NON-LINEAR OSCILLATION OF A MASS ATTACHED TO A STRETCHED ELASTIC WIRE IN A FRACTAL SPACE

by

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The challenge for a non-linear vibration system in a fractal space is more fractal dimensions than frequency-amplitude relationship, the system energy consumption depends upon its fractal property, so its best-case scenario is to establish a relationship among the fractal dimensions, frequency and amplitude. For this purpose, this paper studies a fractal-fractional vibration system of a mass attached to a stretched elastic wire in a fractal space, and its asymptotic periodic property is elucidated, the effect of the fractal dimensions on the vibration system is discussed. This paper offers a new road to fast and reliable analysis of fractal oscillators with high accuracy.

Key words: *fractals calculus, mechanical engineering, frequency formulation, two-scale transform method, non-linear oscillator*

Introduction

Vibration systems occur everywhere, the periodic property is the focus of many practical applications, for examples, a plate vibration [1], energy harvesting devices [2-4], a nanobeam vibration [5-7] and micro-electromechanical systems [8-12]. The periodic property and damping characteristic are widely studied in the non-linear vibration theory, in order to give a best-case scenario of a damped vibration system, much effort had been paid, and new theories were appeared frequently, for examples, the fractional model [13, 14] and viscoelastic model [15, 16].

Vibration properties in air or water or other porous media depends upon the porosity and its distribution. A pendulum [17] on the Earth surface and in a microgravity environment [18] and in a vacuum behaves quite differently, the vacuum can be considered as a continuum space, where Newton's second law can be applied to establish a governing equation, and the air can be considered as a fractal space, its fractal dimensions matter the vibration properties greatly, and it is pivotally important to tackle the fractal space. It was reported that the fractal dimensions affect greatly porous material mechanical property and thermal response [19-21].

The fractal vibration theory was raised as a promising tool to tackle the intractable problems. Much achievement was obtained for fractal Duffing equation [22-24] and forced fractal vibration systems [25, 26]. In this paper we will consider a more complex fractal vibration equation with strongly non-linearity.

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Fractal vibration system

We consider the following fractal vibration system:

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t^{2\alpha}} + \lambda\vartheta - \frac{\kappa\vartheta^n}{\sqrt{1+\vartheta^2}} = 0\,, \quad \vartheta(0) = A, \quad \frac{\mathrm{d}\vartheta}{\mathrm{d}t^{\alpha}}(0) = 0 \tag{1}$$

where ϑ is the displacement, κ and λ and n – the constants, A – the amplitude, $\mathrm{d}\vartheta/\mathrm{d}t^{\alpha}$ – the fractal derivative defined [27-30]:

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t^{\alpha}}(t_0) = \Gamma(1+\alpha) \lim_{\substack{t-t_0 \to \Delta t \\ \Delta t \neq 0}} \frac{\vartheta - \vartheta_0}{(t-t_0)^{\alpha}} \tag{2}$$

Applications of the two-scale fractal derivative are referred in [31-35]. When $\alpha = 1$, the system was studied in [36] by the variational iteration method [37, 38], it is an oscillation of a mass attached to the center of a stretched elastic wire [39].

Equation (1) describes the vibrating process in a fractal space, when it vibrates in a vacuum $\alpha = 1$, the fractal dimensions can be calculated by He-Liu formulation [19], and it mainly depends upon the air density.

Frequency-amplitude relationship

In order to elucidate the *best-case scenario* of the fractal vibration system, we use He's frequency formulation [40, 41]. Due to it simple and reliability, it can give a fact and profound insight into the vibrating properties, and various modifications were appeared in literature, see for examples [42-44]. He and Liu [45] gave a mathematical explanation of the formulation.

To illustrate the formulation, we consider the following general fractal oscillator:

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}t^{2\alpha}} + f(\vartheta) = 0 \tag{3}$$

where f is a non-linear function, and it requires $f(0) = 0$ and $f(\vartheta)/\vartheta > 0$.

