# ANALYSIS OF A FRACTAL MODIFICATION OF ATTACHMENT OSCILLATOR

by

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*In this paper, we consider a combined technique for a fractal modification of the attachment oscillator arising from nanotechnology. This technique is called as TSFT-GRHBM by coupling the two-scale fractal transformation and the global residue harmonic balance method. The approximations and frequencies of this fractal attachment oscillator are given without linearization. Numerical results are provided to confirm its efficiency.*

Key words: *attachment oscillator, fractional complex transformation, global residue harmonic balance method*

## Introduction

Non-linear oscillation has wide applications in science and engineering areas, especially, the temeperture oscillation under a sudden thermal shock plays an important role in the cocoon's biofunction [1], the release oscillation in a hollow fiber gives a new release phenonmon of ions [2], a spring-pendulum system can be used as a tranducer for energy harvesting [3, 4], and a nano/micro beam vibration can be used as a sensor with extremenly high sensitivity [5-10].

Attachment oscillation is a new branch of both physics and mathematics, it can model a nanofiber vibration [11] and nanofibers formation mechanism [12]. The nanofibers can be obtained by either the electrospinning [13-15] or the bubble electrspinning [16, 17]. However, due to fast solvent evaporation, the porous nanofibers have to be considered, so a fractal modification of the attachment oscillation has be adopted:

$$\frac{\mathrm{d}^{2\alpha}u}{\mathrm{d}t^{2\alpha}}+u+\varepsilon_1 u^3+\frac{\varepsilon_2}{u^3}=0 \tag{1}$$

where the fractal derivative $\mathrm{d}^{\alpha}u/\mathrm{d}t^{\alpha}$ is defined by [18-22]:

$$\frac{\mathrm{d}^{\alpha}u}{\mathrm{d}t^{\alpha}}=\Gamma(1+\alpha)\lim_{\substack{t-t_0\to\Delta t\\ \Delta t\neq 0}}\frac{u(t)-u(t_0)}{(t-t_0)^{\alpha}} \tag{2}$$

with a fractal order $0<\alpha\le 1$. When $\alpha$ = 1, the fractal eq. (1) reduces to the original attachment oscillator in [11, 12].

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In recent years, the fractal modifications of non-linear oscillators have been paid much attention, including the fractal N/MEMS system [23], the fractal Duffing oscillator [24-27], the fractal Toda oscillator [28, 29], the fractal Chen-Lee-Liu equation [30], and the fractal Yao-Cheng oscillator [31].

When $\alpha$ = 1, eq. (1) was solved by the homotopy perturbation method [11] and He's frequency formulation [12]. Though both methods are widely used in the non-linear vibration theory, the homotopy perturbation method is extremely suitable for various non-linear vibration systems [32-34], while He's frequency formulation, though simple, is valid for non-linear oscillators without secular terms [35-40]. The secularity in the fractal attachment oscillation makes it difficult to give an exact solution, thus, we focus on the investigation of the numerical approximation to eq. (1).

We consider a combined technique based upon the two-scale fractal transformation [41] and the global residue harmonic balance method [31, 42] (named as TSFT-GRHBM). It was already used for solving the fractal Yao-Cheng oscillator [31]. We will consider an initial value problem of the fractal attachment oscillator (1) as an example to illustrate the efficiency of the combined technology. To remove the difficulty arising from the fractal operator in eq. (1), we first transform eq. (1) to the classical attachment oscillator by using the two-scale fractal transformation [41]. The first and second approximations are given with the help of the global residue harmonic balance method [42-44].

To further illustrate the efficiency of this method, numerical sensitivity analysis of the approximations and frequencies with respect to different parameters is provided. Numerical comparisons with Runge-Kutta method are also presented to confirm its efficiency. The numerical behavior of the fractal approximations to eq. (1) with different orders is finally considered.

## The TSFT-GRHBM technique

We briefly introduce the main idea of the TSFT-GRHBM technique proposed by Lu and Chen [31]. It is a coupling method based on the two-scale fractal transformation and the global residue harmonic balance method. The details of this technique are illustrated below.

*Two-scale fractal transformation (TSFT).* Consider the following fractal PDF:

$$f(u,u_t^{\alpha},u_x^{\beta},u_t^{2\alpha},u_x^{2\beta},\cdots)=0 \tag{3}$$

where

$$u_t^{\alpha}=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} \text{ and } u_x^{\beta}=\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}$$

are two fractal operators defined by He's fractal derivative given in eq. (2) with $0<\alpha\le 1$ and $0<\beta\le 1$, respectively.

