A VARIABLE COEFFICIENT mKdV DYNAMIC MODEL FOR NON-LINEAR LONG WAVE

by

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In this paper, we obtained a variable coefficient partial differential model that characterizes non-linear long waves with topography effects through the multi-scale perturbation expansion method, especially the new model caused by the variation of background shear flow over time. Next, the expansion Jacobian elliptic function method is used to provide an analytical solution for the model and analyze its wave characteristics.

Key words: *non-linear long wave, expansion Jacobi elliptic function, variable coefficient mKdV*

Introduction

The non-linear long wave theory has always received a lot of attention in large-scale atmospheric and oceanic dynamics. In recent years, models such as the Quartic-KdV model, ZK-Burgers model, ZK-mZK-BBM equation have been used to characterize such non-linear waves [1-4]. In addition these 2-D and 3-D wave models, variable coefficient non-linear models will be more suitable for explaining actual atmospheric and oceanic non-linear phenomena. Fu *et al*. [5] derived the extended variable coefficient KdV (VC-KdV) equation for large amplitude equatorial Rossby solitary wave under an external forcing in a shear flow. A non-local constant coefficient KdV (CC-KdV) equation with shifted parity and delayed time reversal is derived and two kinds of non-linear wave excitations are presented explicitly and graphically [6]. A variable coefficient Schrödinger equation derived from vorticity equation [7]. The aforementioned results show that time-dependent variable coefficients affect the amplitude, direction, and velocity of waves which considered as an important factor leading to the development of the dynamical system.

For partial differential models, some scholars have provided various analytical solutions, including the Solitary and periodic solutions [8, 9], M-lump solution and N-soliton [10]. In fact, the models of the solution can be derived as the improved tan-expansion method [11], generalized homogeneous balance method [12], and Hirota's bilinear method [13, 14]. For the variable coefficient partial differential mode, the aforementioned method is not suitable, some solutions to the variable-coefficient equations are obtained with the help of Jacobi elliptic functions for the non-linear long waves [15]. In this paper, based on the quasi geostrophic potential vorticity mode, a new VC-mKdV model for long wave is obtained by using multi-scale analysis

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and perturbation methods, and we give an analytical solution through the expansion Jacobian elliptic function.

The derivation of variable coefficient mKdV model

Firstly, the quasi-geostrophic model with topographic effect is given [16]:

$$\left(\frac{\partial}{\partial t}+\frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}-\frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}\right)\left[\beta(y)y+\nabla^2\Psi+h(y)\right]=0 \tag{1}$$

where $\beta(y)$ is the variable with beta effect of linear variation with dimension in Coriolis force, and the side boundary condition in dimensionless form is:

$$\frac{\partial \Psi}{\partial x}=0,\ \ y=0,1 \tag{2}$$

Taking the base flow as the shear zonal flow:

$$\hat{\Psi}(y)=-\int_0^y[u(y,t)-c_0+\varepsilon\gamma]\mathrm{d}y \tag{3}$$

where $u(y, t)$ is the time-varying shear fundamental flow, which corresponds to the phase velocity of a linear long wave in the shear fundamental flow, equivalent to doing the transformation $x = x - c_0t$, $0 < \varepsilon \ll 1$, and κ – the detuned parameter and the total stream function is:

$$\Psi(x,y,t)=-\int_0^y[u(y,t)-c_0+\varepsilon\kappa]\mathrm{d}y+\varepsilon\psi(x,y,t) \tag{4}$$

where $\psi(x, y, t)$ is the disturbed stream function. Substituting eq. (4) into eq. (1) combined with the boundary condition eq. (2), we obtain the equations about the perturbed stream function:

$$\begin{aligned}&\frac{\partial^3\psi(x,y,t)}{\partial t\partial x^2}-\frac{\partial^2}{\partial t\partial y}u(y,t)+\frac{\partial^3\psi(x,y,t)}{\partial t\partial y^2}+\\&+\left[u(y,t)-c_0+\delta\kappa-\frac{\partial\psi(x,y,t)}{\partial y}\right]\left[\frac{\partial^3\psi(x,y,t)}{\partial x^3}+\frac{\partial^3\psi(x,y,t)}{\partial x\partial y^2}\right]+\\&+\frac{\partial\psi(x,y,t)}{\partial x}\left[\frac{\partial^3\psi(x,y,t)}{\partial x^2\partial y}-\frac{\partial^2}{\partial y^2}u(y,t)+\frac{\partial^3\psi(x,y,t)}{\partial y^3}+y\frac{\mathrm{d}\beta(y)}{\mathrm{d}y}+\beta(y)+\frac{\mathrm{d}h(y)}{\mathrm{d}y}\right]=0\end{aligned} \tag{5}$$