The original frequency formulation reads [40, 41]:

$$\omega^2 = \left.\frac{\mathrm{d}f(\vartheta)}{\mathrm{d}\vartheta}\right|_{u=A/2} \tag{4}$$

Lyu, *et al*. [44] suggested the following modification:

$$\omega^2 = \frac{1}{m}\sum_{m=1}^{m} \left.\frac{\mathrm{d}f(\vartheta)}{\mathrm{d}\vartheta}\right|_{\vartheta=A/m} \tag{5}$$

In our work, $f(\vartheta)$ is:

$$f(\vartheta) = \lambda\vartheta - \frac{\kappa\vartheta^n}{\sqrt{1+\vartheta^2}} \tag{6}$$

When $\lambda = 1$, $n = 1$, we have:

$$\frac{\mathrm{d}f(\vartheta)}{\mathrm{d}\vartheta} = 1 - \frac{\kappa}{(1+\vartheta^2)^{3/2}} \tag{7}$$

Consequently, the approximate frequency-amplitude relationship is obtained:

$$\omega^2 = \frac{1}{3}\left(3 - \frac{\kappa}{1+\frac{1}{4}A^2} - \frac{\kappa}{1+\frac{1}{9}A^2} - \frac{\kappa}{1+\frac{1}{16}A^2}\right) \tag{8}$$

The approximate solution is:

$$\vartheta = A\cos(\omega t^{\alpha}) \tag{9}$$

Equation (9) gives the best-case scenario of the relationship among the fractal dimensions, frequency and amplitude. At the initial stage when time tends to zero, the low frequency property is found as discussed in [46].

When $\alpha = 1$, the period is:

$$\mathrm{T} = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{1}{3}\left(3 - \frac{\kappa}{1+\frac{1}{4}A^2} - \frac{\kappa}{1+\frac{1}{9}A^2} - \frac{\kappa}{1+\frac{1}{16}A^2}\right)}} \tag{10}$$

Table 1 shows the comparison of the approximate period with exact period [36, 39], a good agreement is observed.

Table 1. Comparison of the approximate period with exact period when $\lambda = 1$, $n = 1$

	k = 0.1			k = 0.3		
A	T	T_{ex}	Error percentage	T	T_{ex}	Error percentage
1	6.579483268	6.537507892	0.006420699	7.324461060	7.155651996	0.023591010
2	6.502821496	6.452446504	0.007807115	7.021439852	6.837327148	0.026927584
25	6.287957148	6.299196556	0.001784260	6.297533546	6.331607060	0.005381495
	k = 0.25			k = 0.65		
0.2	7.248427556	7.237568032	0.001500438	10.56565873	10.47921219	0.008249335
3	6.700338880	6.604323916	0.014538197	7.584244452	7.243648216	0.047019985
20	6.301688874	6.333569396	0.005033579	6.331637639	6.416892452	0.013285996

Conclusion

In this work, a robust strategy is proposed to unlock the relationship among the fractal dimensions, frequency and amplitude by He's frequency formulation, and it can be used as an illustrating example of other sophisticated applications to fractal vibration systems. The investigation reveals the low frequency property at the initial stage of the vibrating process, raising the promising possibility of developing fractal vibration devices for advanced applications.

References

[1] Hoang, L. T. T., A New C_0 Third-Order Shear Deformation Theory for the Non-linear Free Vibration Analysis of Stiffened Functionally Graded Plates, *Facta Universitatis Series: Mechanical Engineering, 19* (2021), 2, pp. 285-305

[2] He, C. H., *et al*., Hybrid Rayleigh -Van der Pol-Duffing Oscillator (HRVD): Stability Analysis and Controller, *Journal of Low Frequency Noise, Vibration & Active Control, 41* (2022), 1, pp. 244-268

[3] He, C. H., *et al*., Controlling the Kinematics of a Spring-Pendulum System Using an Energy Harvesting Device, *Journal of Low Frequency Noise, Vibration & Active Control, 41* (2022), 3, pp. 1234-1257

[4] He, C. H., El-Dib, Y. O., A Heuristic Review on the Homotopy Perturbation Method for Non-conservative Oscillators, *Journal of Low Frequency Noise Vibration and Active Control, 41* (2022), 2, pp. 572-603