By introducing the two-scale fractal transformation proposed by He [22, 41]:

$$T=t^{\alpha},\quad X=x^{\beta} \tag{4}$$

we have:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial u}{\partial T} \tag{5}$$

$$\frac{\partial^{\beta}u(x,t)}{\partial x^{\beta}}=\frac{\partial u}{\partial X} \tag{6}$$

Then the original fractional PDE, eq. (3), can be transformed to an ordinary PDE. The applications of the two-scale fractal transformation can be seen in [45-52].

*Global residue harmonic balance method (GRHBM).* Consider the following differential equation:

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = f(u), \quad u(0) = A, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = 0 \tag{7}$$

with a given constant $A$ and a non-linear function $f$ satisfying $f(-u) = -f(u)$.

We consider an auxiliary variable $\tau = \omega\tau$ with a unknown frequency $\omega$, and rewrite eq. (7):

$$\omega^2 \frac{\mathrm{d}^2u}{\mathrm{d}\tau^2} = f(u), \quad u(0) = A, \quad \frac{\mathrm{d}u}{\mathrm{d}\tau}(0) = 0 \tag{8}$$

By GRHBM [31], we assume that the periodic solution exists and can be formulated:

$$u(\tau) = \sum_{k=1}^{N} A_k \cos(k\tau) \tag{9}$$

with

$$A_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(kt)\mathrm{d}t$$

By eq. (9), the first order approximation to eq. (7) is given by:

$$u_1(\tau) = A\cos(\tau), \quad \omega^2 = \omega_1^2 \tag{10}$$

with an unknown frequency $\omega_1$. We can substitute the initial approximation into eq. (8), and remove the secular term, which results in a non-linear system about $\omega_1$.

The $k^{\mathrm{th}}$ order approximation is further assumed:

$$u(\tau) = \sum_{i=1}^{k-1} u_{(i)}(\tau) + pu_k(\tau), \qquad \omega^2 = \omega_1^2 + \sum_{i=2}^{k-1}\omega_{(i)} + p\omega_k \tag{11}$$

where $p$ is an order parameter, $u_{(i)}(\tau)$ and $\omega_{(i)}$ are defined:

$$u_{(1)}(\tau) = u_1(\tau), \quad u_{(i)}(\tau) = \sum_{j=1}^{i-1} u_{(j)}(\tau) + u_i(\tau)$$

$$u_{(i)}(\tau) = \rho_{3,i}[\cos(\tau) - \cos(3\tau)] + \rho_{5,i}[\cos(\tau) - \cos(5\tau)] + \cdots + \rho_{2i+1,i}\{\cos(\tau) - \cos[(2i+1)\tau]\}$$

$$\omega_{(1)} = \omega_1, \quad \omega_{(i)}^2 = \omega_1^2 + \sum_{j=2}^{i-1}\omega_{(j)} + \omega_i, \quad i = 2, \cdots, k \tag{12}$$

By substituting eq. (11) into eq. (8), and calculating the coefficients of $p$-term, it follows a non-linear system denoted by $F_k(\tau, \rho_{3,k}, \rho_{5,k}, \cdots, \rho_{2k+1,k}, \omega_k)$. In this iteration procedure, the residual part of the $(k–1)^{\text{th}}$ order approximation is defined:

$$R_{k-1}(\tau) = \omega_{(k-1)}^2 \frac{\mathrm{d}^2 u_{(k-1)}}{\mathrm{d}\tau^2} - f[u_{(k-1)}] \tag{13}$$

The GRHBM method [31] suggests the coupling approach based on two non-linear functions $F_k(\tau, \rho_{3,k}, \rho_{5,k}, \cdots, \rho_{2k+1,k}, \omega_k)$ and $R_{k-1}(\tau)$:

$$F_k(\tau, \rho_{3,k}, \rho_{5,k}, \cdots, \rho_{2k+1,k}, \omega_k) + R_{k-1}(\tau) = 0 \tag{14}$$

The unknown parameters in eq. (11) can be obtained from the linear equations by letting the coefficients of the harmonic terms as zero.