Considering that u is the background flow, and $u(y, t)$ is slowly varying about time, t, which can be set:

$$u(y,t)=u(y)+\delta u(t),\ \ 0<\delta\ll 1 \tag{6}$$

We perform G-M transformation on the variable x, t, $X = \varepsilon^{1/2}x$, $T = \varepsilon^{3/2}t$, and in order to balance the non-linear effect in the equation with the dispersion effect:

$$\frac{\partial}{\partial x}=\varepsilon^{1/2}\frac{\partial}{\partial x},\ \frac{\partial}{\partial t}=\varepsilon^{3/2}\frac{\partial}{\partial T} \tag{7}$$

We set $\delta u(t) = \varepsilon u(T)$, and substitute eqs. (6) and (7) into eq. (5):

$$\begin{aligned}&\varepsilon^{5/2}\frac{\partial^3\psi}{\partial T\partial X^2}+\varepsilon^{3/2}\left(\frac{\partial^3\psi}{\partial T\partial y^2}+\frac{\partial\psi}{\partial X}\frac{\partial^3\psi}{\partial X^2\partial y}\right)-u''(y)+\frac{\partial^3\psi}{\partial y^3}+\beta'(y)+\beta(y)+h'(y)+\\&+\left[u(y)+\varepsilon u(T)-c_0+\delta\kappa-\frac{\partial\psi}{\partial y}\right]\left(\varepsilon^{\frac{3}{2}}\frac{\partial^3\psi}{\partial X^3}+\sqrt{\varepsilon}\frac{\partial^3\psi}{\partial X\partial y^2}\right)\end{aligned} \tag{8}$$

We perform a small parameter expansion on the disturbed stream function:

$$\psi = \psi_1(X,y,T) + \varepsilon\psi_2(X,y,T) + \varepsilon^2\psi_3(X,y,T) + \cdots \tag{9}$$

Substituting eq. (9) into eq. (8), we obtain the expression for ε from low to high order:

$$\varepsilon^{\frac{3}{2}} : [u(y)-c_0]\frac{\partial^2}{\partial y^2}\left(\frac{\partial \psi_1}{\partial X}\right) + [\beta(y) + \beta'(y)y - u''(y) + h'(y)]\frac{\partial \psi_1}{\partial X} = 0,\ \frac{\partial \psi_1}{\partial X} = 0,\ y = 0,1 \tag{10}$$

Let $\psi_1 = A(X, T)\phi_1(\mathrm{y})$ satisfy the eigenvalues:

$$\frac{\mathrm{d}^2\phi_1(y)}{\mathrm{d}y^2} + \frac{\beta(y) + \beta'(y)y - u''(y) + h'(y)}{u(y)-c_0}\phi_1(y) = 0,\ \phi_1(0) = \phi_1(1) = 0 \tag{11}$$

$$\phi_1(0) = \phi_1(1) = 0 \tag{12}$$

In general:

$$u(y) - c_0 < 0,\ \text{if}\ \ \beta(y) + \beta'(y)y - u''(y) \neq 0$$

in order to determine the amplitude $A(X, T)$, we also have to solve the higher order problem for the case of order $\varepsilon^{5/2}$, *e.g.*:

$$\begin{gathered}\varepsilon^{5/2} : \frac{\partial \psi_2}{\partial X}\left[y\beta'(y) - u''(y) + \beta(y) + h'(y)\right] + \frac{\partial^3 \psi_2}{\partial X \partial y^2}\left[u(y) - c_0\right] = \\ = -\left\{\frac{\partial^3 \psi_1}{\partial T \partial y^2} - u(T)\frac{\partial^3 \psi_1}{\partial X \partial y^2} + \frac{\partial^3 \psi_1}{\partial X \partial y^2}\frac{\partial \psi_1}{\partial y} - \frac{\partial^3 \psi_1}{\partial X \partial y^2}\kappa + \frac{\partial^3 \psi_1}{\partial X^3}\left[u(y) - c_0\right] + \frac{\partial \psi_1}{\partial X}\frac{\partial^3 \psi_1}{\partial y^3}\right\} = F\end{gathered} \tag{13}$$