[5] Jankowski, P., Detection of Non-local Calibration Parameters and Range Interaction for Dynamics of FGM Porous Nanobeams Under Electro-Mechanical Loads, *Facta Universitatis Series: Mechanical Engineering, 20* (2022), 3, pp. 457-478

[6] Faghidian, S. A., Tounsi, A., Dynamic Characteristics of Mixture Unified Gradient Elastic Nanobeams, *Facta Universitatis Series: Mechanical Engineering, 20* (2022), 3, pp. 539-552

[7] Limkatanyu, S., *et al*., Bending, Bucking and Free Vibration Analyses of Nanobeam-Substrate Medium Systems, *Facta Universitatis Series: Mechanical Engineering, 20* (2022), 3, pp. 561-587

[8] He, J.-H., *et al*., Pull-in Stability of a Fractal MEMS System and Its Pull-in Plateau, *Fractals, 30* (2022), 9, 2250185

[9] Tian, D., *et al*., Fractal N/MEMS: From Pull-in Instability to Pull-in Stability, *Fractals*, *29* (2021), 2, 2150030

[10] Tian, D., He, C. H., A Fractal Micro-Electromechanical System and Its Pull-in Stability, *Journal of Low Frequency Noise Vibration and Active Control, 40* (2021), 3, pp. 1380-1386

[11] He, C. H., A Variational Principle for a Fractal Nano/Microelectromechanical (N/MEMS) System, *International Journal of Numerical Methods for Heat & Fluid Flow, 33* (2023), 1, pp. 351-359

[12] He, J.-H., Fast Identification of the Pull-in Voltage of a Nano/Micro-Electromechanical System, *Journal of Low Frequency Noise Vibration and Active Control, 41* (2022), 2, pp. 566-571

[13] Srivastava, H. M., *et al*., An Efficient Analytical Technique for Fractional Model of Vibration Equation, *Applied Mathematical Modelling*, *45* (2017), May, pp. 192-204

[14] Kuo, P. H., *et al*., Novel Fractional-Order Convolutional Neural Network Based Chatter Diagnosis Approach in Turning Process with Chaos Error Mapping, *Non-linear Dynamics*, *111* (2023), 8, pp. 7547-7564

[15] Liang, Y. H., Wang, K. J., A New Fractal Viscoelastic Element: Promise and Applications to Maxwell-Rheological Model, *Thermal Science, 25* (2021), 2B, pp. 1221-1227

[16] Xiu, G. Z., *et al*., Integral Representation of the Viscoelastic Relaxation Function, *Journal of Shanghai Normal University* (*Natural Sciences*), *48* (2019), 3, pp. 242-251

[17] He, J.-H., *et al*. Periodic Property and Instability of a Rotating Pendulum System, *Axioms, 10* (2021), 3, 191

[18] Wang, K. L., He's Frequency Formulation for Fractal Non-linear Oscillator Arising in a Microgravity Space, Numer, *Meth. Part. D. E., 37* (2020), 2, pp. 1374-1384

[19] He, C. H., Liu, C., Fractal Dimensions of a Porous Concrete and Its Effect on the Concrete's Strength, *Facta Universitatis Series: Mechanical Engineering, 21* (2023), 1, pp. 137-150

[20] He, C. H., *et al*., A Fractal Model for the Internal Temperature Response of a Porous Concrete, *Applied and Computational Mathematics*, *21* (2022), 1, pp. 71-77

[21] Liu, F. J., *et al*., Thermal Oscillation Arising in a Heat Shock of a Porous Hierarchy and Its Application, *Facta Universitatis Series: Mechanical Engineering*, *20* (2022), 3, pp. 633-645

[22] Feng, G. Q., He's Frequency Formula to Fractal Undamped Duffing Equation, *Journal of Low Frequency Noise Vibration and Active Control, 40* (2021), 4, pp. 1671-1676

[23] El-Dib, Y. O., *et al*., An Innovative Technique to Solve a Fractal Damping Duffing-Jerk Oscillator, *Communications in Theoretical Physics, 75* (2023), 5, 055001

[24] He, J.-H., *et al*., Homotopy Perturbation Method for Fractal Duffing Oscillator with Arbitrary Conditions, *Fractals, 30* (2022), 9, 22501651