## Analysis of fractal attachment oscillator by TSFT-GRHBM

We consider the initial value problem of the fractal attachment oscillator (1) with the following initial conditions:

$$\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}(0) = 0, \quad u(0) = A \tag{15}$$

By using the two-scale fractal transformation $T = t^\alpha$, we transform (1) as the following non-linear equation:

$$u^4 \frac{\mathrm{d}^2 u}{\mathrm{d}T^2} + u^5 + \varepsilon_1 u^7 + \varepsilon_2 u = 0 \tag{16}$$

The initial conditions can be reformulated:

$$\frac{\mathrm{d}u}{\mathrm{d}T}(0) = 0, \quad u(0) = A \tag{17}$$

Obviously, the constrained condition for GRHBM holds by eq. (16). Thus, we introduce an auxiliary variable $\tau = \omega\tau$, and reformulate eq. (16):

$$u^4 \omega^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} + u^5 + \varepsilon_1 u^7 + \varepsilon_2 u = 0 \tag{18}$$

with the initial conditions defined by:

$$\frac{\mathrm{d}u}{\mathrm{d}\tau}(0) = 0, \quad u(0) = A \tag{19}$$

We assume that the initial approximation to eq. (18) is given by:

$$u_1(\tau) = A\cos(\tau), \quad \omega^2 = \omega_1^2 \tag{20}$$

By substituting eq. (20) into eq. (18), it follows:

$$\left(\frac{5}{8}A^5 + \frac{35}{64}A^7\varepsilon_1 + A\varepsilon_2 - \frac{5}{8}A^5\omega_1^2\right)\cos\tau + \left(\frac{5}{16} + \frac{21}{64}A^2\varepsilon_1 - \frac{5}{16}\omega_1^2\right)A^5\cos 3\tau +$$
$$+\left(\frac{1}{16} + \frac{7}{64}A^2\varepsilon_1 - \frac{1}{16}\omega_1^2\right)A^5\cos 5\tau + \frac{1}{64}A^7\varepsilon_1\cos 7\tau = 0 \tag{21}$$

In order to remove the secular term in eq. (21), the coefficients of the harmonic term $\cos\tau$ should be equal to zero. This implies that:

$$\frac{5}{8}A^5+\frac{35}{64}A^7\varepsilon_1+A\varepsilon_2-\frac{5}{8}A^5\omega_1^2=0 \tag{22}$$

It follows that the first order approximated frequency is given by:

$$\omega_1=\sqrt{1+\frac{7}{8}A^2\varepsilon_1+\frac{8\varepsilon_2}{5A^4}} \tag{23}$$

Different with eq. (20), the second order approximation to eq. (18) is defined by:

$$u(\tau)=u_1(\tau)+pu_2(\tau),\quad \omega^2=\omega_1^2+p\omega_2 \tag{24}$$

where $u_2(\tau)=\rho[\cos(\tau)-\cos(3\tau)]$ and $p$ is a perturbation parameter. The key point of GRHBM lies in the determination of two unknown parameters $\rho$ and $\omega_2$. We substitute eq. (24) into eq. (18), and collect the coefficients of the $p$-term, which results in a non-linear function denoted by $F_k(\tau,\rho,\omega_k)$. Recalling the residual part of eq. (21), we denote it by:

$$R_0(\tau)=\left(\frac{5}{16}+\frac{21}{64}A^2\varepsilon_1-\frac{5}{16}\omega_1^2\right)A^5\cos3\tau+$$
$$+\left(\frac{1}{16}+\frac{7}{64}A^2\varepsilon_1-\frac{1}{16}\omega_1^2\right)A^5\cos5\tau+\frac{1}{64}A^7\varepsilon_1\cos7\tau \tag{25}$$

We couple these two non-linear functions as the following equation:

$$F_k(\tau,\rho,\omega_k)+R_0(\tau)=0 \tag{26}$$

Again, the approximation in GRHBM requires no secular term, and we set the coefficients of two harmonic terms including $\cos(\tau)$ and $\cos(3\tau)$ as zero. This results in the following equations:

$$\Gamma_1\rho+\Gamma_2\omega=0 \tag{27}$$

$$\Gamma_3\rho+\Gamma_4\omega=\Gamma_5 \tag{28}$$

where $\Gamma_i\,(i=1,\cdots,5)$ are defined by:

$$\begin{gathered}\Gamma_1=\frac{25}{16}A^4+\frac{49}{32}A^6\varepsilon_1+\varepsilon_2+\frac{15}{16}A^4\omega_1^2\\ \Gamma_2=-\frac{5}{8}A^5\\ \Gamma_3=-\frac{5}{16}A^4-\varepsilon_2+\frac{53}{16}A^4\omega_1^2\\ \Gamma_4=-\frac{5}{16}A^5\\ \Gamma_5=-\frac{5}{16}A^5-\frac{21}{64}A^7\varepsilon_1+\frac{5}{16}A^5\omega_1^2\end{gathered} \tag{29}$$

