LOCAL FRACTIONAL DUFFING EQUATION
Its Periodic Property and its Application to Energy Harvesting

by

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A local fractional modification of the Duffing equation is considered, and the homotopy perturbation method is employed to reveal its frequency-amplitude relationship, which is of paramount importance in the optimal design of the energy harvesting devices and chatter detection. Effects of the initial conditions on the periodic property is also discussed.

Key words: *Duffing equation, homotopy perturbation method, chatter signal*

Introduction

Duffing equation is a ubiquitous model for non-linear vibration systems [1-4] because many complex vibration problems can be finally simplified to the well-known Duffing equation or its modification. A nano/micro beam vibration plays an import role in engineering applications [5-7] and micro/nano electromechanical systems [8-13], furthermore a vibration system can be used for analysis of the energy harvesting efficiency [14-16], and thermal property of the cocoon-like porous medium [17]. In literature, Duffing equation has been widely used as a good paradigm for verification of some an analytical method, *e.g.*, the fractional residual method [18], the variational iteration method [19, 20], the variational approach [21], the homotopy perturbation method [22, 23] and its various modifications [24, 25], and He's frequency formulation [26-31].

In this paper, the homotopy perturbation method [22, 23] will be applied to reveal the frequency-amplitude relationship of Duffing equation with local fractional derivatives [32-34], and its application to energy harvesting will be discussed.

Preliminaries of local fractional calculus

In this section, we introduce some mathematical preliminaries of the local fractional calculus theory in fractal space for our subsequent development.

Definition 1. Suppose that there is [32, 33]:

$$\left|u(t)-u(t_0)\right|<\varepsilon^{\alpha} \tag{1}$$

with $\left|t-t_0\right|<\delta,\ \text{for}\ \varepsilon,\delta>0$ and $\varepsilon,\delta\in R$, then $u(t)$ is called local fractional continuous at $t = t_0$ and it is denoted by $\lim u(t) = u(t_0)$.

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Definition 2. Suppose that the function $u(t)$ is satisfied the condition (1) for $t \in (a, b)$, it is called local fractional continuous on the interval (a, b), denoted by [32, 33]:

$$u(t) \in C_\alpha(a,b) \tag{2}$$

Definition 3. In fractal space, let $u(t) \in C_\alpha(a, b)$, the local fractional derivative of $u(t)$ of order α at $t = t_0$ is given by [32, 33]:

$$\mathrm{D}_t^{(\alpha)} u(t_0) = u^{(\alpha)}(t_0) = \frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha}\Big|_{t=t_0} = \lim_{t\to t_0} \frac{\Delta^\alpha [u(t)-u(t_0)]}{(t-t_0)^\alpha} \tag{3}$$

where $\Delta^\alpha [u(t)-u(t_0)] \cong \Gamma(1+\alpha)\Delta[u(t)-u(t_0)]$.

Definition 4. Let the function $u(t)$ satisfied the condition (2), the local fractional integral of $u(t)$ of order α in the interval $[a, b]$ is defined by [32, 33]:

$${}_aI_b^{(\alpha)} u(t) = \frac{1}{\Gamma(1+\alpha)} \int_a^b u(t)(\mathrm{d}t)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t\to 0} \sum_{j=0}^{j=N-1} u(t_j)(\Delta t_j)^\alpha \tag{4}$$

where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\left\{\Delta t_1, \Delta t_2, \Delta t_j, ...\right\}$, and $[t_j, t_{j+1}]$, $j = 0,..., N-1$, $t_0 = a, t_N = b$, is a partition of the interval $[a,b]$.

Homotopy perturbation method for local fractional Duffing equation

A non-linear vibration system with local fractional derivatives [32, 33] can be generally written in the form:

$$\frac{\mathrm{d}^{2\alpha} u(t)}{\mathrm{d}t^{2\alpha}} + g(u) = 0, \quad u(0) = A, \quad \frac{\mathrm{d}^\alpha u(0)}{\mathrm{d}t^\alpha} = B \tag{5}$$

where A and B are constants and g is a non-linear restoring force and $g(u)/u > 0$.

