LOCAL FRACTIONAL DAMPED NON-LINEAR OSCILLATION
Frequency Estimation and Energy Consumption

by

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This paper studies a local fractional vibration system with a damped non-linear term to reveal its frequency property and its energy consumption. A modification of He's frequency formulation is recommended for this purpose. Some examples are given to illustrate the solving process and the reliability of the method. Additionally, the effect of the initial conditions on the vibrating properties is elucidated. This paper offers a new window for fast and effective insight into local fractional vibration systems.

Key words: *local fractional derivative, singular oscillator, tangent oscillator, non-conservative system*

Introduction

The damped non-linear oscillator is an important model in science and engineering [1-4], the damped term consumes the energy of the vibrating system at each cycle, and finally the system will stop its periodic motion when the system runs out of its energy. Energy exhausting and energy harvesting have been the focus of the modern thermal science. Due to the serious energy crisis, it is of paramount importance to reduce energy consumption and to improve energy harvesting efficiency. Nowadays the nanotechnology is widely used for energy harvesting [5-7] to enhance thermal conduction through the boundary-layer. A suitable boundary layer can also lead to a minimal energy consumption as shown in the lotus effect or the surface wetting property of nanofiber membrane [8-10], and the sand dunes motion in desert [11]. The energy harvesting is to convert the vibration energy to electronic energy, and frequency property affects greatly its energy harvesting efficiency [12-16].

The frequency estimation and its energy consumption are two main factors in practical applications. How to solve the damped non-linear oscillator is a hot research topic in both mathematics and engineering, and a fast and effective estimation of its frequency property is much needed in engineering applications, though there are some analytical methods for studying the frequency property, *e.g.*, the residual method coupled with Adomian decomposition method [17], the variational iteration method [18, 19], the variational approach [20], the homotopy perturbation method [21-24], and He's frequency formulation [25-31]. However, the methods for local fractional vibration systems with damped terms were rare in literature.

The local fractional calculus [32-34] provides a new mathematical tool for the study of non-absolutely continuous and non-differentiable functions. Because of the quantitative re-

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lation between the calculus order of a fractal function and the Hausdorff dimension of its set, the local fractional calculus has potential advantages in describing fractal phenomena [32, 33]. Local differential operators are widely used in mathematical theory and engineering technology to explore the differential properties of no differentiable functions and fractal objects. Such as water layer, turbulence, porous media, *etc*. In this paper, the local fractional damping non-linear oscillator is applied to reveal the characteristics of vibration problems in a Cantor stepped fractal medium. Moreover, image simulation analysis can be carried out intuitively.

This paper introduces a new method for estimating the frequency of local fractional damped oscillation under specific initial conditions. According to this method, the new frequency estimation formulas of the damped tangent oscillator, the damped singular oscillator, the damped hyperbolic tangent oscillator and the oscillation with non-linear damping force are all given.

Local fractional damped non-linear oscillation and its energy consumption

In this paper, we consider the following local fractional damped oscillation:

$$\frac{\mathrm{d}^{2\alpha}\Theta}{\mathrm{d}t^{2\alpha}} + \mu\frac{\mathrm{d}^{\alpha}\Theta}{\mathrm{d}t^{\alpha}} + f(\Theta) = 0, \quad \Theta(0) = a, \quad \Theta^{(\alpha)}(0) = b \tag{1}$$

where μ, a, and b are all constants in fractal space, f is a non-linear function of Θ, and $\mathrm{d}^{\alpha}\Theta/\mathrm{d}t^{\alpha}$ is the local fractional derivative of order α [32, 33], the physical explanation of local fractional calculus is available in [35-37].