Along the same lines $\psi_2 = B(X, T)\phi_2(\mathrm{y})$, and with:

$$\phi_1\frac{\partial^2 \phi_2}{\partial y^2} = \frac{\partial}{\partial y}\left(\phi_1\frac{\partial \phi_2}{\partial y}\right) - \frac{\partial}{\partial y}\left(\phi_2\frac{\partial \phi_1}{\partial y}\right) + \phi_2\frac{\partial^2 \phi_1}{\partial y^2}$$

we ultimately simplified from eqs. (11) and (13) to:

$$\int_0^1 \frac{\phi_1(y)F}{u(y)-c_0}\mathrm{d}y = 0 \tag{14}$$

We get:

$$\begin{gathered}\int_0^1 \frac{\phi_1(y)}{u(y)-c_0}\frac{\partial^2\phi_1(y)}{\partial y^2}\mathrm{d}y\frac{\partial A}{\partial T} - u(T)\frac{\partial A}{\partial X}\int_0^1 \frac{\phi_1(y)}{u(y)-c_0}\frac{\partial^2\phi_1(y)}{\partial y^2}\mathrm{d}y - \kappa\int_0^1 \frac{\phi_1(y)}{u(y)-c_0}\frac{\partial^2\phi_1(y)}{\partial y^2}\mathrm{d}y\frac{\partial A}{\partial X} + \\ + \int_0^1 \frac{\phi_1(y)}{u(y)-c_0}\frac{\partial\phi_1(y)}{\partial y}\frac{\partial{\phi_1}^2(y)}{\partial y^2}\mathrm{d}yA\frac{\partial A}{\partial X} + \int_0^1 \frac{{\phi_1}^2(y)}{u(y)-c_0}\frac{\partial^3\phi_1(y)}{\partial y^3}\mathrm{d}y\frac{\partial A}{\partial X}A + \int_0^1 {\phi_1}^2(y)\mathrm{d}y\frac{\partial^3 A}{\partial X^3} = 0\end{gathered} \tag{15}$$

Finally, we obtain:

$$\frac{\partial A}{\partial T} + [u(T) + \alpha_0]\frac{\partial A}{\partial X} + \alpha_1 A\frac{\partial A}{\partial X} + \alpha_2\frac{\partial^3 A}{\partial X^3} = 0 \tag{16}$$

where

$$\begin{gathered}I = \int_0^1 \frac{\phi_1(y)}{u(y)-c_0}\frac{\partial^2\phi_1(y)}{\partial y^2}\mathrm{d}y,\ \alpha_0 = -\frac{\kappa}{I}\int_0^1 \frac{\phi_1(y)}{u(y)-c_0}\frac{\partial^2\phi_1(y)}{\partial y^2}\mathrm{d}y \\ \alpha_1 = \frac{1}{I}\int_0^1\left[\frac{\phi_1(y)}{u(y)-c_0}\frac{\partial\phi_1(y)}{\partial y}\frac{\partial{\phi_1}^2(y)}{\partial y^2} + \frac{{\phi_1}^2(y)}{u(y)-c_0}\frac{\partial^3\phi_1(y)}{\partial y^3}\right]\mathrm{d}y \\ \alpha_2 = -\frac{1}{I}\int_0^1 \phi_1^2(y)\mathrm{d}y\end{gathered} \tag{17}$$

Equation (16) degenerates into the KdV equation without the latitudinal background flow change slowly with time $u(T)$:

$$\frac{\partial A}{\partial T}+\alpha_1 A\frac{\partial A}{\partial X}+\alpha_2\frac{\partial^3 A}{\partial X^3}=0 \tag{18}$$