[25] Elias-Zuniga, A., *et al*., Investigation of the Steady-State Solution of the Fractal Forced Duffing's Oscillator Using Ancient Chinese Algorithm, *Fractals, 29* (2021), 6, 2150133

[26] He, J.-H., *et al*., Forced Non-linear Oscillator in a Fractal Space, *Facta Universitatis, Series: Mechanical Engineering, 20* (2022), 1, pp. 1-20
[27] He, J.-H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, *Int. J. Theor. Phys., 53* (2014), June, pp. 3698-718
[28] He, J.-H., Fractal cAlculus and Its Geometrical Explanation, *Results.Phys., 10* (2018), Sept., pp. 272-276
[29] He J. H., *et al.*, A Tutorial Introduction to the Two-Scale Fractal Calculus and Its Application to the Fractal Zhiber-Shabat Oscillator, *Fractals, 29* (2021), 8, 2150268
[30] He, J.-H., Ain, Q. T., New Promises and Future Challenges of Fractal Calculus: From Two-Scale Thermodynamics to Fractal Variational Principle, *Thermal Science, 24* (2020), 2A, pp. 659-681
[31] Wang, K. L., *et al*., Physical Insight of Local Fractional Calculus And Its Application to Fractional KdV-Burgers-Kuramoto Equation, *Fractals, 27* (2019), 7, 1950122
[32] Wang, K. L., He, C. H., A Remark on Wang's Fractal Variational Principle, *Fractals, 27* (2019), 8, 1950134
[33] Ain, Q. T., He, J.-H., On Two-Scale Dimension and Its Applications, *Thermal Science, 23* (2019), 3B, pp. 1707-1712
[34] He, J.-H., Ji, F. Y., Two-Scale Mathematics and Fractional Calculus for Thermodynamics, *Thermal Science, 23* (2019), 4, pp. 2131-2133
[35] Qian M. Y., He J. H., Two-Scale Thermal Science for Modern Life –Making the Impossible Possible, *Thermal Science, 26* (2022), 3B, pp. 2409-2412
[36] Kachapi, S. H., *et al*., Periodic Solution for Strongly Non-linear Vibration Systems by He's Variational Iteration Method, *Math. Method. Appl. Sci., 32* (2009), 18, pp. 2339-2349
[37] Wang, S. Q., He, J.-H., Variational Iteration Method for Solving Integro-Differential Equations, *Physics letters A, 367* (2007), 3, pp. 188-191
[38] Deng, S. X., Ge, X. X., The Variational Iteration Method for Whitham-Broer-Kaup System with Local Fractional Derivatives, *Thermal Science*, *26* (2022), 3B, pp. 2419-2426
[39] Sun, W. P., *et al*., Approximate Analytical Solutions for Oscillation of a Mass Attached to a Stretched Elastic Wire, *Journal of Sound & Vibration, 300* (2007), 3-5, pp. 1042-1047
[40] He, J.-H., The Simplest Approach to Non-linear Oscillators, *Results in Physics, 15* (2019), 102546
[41] He, J.-H., The Simpler, the Better: Analytical Methods for Non-linear Oscillators and Fractional Oscillators, *Journal of Low Frequency Noise Vibration and Active Control, 38* (2019), 3-4 pp. 1252-1260
[42] Ma, H. J., Simplified Hamiltonian-Based Frequency-Amplitude Formulation for Non-linear Vibration Systems, *Facta Universitatis-Series Mechanical Engineering, 20* (2022), 2, pp. 445-45.
[43] Tian, Y., Frequency Formula for a Class of Fractal Vibration System, *Reports in Mechanical Engineering, 3* (2022), 1, pp. 55-61
[44] Lyu, G. J., *et al*., Straightforward Method for Non-linear Oscillators, *Journal of Donghua University* (*English Edition*), *40* (2023), 1, pp. 105-109
[45] He, C. H., Liu, C., A Modified Frequency-Amplitude Formulation for Fractal Vibration Systems, *Fractals, 30* (2022), 3, 2250046
[46] He, C. H., *et al*., Low Frequency Property of a Fractal Vibration Model for a Concrete Beam, *Fractals, 29* (2021), 5, 2150117

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