)\mathrm{d}\tau - c\int_0^t R_3 \mathrm{e}^{-\gamma(t-\tau)}\psi(\tau)\mathrm{d}\tau =$$

$$= -c(R_1 + R_2)\int_0^t \mathrm{e}^{-\alpha(t-\tau)}\cos\beta(t-\tau)\psi(\tau)\mathrm{d}\tau - cR_3\int_0^t \mathrm{e}^{-\gamma(t-\tau)}\psi(\tau)\mathrm{d}\tau = I_1 + I_2 \tag{17}$$

Let's call that:

$$I_1 = -c(R_1 + R_2)\int_0^t \mathrm{e}^{-\alpha(t-\tau)}\cos\beta(t-\tau)\psi(\tau)\mathrm{d}\tau \tag{18}$$

$$I_2 = -cR_3\int_0^t \mathrm{e}^{-\alpha(t-\tau)}\psi(\tau)\mathrm{d}\tau \tag{19}$$

By substituting eq. (5) into eq. (18), we can get:

$$|I_1| = c(R_1 + R_2)\left|\int_0^t \mathrm{e}^{-\alpha(t-\tau)}\cos\beta(t-\tau)\psi(\tau)\mathrm{d}\tau\right| =$$

$$= c(R_1 + R_2)\left|\int_0^t \mathrm{e}^{-\alpha(t-\tau)}\cos\beta(t-\tau)\mathrm{d}\tau\int_{-a}^0 G(\tau-\tau_1)v(\tau_1)\mathrm{d}\tau_1\right| \tag{20}$$

Assume that the response speed before the initial time is bounded and reasonable, namely:

$$|v(t)| \le M, \quad t \in [-a, 0]$$

Note that $|\cos(\beta t)| \le 1$, it can be obtained from eq. (20):

$$|I_1| \le c(R_1 + R_2)M\left|\int_0^t \mathrm{e}^{-\alpha(t-\tau)}\mathrm{d}\tau\int_{-a}^0 G(\tau-\tau_1)\mathrm{d}\tau_1\right| \tag{21}$$

Substituting $G(t) = \mu \mathrm{e}^{-\mu t}$ into eq. (21), we can get:

$$|I_1| \le c(R_1 + R_2)M(1-\mathrm{e}^{-\mu a})\left|\int_0^t \mathrm{e}^{-\alpha(t-\tau)-\mu\tau}\mathrm{d}\tau\right| = c(R_1 + R_2)M(1-\mathrm{e}^{-\mu a})\mathrm{e}^{-\alpha t}\left|\int_0^t \mathrm{e}^{(\alpha-\mu)\tau}\mathrm{d}\tau\right| =$$

$$= \frac{c(R_1 + R_2)M(1-\mathrm{e}^{-\mu a})}{|\alpha-\mu|}\left|\mathrm{e}^{-\mu t} - \mathrm{e}^{-\alpha t}\right| \tag{22}$$

Because of $\mu, \alpha > 0$, it is obvious that:

$$\lim_{t\to\infty}|I_1| = 0 \tag{23}$$

By substituting eq. (5) into eq. (19), we can get:

$$|I_2| = cR_3\left|\int_0^t \mathrm{e}^{-\gamma(t-\tau)}\psi(\tau)\mathrm{d}\tau\right| = cR_3\left|\int_0^t \mathrm{e}^{-\gamma(t-\tau)}\mathrm{d}\tau\int_0^t \mathrm{e}^{-\gamma(t-\tau)}(\tau)\mathrm{d}\tau\int_{-a}^0 G(\tau-\tau_1)v(\tau_1)\mathrm{d}\tau_1\right| \le$$

$$\le McR_3(1-\mathrm{e}^{-\mu a})\left|\int_0^t \mathrm{e}^{-\gamma(t-\tau)\mu\tau}\mathrm{d}\tau\right| = McR_3(1-\mathrm{e}^{-\mu a})\mathrm{e}^{-\gamma t}\left|\int_0^t \mathrm{e}^{-\gamma(t-\tau)\tau}\mathrm{d}\tau\right| =$$

$$= \frac{McR_3(1-\mathrm{e}^{-\mu a})}{|\gamma-\mu|}\left|\mathrm{e}^{-\mu t}-\mathrm{e}^{-\gamma t}\right| \tag{24}$$

Because of $\mu, \gamma > 0$, it is obvious that:

$$\lim_{t\to\infty}|I_2| = 0 \tag{25}$$

So far, we have shown that:

$$\lim_{t\to\infty}|I_1| = 0 \quad \text{and} \quad \lim_{t\to\infty}|I_2| = 0$$

which means that the first term of eq. (14) also gradually decreases to zero over time, *i.e.*:

$$-c\int_0^t h(t-\tau)\psi(\tau)\mathrm{d}\tau \to 0, \quad \text{when} \quad t\to\infty$$

Obviously, we get that the last three terms of eq. (14) also gradually decrease to zero over time. Therefore, the solution of the equation of motion in eq. (14) converges to zero, *i.e.*:

$$x(t) \to 0, \quad \text{when} \quad t\to\infty \tag{26}$$

In other words, the initial response of an exponentially damped oscillator will gradually decay with time until it reaches stability, this is called as the pull-down stability in [33].