Recently fractional vibration systems [38, 39] and fractal vibration systems [40-48] were appeared in literature to model various vibration systems where the traditional vibration theory becomes invalid. Though eq. (1) can be effectively solved by the homotopy perturbation method [49-55], here is suggested a modification of He's frequency formulation [25-31] to insight into its frequency property

First, we linearize eq. (1) into the following equation:

$$\frac{\mathrm{d}^{2\alpha}\Theta}{\mathrm{d}t^{2\alpha}} + \mu\frac{\mathrm{d}^{\alpha}\Theta}{\mathrm{d}t^{\alpha}} + \varpi^2\Theta = 0, \quad \Theta(0) = a, \quad \Theta^{(\alpha)}(0) = b \tag{2}$$

where ϖ is an undetermined constant. Its solution has one following form:

$$\Theta(t) = AE_{\alpha}\left(-\frac{\mu t}{2}\right)\xi(t) \tag{3}$$

where E_{α} is a modification of exponential function in the local fractional sense [32, 33] and A – the amplitude, it can be approximately calculated as:

$$A = \sqrt{a^2 + \frac{1}{\Omega^2}\left(b + \frac{\mu}{2}\right)^2} \tag{4}$$

and $\xi(t)$ is the function determined by the following equation:

$$\xi^{(2\alpha)}(t) + \Omega^2\xi(t) = 0 \tag{5}$$

and Ω is given by:

$$\Omega^2 = \varpi^2 - \frac{\mu^2}{4} \tag{6}$$

According to eqs. (1) and (3), we have:

$$f(\Theta) - \varpi^2 \Theta = 0 \tag{7}$$

We can take the derivative of eq. (7) with respect to the Θ and use the frequency formulation [25-31]. Then, we can get:

$$\varpi^2 = \frac{\mathrm{d}^\alpha f(\Theta)}{\mathrm{d}^\alpha \Theta}\Big|_{\Theta=\pm NA} \tag{8}$$

where N is the constant.

By substituting the solution of eq. (7) into eq. (3), the solution of eq. (2) can be written:

$$\Theta(t) = AE_\alpha\left(-\frac{\mu t}{2}\right)\cos_\alpha(\Omega t + \varphi) = \sqrt{a^2 + \frac{1}{\Omega^2}\left(b + \frac{\mu}{2}\right)^2} E_\alpha\left(-\frac{\mu t}{2}\right)\cos_\alpha(\Omega t + \varphi) \tag{9}$$

where ϕ is determined by the initial conditions:

$$\varphi = \arctan_\alpha\left[-\frac{1}{a\Omega}\left(b + \frac{\mu}{2}\right)\right] \tag{10}$$

Hereby $\cos_\alpha t$ and $\arctan_\alpha t$ are modified functions of cos(t) and arctan(t) in the local fractional sense, respectively [32, 33].

The energy consumption is shown in the attenuation of the amplitude, see eq. (3). The amplitude, $AE_\alpha(-\mu t/2)$, reduces exponentially with respect to time. When the amplitude is zero, the system consumes all its vibration energy, and it forbids a periodic motion. This property is similar to the pull-down stability of a non-linear vibration system with even non--linearities [56].

Illustrating examples

In order to elucidate the solving process, three examples are given here.

Damped tangent oscillator

The damped tangent oscillator with local fractional derivative is:

$$\frac{\mathrm{d}^{2\alpha}\Theta}{\mathrm{d}t^{2\alpha}} + \mu\frac{\mathrm{d}^\alpha\Theta}{\mathrm{d}t^\alpha} + \omega_0 \tan_\alpha(\Theta) = 0, \quad \Theta(0) = a, \quad \Theta^{(\alpha)}(0) = b \tag{11}$$

When $\alpha = 1$, eq. (11) turns out to be traditional tangent oscillator [57, 58]. According to eqs. (8) and (11), we can choose $N = \sqrt{3}/2$ in eq. (8), then we have:

$$\varpi^2 = \omega_0 \frac{\mathrm{d}^\alpha \tan_\alpha(\Theta)}{\mathrm{d}\Theta^\alpha}\Big|_{u=\sqrt{3}A/2} \tag{12}$$

where A is given in eq. (4).