When $g(u) = au + bu^3$, we have the local fractional Duffing equation:

$$\frac{\mathrm{d}^{2\alpha} u(t)}{\mathrm{d}t^{2\alpha}} + au + bu^3 = 0, \quad u(0) = A, \quad \frac{\mathrm{d}^\alpha u(0)}{\mathrm{d}t^\alpha} = B \tag{6}$$

When $\alpha = 1$, eq. (6) was studied in [35], and the fractal modification of the Duffing equation was given in [36, 37]. The physical explanation of the local fractional calculus was given in [38-40], and the fractional order is relative to the two-scale fractal dimensions [41-43], and can be calculated by He-Liu fractal dimension formulation [44].

In order to illustrate the process of the Homotopy perturbation method [22, 23], we explore the following Homotopy equation [22, 23]:

$$\frac{\mathrm{d}^{2\alpha} u(t)}{\mathrm{d}t^{2\alpha}} + \varpi^2 u + p(au + bu^3 - \varpi^2 u) = 0 \tag{7}$$

where ϖ is a homotopy parameter. When $p = 0$, eq. (7) is a linearized one with a frequency of ϖ, and when $p = 1$, eq. (7) becomes eq. (6). There are other approaches to construction of a needed homotopy equation for a special non-linear equation, details were discussed in [44-46].

Following what is requested by the Homotopy perturbation method [22, 23], we assume:

$$u = u_0 + pu_1 + p^2 u_2 + ... \tag{8}$$

Equation (7) is decomposed into a series of linear equations. The first two linear differential equations are:

$$\frac{d^{2\alpha}u_0}{dt^{2\alpha}} + \varpi^2 u_0 = 0, \quad u_0(0) = A, \quad \frac{d^{\alpha}u_0}{dt^{\alpha}}(0) = B \tag{9}$$

and

$$\frac{d^{2\alpha}u_1}{dt^{2\alpha}} + \varpi^2 u_1 + a u_0 + b u_0{}^3 - \varpi^2 u_0 = 0 \tag{10}$$

Solving eq. (9), we can get the solution:

$$u_0(t) = C\cos_\alpha(\varpi t + \varphi)^\alpha \tag{11}$$

$$C^2 = B^2 + \frac{A^2}{\varpi^2}, \quad \varphi = \arctan_\alpha\left(-\frac{A}{B\varpi}\right)^\alpha - \frac{\pi}{2} < \varphi < 0 \tag{12}$$

where $\cos_\alpha t$ and $\arctan_\alpha t$ are, respectively, modified functions of cos(t) and arctan(t) in the local fractional sense [32, 33]. Now eq. (10) becomes:

$$\frac{d^{2\alpha}u_1}{dt^{2\alpha}} + \varpi^2 u_1 + bC^3\cos^3{}_\alpha(\varpi t + \varphi)^\alpha + (a - \varpi^2)C\cos_\alpha(\varpi t + \varphi)^\alpha = 0 \tag{13}$$

After a simple calculation, eq. (13) becomes:

$$\frac{d^{2\alpha}u_1}{dt^{2\alpha}} + \varpi^2 u_1 + C\left(a - \omega^2 + \frac{3}{4}bC^2\right)\cos_\alpha(\varpi t + \varphi)^\alpha + \frac{1}{4}bC^3\cos_\alpha(3\varpi t + 3\varphi)^\alpha = 0 \tag{14}$$

In order to be a periodic solution of u_1, the coefficient of $\cos_\alpha(\varpi t + \varphi)$ must be zero, that is:

$$C\left(a - \omega^2 + \frac{3}{4}bC^2\right) = 0 \tag{15}$$

Solving ω from eq. (15) results in:

$$\varpi^2 = a + \frac{3}{4}bC^2 = a + \frac{3}{4}b\left(B^2 + \frac{A^2}{\varpi^2}\right) \tag{16}$$

or

$$\varpi^4 - \left(a + \frac{3}{4}bB^2\right)\varpi^2 - \frac{3}{4}A^2 b = 0 \tag{17}$$