In practical applications, it reveals the fact that the historical response of past motion does not affect the stability of such systems.

Numerical simulation

As the exponential damping model reflects the memory of mechanical properties of viscoelastic materials, we conduct numerical simulation on the initial response of the exponential damping oscillator to verify the stability of the initial response of such systems.

Assume the mass of oscillator m = 5 kg, stiffness k = 500 N/s, damping coefficient c = 40 Ns/m, other parameters are μ = 5, a = 4 seconds.

By introducing the following new variables:

$$x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = \int_0^t \mathrm{e}^{-\mu(t-\tau)}\dot{x}(\tau)\mathrm{d}\tau$$

Equation (7) can be converted into:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c\mu}{m}x_3 - \frac{c}{m}\psi(t) \\ \dot{x}_3 &= x_2 - \mu x_3 \end{aligned} \tag{27}$$

The vibration motion in the time period $[-a, 0]$ is specified by the following two specific historical functions:

$$v_1(t) = 0.5\cos 5t, \quad v_2(t) = \sin 5t$$

According to eq. (5), the initialization functions are calculated:

$$\psi_1(t) = 0.25\text{e}^{-\mu t}, \quad \psi_2(t) = -0.5\text{e}^{-\mu t}$$

The initial conditions of eq. (27) are, respectively:

$$(0.5 \quad 0 \quad 0)^{\text{T}}, \quad (0 \quad -5 \quad 0)^{\text{T}}$$

Figure 2 shows the initial responses of exponentially damped oscillators with different historical responses. It can be clearly seen from the simulation diagrams that the exponentially damped oscillator adopts four specific historical functions, respectively. After the initial time, the initial response gradually weakens until it reaches a stable state. Therefore, we can draw a conclusion that although the past motion history is different, the initial response of the oscillator from the initial time always decays gradually with the increase of time, and finally reaches stability. The numerical simulation results verify the stability of the initial response of the oscillator.

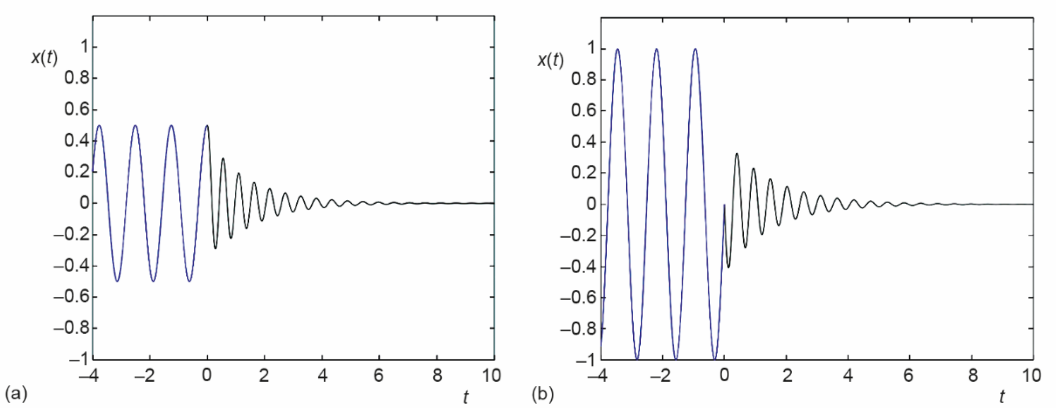

Figure 2. Initialization response with past history; (a) $v_1 = 0.5\cos 5t$, (b) $v_2 = \sin 5t$

Conclusion

In this paper, the stability of the initial response of an exponentially damped oscillator is proved. Both theoretical analysis and numerical simulation show that under different vibration history conditions, the initial response of such a system decreases gradually with the increase of time until it reaches a stable state. This means that while past movements affect the dynamical behavior of these systems, they have no effect on stability. This phenomenon reveals the fact that for a single degree of freedom exponentially damped oscillator without external force, the vibration always stops due to internal damping, regardless of the history of past vibration motion.

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