For given parameters of ω_0, a, and b, the frequency can be calculated easily from eq. (11).

For small amplitude, by virtue of the following equation:

$$\tan_\alpha \Theta \approx \frac{\Theta^\alpha}{\Gamma(1+\alpha)} + \frac{\Theta^{3\alpha}}{3\alpha} \tag{13}$$

Equation (12) becomes:

$$\varpi^2 = \omega_0 \frac{\mathrm{d}^\alpha \tan_\alpha(\Theta)}{\mathrm{d}\Theta^\alpha}\Big|_{u=\sqrt{3}A/2} \approx \omega_0\left(1+\frac{3A^2}{4}\right) \tag{14}$$

In view of eq. (4), we can convert eq. (14) into the following form:

$$\varpi^2 = \omega_0\left(1+\frac{3A^2}{4}\right) = \omega_0\left\{1+\frac{3}{4}\left[a^2+\frac{1}{\Omega^2}\left(b+\frac{\mu}{2}\right)^2\right]\right\} \tag{15}$$

This is:

$$\Omega^2 = \varpi^2 - \frac{\mu^2}{4} = \omega_0\left\{1+\frac{3}{4}\left[a^2+\frac{1}{\Omega^2}\left(b+\frac{\mu}{2}\right)^2\right]\right\} - \frac{\mu^2}{4}$$

or

$$\Omega^4 - \left[\omega_0\left(1+\frac{3}{4}a^2\right)-\frac{\mu^2}{4}\right]\Omega^2 - \omega_0\left(b+\frac{\mu}{2}\right)^2 = 0 \tag{16}$$

Solving Ω from eq. (16) and ignoring the meaningless root gives:

$$\Omega = \sqrt{\frac{\omega_0\left(1+\frac{3}{4}a^2\right)-\frac{\mu^2}{4}+\sqrt{\left[\omega_0\left(1+\frac{3}{4}a^2\right)-\frac{\mu^2}{4}\right]^2+4\omega_0\left(b+\frac{\mu}{2}\right)^2}}{2}} \tag{17}$$

By virtue of eqs. (4), (9), (10), and (17), the approximate solution of eq. (11) can be expressed clearly and concretely.

Damped hyperbolic tangent oscillator

The damped hyperbolic tangent oscillator with local fractional derivatives reads:

$$\frac{\mathrm{d}^{2\alpha}\Theta}{\mathrm{d}t^{2\alpha}} + \mu\frac{\mathrm{d}^\alpha\Theta}{\mathrm{d}t^\alpha} + \omega_0 \tanh_\alpha(\Theta) = 0, \quad \Theta(0) = a, \quad \Theta^{(\alpha)}(0) = b \tag{18}$$

When $\alpha = 1$, eq. (18) turns out to be traditional hyperbolic tangent oscillator [57, 59]. For small amplitude, $\tanh_\alpha(\Theta)$ can be approximated:

$$\tanh_\alpha \approx \frac{\Theta^\alpha}{\Gamma(1+\alpha)} - \frac{\Theta^{3\alpha}}{3\alpha} \tag{19}$$

According to eqs. (3), (18), and (19), we can choose $N=\sqrt{3}/2$ in eq. (3), then we can soon obtain the following relation:

$$\varpi^2=\omega_0\frac{\mathrm{d}^\alpha\tanh_\alpha(\Theta)}{\mathrm{d}\Theta^\alpha}\Big|_{u=\sqrt{3}A/2}\approx\omega_0\left(1-\frac{3A^2}{4}\right) \tag{20}$$

where A is defined in eq. (4).