By eq. (27) and eq. (28), we have the following results:

$$\omega_2 = \frac{\Gamma_1\Gamma_5}{\Gamma_1\Gamma_4 - \Gamma_2\Gamma_3} \tag{30}$$

$$\rho = \frac{\Gamma_2\Gamma_5}{\Gamma_2\Gamma_3 - \Gamma_1\Gamma_4} \tag{31}$$

Thus, the second order approximated frequency can be formulated:

$$\hat{\omega}_2 = \sqrt{1 + \frac{7}{8}A^2\varepsilon_1 + \frac{8\varepsilon_2}{5A^4} + \frac{\Gamma_1\Gamma_5}{\Gamma_1\Gamma_4 - \Gamma_2\Gamma_3}} \tag{32}$$

By FCT with $T = t^\alpha$, we finally obtain the fractal approximation:

$$u(\tau) = A\cos(\hat{\omega}_2 t^\alpha) + \rho[\cos(\hat{\omega}_2 t^\alpha) - \cos(3\hat{\omega}_2 t^\alpha)] \tag{33}$$

where $\hat{\omega}_2$ and $\rho$ are given by eqs. (32) and (31), respectively. We remark that the higher order approximations can be given by similar procedure of GRHBM. Generally, the second order approximated solutions can guarantee the high accuracy, which will be confirmed by the numerical results in the next section.

## Numerical comparisons

In this section, we consider the initial value problem associated with the fractal attachment oscillator (1). The numerical results including the attachment oscillators with the integer or fractal order derivative are investigated in detail. We test TSFT-GRHBM for this non-linear problem, and compare it with Runge-Kutta method. For simplicity, the first and second order approximations obtained by TSFT-GRHBM are denoted by $\text{GRHBM}_1$ and $\text{GRHBM}_2$, respectively.

We first consider the fractal attachment oscillator (1) with the integer order $\alpha = 1$. In this example, we set $\varepsilon_1 = \varepsilon_2 = 0.01$. The numerical investigation of the approximations with large amplitudes is considered to illustrate the efficiency and stability of TSFT-GRHBM. For this purpose, we will consider four cases including *A* = 10, 20, 30, and 40. Figure 1 shows the numerical results of eq. (1) with *A* = 10. The numerical comparisons of the approximations by TSFT-GRHBM and RK for (1) with *A* = 10 are plotted on the left side of fig. 1. The error curves of the approximations are given on the right side of fig. 1, where the log error is defined:

$$\log_{\text{error}} = \log\left|\frac{u_{RK} - \hat{u}}{u_{RK}}\right| \tag{34}$$

with $\hat{u}$ given by TSFT-GRHBM. By fig. 1, TSFT-GRHBM performs well for the large amplitude case. The accuracy of the approximations is further improved by considering the residual part in GRHBM, since the log error of $\text{GRHBM}_2$ is less than that of $\text{GRHBM}_1$. To further consider the impact of the large amplitude on the approximations by TSFT-GRHBM, we plot in figs. 2-4 the numerical results of the approximated solutions to eq. (1) with rest amplitudes. Again, $\text{GRHBM}_2$ works better than $\text{GRHBM}_1$. We also consider the frequency dependence with respect to the amplitude. The frequency curve is plotted in fig. 5. It is easy to find that the approximated frequency is monotonic increasing about the amplitude A.