As is well known, the traveling wave solution of eq. (18) is:

$$A(X,T)=B_0\text{sech}^2\sqrt{\frac{a_1B_0}{12a_2}}(X-c_aT) \tag{19}$$

where $B_0 = 3c_a/a_1$, c_a is the wave speed of the traveling wave in a slowly varying space time. If there is a stable solitary wave, we can obtain $a_1a_2B_0 > 0$, when $a_1a_2 > 0$, we can get $B_0 > 0$, the system exists peaked solitary wave, and *vice versa*, there is a trough waveform, then a_2 and c_a have the same sign, according to the coefficient

$$\beta(y)+\beta'(y)y-u''(y)+h'(y)$$

in eq. (11), in the northern hemisphere mid-latitude sea, if there is no shear in the background current or the shear is not too strong

$$\beta(y)+\beta'(y)y-u''(y)+h'(y)>0,\ \text{so that}\ \alpha_2<0$$

it can be judged that the solitary wave has the characteristic of propagation the westward propagation.

In order to explore the fluctuation solution of eq. (16), it is observed that the coefficient $u(T)$ of this $u(T)(\partial A/\partial X)$ is a function of time, and considering $u(T)$ as a generalized function, we solve it by the method of variable coefficients Jacobian elliptic functions. Let:

$$A=A(\xi),\ \xi=f(T)X+\text{g}(T) \tag{20}$$

where $f(T)$, $\text{g}(T)$ are the undetermined coefficients of the independent variable t, according to the balance between the non-linear term and the highest order term, the solution is obtained in the form:

$$A(\xi)=\gamma_0(T)+\gamma_1(T)\text{sn}(\xi)+\gamma_2(T)\text{sn}^2(\xi) \tag{21}$$

We substitute eqs. (20)-(21) into eq. (16) to obtain an expression for the coefficients $\gamma_0(T)$, $\gamma_1(T)$, $\gamma_2(T)$:

$$\begin{aligned}&\gamma_0'+\gamma_1'\text{sn}(\xi)+\gamma_2'\text{sn}^2(\xi)+\gamma_1[(u(T)+\alpha_0)f+f'x+g'+\alpha_1 f\gamma_0-(1+m^2)\alpha_2 f^3]\text{cn}\xi\text{dn}\xi+\\&+\{2\gamma_2[f'x+g'+(u(T)+\alpha_0)f]+\alpha_1 f(\gamma_1^2+\gamma_0\gamma_2)-8(1+m^2)\alpha_2 f^3\gamma_2\}\text{sn}(\xi)\text{c}n\xi\text{dn}\xi+\\&+3\gamma_1 f[\alpha_1\gamma_2+2m^2\alpha_2 f^2]\text{sn}^2(\xi)\text{cn}\xi\text{dn}\xi+2\gamma_2 f[\alpha_1\gamma_2+12m^2\alpha_2 f^2]\text{sn}^3(\xi)\text{cn}\xi\text{dn}\xi=0\end{aligned} \tag{22}$$

We obtain the values of coefficients through a system of algebraic equations $\gamma_0 = r_0$, $\gamma_1 = 0$, r_0, r_2 are constants, $f(t) = k_0$, $\gamma_2 = 12m^2k_0^2(\alpha_2/\alpha_1)$, $m(0 < m < 1)$ is the modulus:

$$\text{g}(t)=\int_0^T\left\{4(1+m^2)\alpha_2k_0^3-(u(T)+\alpha_0)k_0\right\}\text{d}t \tag{23}$$

so

$$A(\xi)=r_0-12m^2\frac{\alpha_2}{\alpha_1}k_0^2+12m^2\frac{\alpha_2}{\alpha_1}k_0^2\text{cn}^2\xi \tag{24}$$

$$\xi=k_0\left\{x+k_0^2\int_0^T\left[4(1+m^2)\alpha_2-(u(T)+\alpha_0)\right]\text{d}t\right\} \tag{25}$$

where r_0, k_0 are constants.

Conclusions

In the barotropic Earth fluids, we derive a variable coefficient mKdV model that characterizes non-linear long waves. The time-varying shear flow can affect the linear term of the model. In addition, topography is one of the factors that excite non-linear waves, and topography can change the wave crest (trough) shape of way.

Next, we use the extended Jacobian elliptic function method to calculate the variable coefficient partial differential mode and obtain its theoretical solution. If the u(T) term is ignored, the solution can degenerate into the standard previous study. Finally, the analytical solution of the equation helps us explain the dynamic phenomena in the ocean, which will be further explored in future research.

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Nomenclature

t – time, [s]

x, y, z – co-ordinates, [m]

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