In view of eq. (4), eq. (20) turns out to be:

$$\varpi^2=\omega_0\frac{\mathrm{d}^\alpha\tanh_\alpha(\Theta)}{\mathrm{d}u^\alpha}\Big|_{u=\sqrt{3}A/2}\approx\omega_0\left\{1-\frac{3}{4}\left[a^2+\frac{1}{\Omega^2}\left(b+\frac{\mu}{2}\right)^2\right]\right\} \tag{21}$$

This is:

$$\Omega^2=\varpi^2-\frac{\mu^2}{4}=\omega_0\left\{1-\frac{3}{4}\left[a^2+\frac{1}{\Omega^2}\left(b+\frac{\mu}{2}\right)^2\right]\right\}-\frac{\mu^2}{4}$$

or

$$\Omega^4-\left[\omega_0\left(1-\frac{3}{4}a^2\right)-\frac{\mu^2}{4}\right]\Omega^2-\omega_0\left(b+\frac{\mu}{2}\right)^2=0 \tag{22}$$

Solving Ω from eq. (22) and ignoring the meaningless root gives:

$$\Omega=\sqrt{\frac{\omega_0\left(1-\frac{3}{4}a^2\right)-\frac{\mu^2}{4}+\sqrt{\left[\omega_0\left(1-\frac{3}{4}a^2\right)-\frac{\mu^2}{4}\right]^2+4\omega_0^2\left(b+\frac{\mu}{2}\right)^2}}{2}} \tag{23}$$

By virtue of eqs. (8)-(10), and (23), the approximate solution of eq. (18) can be expressed clearly and concretely.

Damped singular oscillator

Now we consider a damped singular oscillator with local fractional derivatives:

$$\frac{\mathrm{d}^{2\alpha}\Theta}{\mathrm{d}t^{2\alpha}}+\mu\frac{\mathrm{d}^\alpha\Theta}{\mathrm{d}t^\alpha}+\frac{1}{k\Theta}=0,\quad u(0)=a,\quad u^{(\alpha)}(0)=b \tag{24}$$

When $\alpha=1$, eq. (24) turns out to be traditional singular oscillator [60]. According to eqs. (4), (8), and (24), we can soon obtain the following relations:

$$\varpi^2=\frac{1}{k\Theta^2}\Big|_{\Theta=NA}=\frac{1}{kN^2A^2}=\frac{1}{kN^2\left[a^2+\frac{1}{\Omega^2}\left(b+\frac{\mu}{2}\right)^2\right]}$$

and

$$\Omega^2=\varpi^2-\frac{\mu^2}{4}=\frac{1}{kN^2\left[a^2+\frac{1}{\Omega^2}\left(b+\frac{\mu}{2}\right)^2\right]}-\frac{\mu^2}{4} \tag{25}$$

That is:

$$\Omega^2 = \frac{\Omega^2}{kN^2\left[a^2\Omega^2 + \left(b + \frac{\mu}{2}\right)^2\right]} - \frac{\mu^2}{4} \tag{26}$$

Solving Ω from eq. (26) and ignoring the meaningless root gives:

$$\Omega = \sqrt{\frac{-\left[kN^2\left(b+\frac{\mu}{2}\right)^2 + \frac{\mu^2}{4}kN^2a^2 - 1\right] + \sqrt{\left[kN^2\left(b+\frac{\mu}{2}\right)^2 + \frac{\mu^2}{4}kN^2a^2 - 1\right]^2 - k^2N^4a^2\left(b+\frac{\mu}{2}\right)^2\mu^2}}{2kN^2a^2}} \tag{27}$$

By virtue of eqs. (8)-(10), and (27), the approximate solution of eq. (24) can be expressed clearly and concretely.

Conclusion

In this paper, a modified He’s frequency formulation [25-31] is proposed for local fractional damped non-linear oscillators. The given examples show its reliability and simplicity, the frequency property of a local fractional damped oscillator is different from that for traditional damped oscillators, the energy consumption of the local fractional vibration systems has some amazing property and has wide applications to practical engineering vibration systems. This paper opens a new road for the optimal design of a local fractional vibration system, and offers a new window for minimizing the energy consumption by suitable controlling the damped term and detecting chatter in machine process [61], the local fractional vibration system can also incorporate with the machine learning technology [62, 63].

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