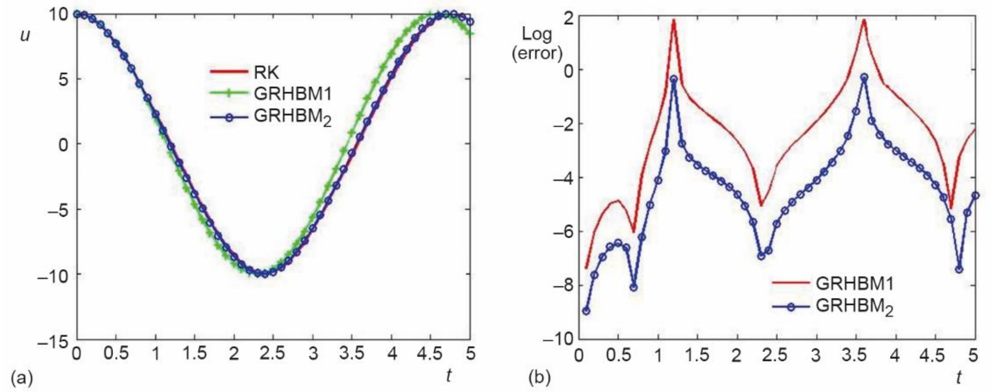


**Figure 1. Results of the approximations by TSFT-GRHBM for eq. (1) with *A* = 10**

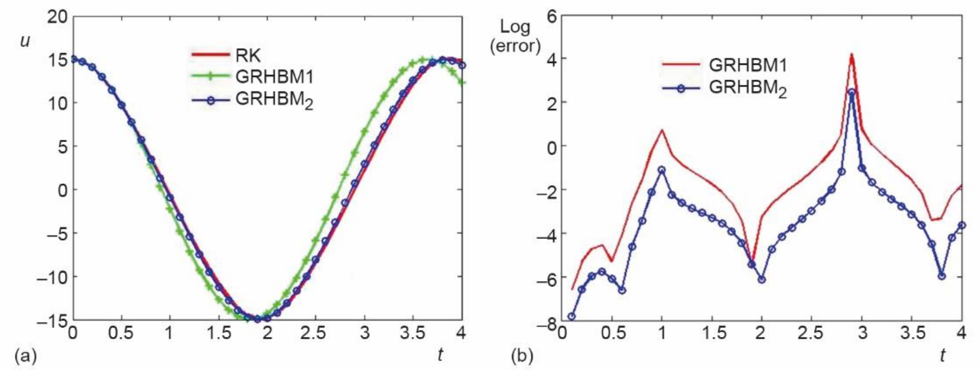


**Figure 2. Results of the approximations by TSFT-GRHBM for eq. (1) with *A* = 15**

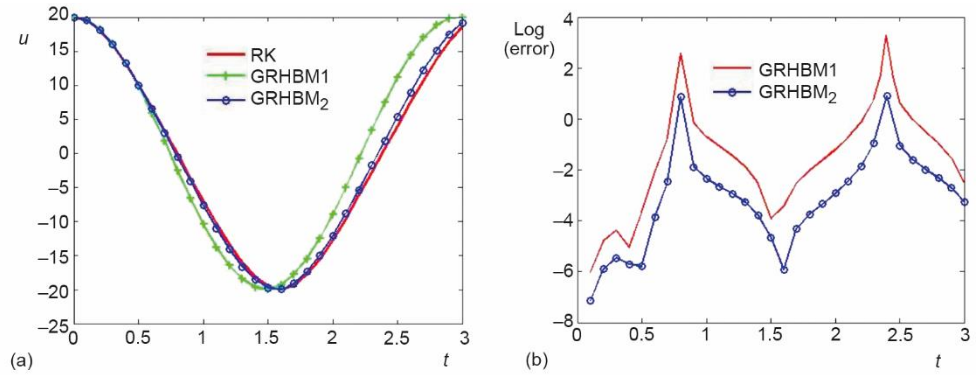


**Figure 3. Results of the approximations by TSFT-GRHBM for eq. (1) with *A* = 20**

We then consider the fractal attachment oscillator with different fractal orders. The parameters are the same as that in previous part. We test the non-linear oscillators with $\alpha = 0.2, 0.4, 0.6, 0.8, 1.$ By the left side of fig. 6, we see that the oscillation behavior becomes more complex when the value of $\alpha$ approaches to a small value. For the rest large amplitudes

*A*, the oscillation behavior is similar (see figs. 6 and 7). By the provided results, we can conclude that TSFT-GRHBM is efficient and stable for the fractal attachment oscillator.