That is:

$$\varpi = \sqrt{\frac{\left(a + \frac{3}{4}bB^2\right) + \sqrt{\left(a + \frac{3}{4}bB^2\right)^2 + 3A^2 b}}{2}} \tag{18}$$

Now eq. (13) becomes:

$$\frac{d^{2\alpha}u_1}{dt^{2\alpha}} + \varpi^2 u_1 + \frac{1}{4}bC^3\cos_\alpha(3\varpi t + 3\varphi)^\alpha = 0 \tag{19}$$

The solution of eq. (19) is:

$$u_1 = \frac{bC^3}{32\varpi^2}[\cos_\alpha(3\varpi t + 3\varphi)^\alpha - \cos_\alpha(\varpi t + \varphi)^\alpha] \tag{20}$$

The approximate solution is:

$$u(t) = u_0 + u_1 = C\cos_\alpha(\varpi t + \varphi)^\alpha + \frac{bC^3}{32\varpi^2}[\cos_\alpha(3\varpi t + 3\varphi)^\alpha - \cos_\alpha(\varpi t + \varphi)^\alpha] \tag{21}$$

where ϖ is given in eq. (18).

The approximate period can be calculated:

$$T = \frac{2\pi}{\varpi} = \frac{2\pi}{\sqrt{\dfrac{\left(a + \dfrac{3}{4}bB^2\right) + \sqrt{\left(a + \dfrac{3}{4}bB^2\right)^2 + 3A^2 b}}{2}}} \tag{22}$$

When $\alpha = 1$, our result is same as that in [35]. In order to obtain higher order approximations, we expand the coefficient of the linear term in the form [45-47]:

$$a = \varpi^2 + p\varpi_1 + p^2\varpi_2 + ... \tag{23}$$

Following the solving process of the homotopy perturbation method [45-47], we have:

$$\frac{d^{2\alpha}u_0}{dt^{2\alpha}} + \varpi^2 u_0 = 0, \quad u_0(0) = A, \quad \frac{d^\alpha u_0}{dt^\alpha}(0) = B \tag{24}$$

$$\frac{d^{2\alpha}u_1}{dt^{2\alpha}} + \varpi^2 u_1 + \frac{1}{4}bC^3\cos_\alpha(3\varpi t + 3\varphi)^\alpha = 0 \tag{25}$$

and

$$\frac{d^{2\alpha}u_2}{dt^{2\alpha}} + \varpi^2 u_2 + \varpi_2 u_0 + \varpi_1 u_1 + 3bu_0^2 u_1 = 0 \tag{26}$$

Solving u_0 and u_1 from eqs. (24) and (25) respectively, we have:

$$u_0(t) = C\cos_\alpha(\varpi t + \varphi)^\alpha \tag{27}$$

where

$$C^2 = B^2 + \frac{A^2}{\varpi^2}, \quad \varphi = \arctan_\alpha\left(-\frac{A}{B\varpi}\right)^\alpha - \frac{\pi}{2} < \varphi < 0$$

and

$$u_1 = \frac{bC^3}{32\varpi^2}[\cos_\alpha(3\varpi t + 3\varphi)^\alpha - \cos_\alpha(\varpi t + \varphi)^\alpha] \tag{28}$$

And ω_1 is obtained in a similar way:

$$\omega_1 = -\frac{3}{4}bC^2 \tag{29}$$

Now eq. (26) becomes:

$$u_2'' + \varpi^2 u_2 + \varpi_2 C\cos(\varpi t+\varphi)^\alpha + \frac{\varpi_1 bC^3}{32\varpi^2}[\cos_\alpha(3\varpi t+3\varphi)^\alpha - \cos_\alpha(\varpi t+\varphi)^\alpha] +$$

$$+3bC^2\frac{bC^3}{32\varpi^2}\cos^2(\varpi t+\varphi)^\alpha[\cos_\alpha(3\varpi t+3\varphi)^\alpha - \cos_\alpha(\varpi t+\varphi)^\alpha] = 0 \tag{30}$$

That is:

$$u_2'' + \varpi^2 u_2 + \left(\varpi_2 C - \frac{3b^2C^5}{128\varpi^2}\right)\cos(\varpi t+\varphi)^\alpha + \frac{3b^2C^5}{128\varpi^2}\cos(5\varpi t+5\varphi)^\alpha = 0 \tag{31}$$

The coefficient of $\cos(\varpi t+\varphi)$ in eq. (31) has to be zero, that is:

$$\varpi_2 C - \frac{3b^2C^5}{128\varpi^2} = 0 \tag{32}$$

or

$$\varpi_2 = \frac{3b^2C^4}{128\varpi^2} \tag{33}$$

The solution of eq. (31) is:

$$u_2 = \frac{1}{24}\left(\frac{3b^2C^5}{128\varpi^2}\right)[\cos(5\varpi t+5\varphi)^\alpha - \cos(\varpi t+\varphi)^\alpha] \tag{34}$$

The second-order approximate solution is:

$$u = \lim_{p\to 1}(u_0 + pu_1 + p^2u_2) = u_0 + u_1 + u_2 \tag{35}$$

where u_0, u_1, and u_2 are given, respectively in eqs. (27), (28), and (34).

The frequency can be calculated according to eq. (31):

$$a = \lim_{p\to 1}(\varpi^2 + p\varpi_1 + p^2\varpi_2) = \varpi^2 + \varpi_1 + \varpi_2 \tag{36}$$

where ϖ_1 and ϖ_2 are given, respectively in eqs. (29) and (33), so we have:

$$\varpi^2 = a + \frac{3}{4}bC^2 - \frac{3b^2C^4}{128\varpi^2} \tag{37}$$

or

$$\varpi^4 = a\varpi^2 + \frac{3}{4}b(B^2\varpi^2 + A^2) - \frac{3b^2}{128}\left(B^2 + \frac{A^2}{\varpi^2}\right)^2 \tag{38}$$

or

$$\varpi^4 = a\varpi^2 + \frac{3}{4}b(B^2\varpi^2 + A^2) - \frac{3b^2}{128}\left(B^4 + \frac{A^4}{\varpi^4} + \frac{2A^2B^2}{\varpi^2}\right) \tag{39}$$

Finally, we obtain:

$$\varpi^8 - \left(a + \frac{3}{4}bB^2\right)\varpi^6 - \left(\frac{3}{4}bA^2 - \frac{3b^2B^4}{128}\right)\varpi^4 + \frac{3b^2}{64}A^2B^2\varpi^2 + \frac{3b^2A^4}{128} = 0 \tag{40}$$

Equation (40) shows that the frequency depends upon strongly the initial conditions.

Vibration energy harvesting

The vibration energy harvesting [48-50] is to convert the vibration energy to electrical energy, and the frequency property of the transducer plays an important role in the energy harvesting efficiency [14-16]. Equation (22) shows the amplitude-period relation, it is obvious that a low frequency leads to a large amplitude. The low-frequency property [37, 51, 52] affects greatly the energy harvesting efficiency. Equation (22) also reveals that the initial conditions also affect the frequency property. Forced vibrating systems were discussed in [53], and when $g(u)$ in eq. (1) involves even non-linear term, the pull-down instability [54] has to be considered.

The chatter signal during the machining process can be also described by a fractal vibration system, and it has obvious advantages over the fractional convolutional neural network [55] and deep learning system or machine learning technology [56, 57].

Conclusion

In this paper, the approximate frequency of Duffing equation with the local fractional derivatives is given by the homotopy perturbation method, which is proved to be an unprecedented and powerful method for the local fractional calculus. The effect of the initial conditions on the periodic property is also elucidated, eq. (22). The frequency-amplitude relationship gives a fast and effective insight into the periodic property of a local fractional vibration system, and it can be used for the optimal design of the energy harvesting devices and chatter detection of machining process.

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