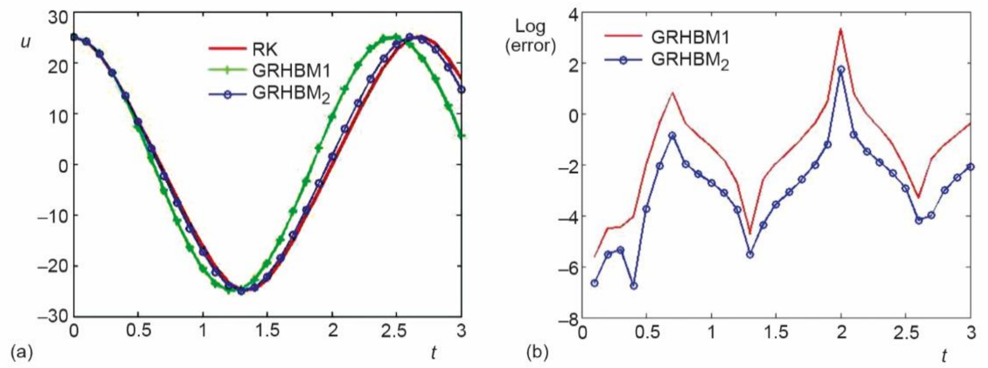


**Figure 4. Results of the approximations by TSFT-GRHBM for eq. (1) with *A* = 25**

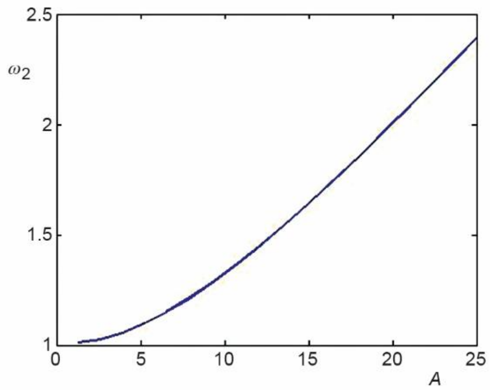


**Figure 5. Approximated frequency curve of eq. (1) with 0 < *A* ≤ 25**

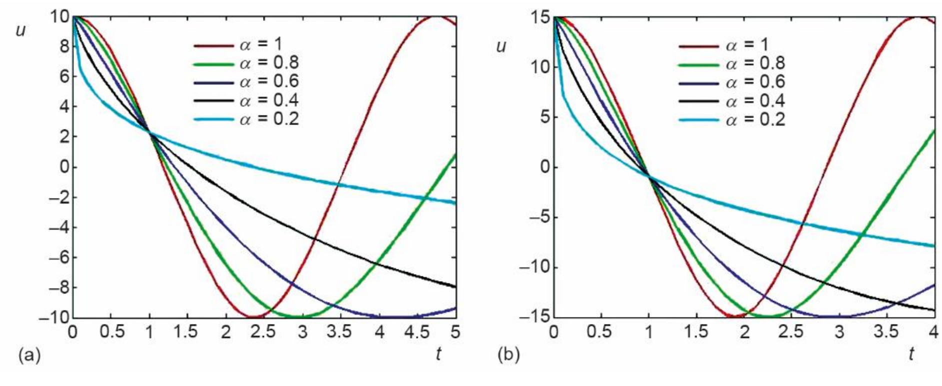


**Figure 6. Numerical behavior of fractal solutions of eq. (1) with *A* = 10 (a) and *A* = 15 (b)**

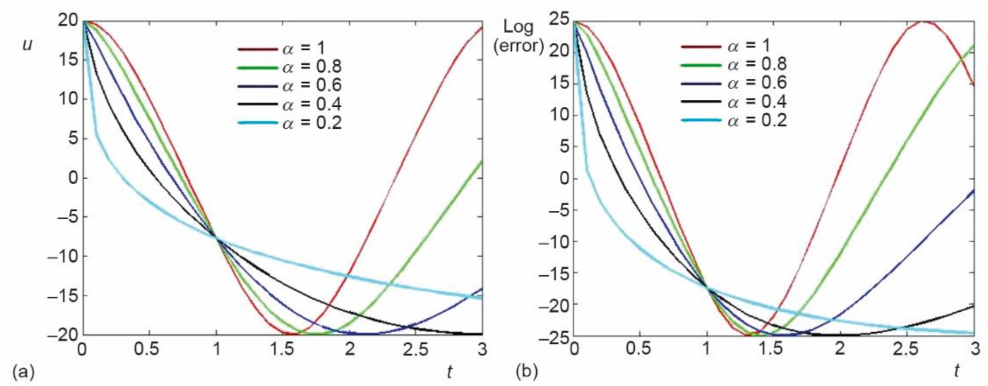


**Figure 7. Numerical behavior of fractal solutions of eq. (1) with $A$ = 20 (a) and $A$ = 25 (b)**

## Conclusion

This paper proposed a numerical approach based on the two-scale fractal transformation and the global residue harmonic balance method for solving the fractal attachment oscillator. The approximations and frequencies are provided without complicated computation. Numerical comparisons and sensitive analysis are presented to confirm the efficiency and stability of the proposed technique. In future work, we will apply this approach to other fractal oscillators and fractal chatter diagnosis [52, 53].

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