SOLUTIONS OF THE KdV-mKdV EQUATIONS ARISING IN NON-LINEAR ELASTIC RODS UNDER FRACTAL DIMENSION

by

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A prediction of rod wave type with great precision is extremely important in theoretical analysis and practical applications. Besides the well-known periodic motion and resonance, this paper studies the rod wave in a fractal space, and a fractal solitary wave is unlocked by the variational approach, the results reveal that the rod strain non-linearity and fractal dimensions affect greatly the wave travelling properties. This paper offers a new window for identifying a solitary wave from periodic motion easily and accurately.

Key words: *non-linear elastic rod, KdV-MKdV equation, He's fractal derivatives, He-Weierstrass function, soliton*

Introduction

It has become a hot topic to identify exactly the wave types of a non-linear elastic rod in academy in recent years [1-5]. Zhuang and Yang [6] used the inverse dispersion method to study the physical performance of the 1-D weak non-linear long rods. Han and Zheng [7] considered the 3-D non-linear elasticity constitutive relation, and further studied the motion of the 1-D non-linear rod and derived the KdV-mKdV equation. Taking the material constant $n = 2$, Hu *et al.* [8] provided a numerical simulation of the wave equation of the non-linear elastic rod by the polysymplectic method, discussing the effects of the non-linear effect and the geometric dispersion effect on the solitary wave propagation. Guo *et al.* [9] took the material constant $n \geq 2$ and transformed the wave equation into the deformed KdV equation by using the reduction perturbation method. Kabir [10] took the material constant, and obtained the exact solution of the wave equation by using the modified Kudryashov method, the G'/G expansion method, and the exp-function method. Celik *et al.* [11] took the material constant to obtain the exact solution of the wave equation by using the Lie group analysis method and the F-expansion method. Guo *et al.* [12] applied the sine-cosine method to the wave equation of the non-linear elastic rods to obtain some new periodic and isolated solutions of the equation (the material constant $n \neq 1$). Ji *et al.* [13] elucidated that the solitary wave of a concrete pillar can be used for reliability and safety design of a bridge or a building. In this paper, we will study the solitary wave in a fractal space arising in a rod vibration.

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The KdV-mKdV equation related to strain in non-linear elastic rods

As shown in fig. 1, it is a non-linear infinite-long homogeneous, constant section circular rod with a line mass of ρ_l and a radius of R, and the column coordinate system (r, θ, x) is also introduced. According to [7], we assume that the rod is subjected to a sudden axial tensile load, the assumption of a flat section is still valid, and the following basic assumptions are made during the derivation of the motion equation:

– During the loading process, the infinitely long rod is in a uniaxial stress state, the expression is $\sigma_\mathrm{r} = \sigma_\theta = 0$, where σ_r and σ_θ represent radial and circumferential stresses, respectively.
– Consider the influence of lateral effects on deformation, it means that $\varepsilon_\mathrm{r} = -\nu\varepsilon_x$. Meanwhile, using geometric equations, it is easy to obtain $u_r = r\varepsilon_\mathrm{r} = -\nu r\partial u/\partial x$, where ν is the Poisson's ratio, ε_r and ε_x represent radial and axial strain, respectively, u_r and u represent radial and axial displacement, respectively.
– Assuming that the material follows a non-linear elastic constitutive equation:

$$\sigma_x = E\varepsilon_x + E\sum_{i=2}^{n}\alpha_i\varepsilon_x^i$$

where the first term in the equation represents linear elastic stress and the remaining terms represent non-linear elastic stress, E is the elastic modulus, α_i and n are the material constants.

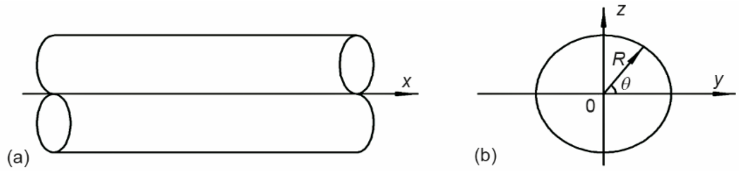

Figure 1. Non-linear infinite rod

Taking into account the transverse effect, the kinetic energy per unit length of an elastic rod is the sum of the longitudinal kinetic energy and the transverse kinetic energy [7]:

$$T = \left(\int_0^R \frac{1}{2}2\pi\rho_l r\mathrm{d}r\right)\left(\frac{\partial u}{\partial t}\right)^2 + \left(\int_0^R \frac{1}{2}2\pi\rho_l r\mathrm{d}r\right)\left(-\nu r\frac{\partial^2 u}{\partial t\partial x}\right)^2 = \frac{1}{2}\rho_l S\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{4}\rho_l SR^2\nu^2\left(\frac{\partial^2 u}{\partial t\partial x}\right) \quad (1)$$

where $S = \pi R^2$.

Based on the assumption of uniaxial stress, the elastic rod strain energy per unit length is:

$$W = \int_0^R\left(\int_0^\varepsilon \sigma\mathrm{d}\varepsilon\right)2\pi r\mathrm{d}r = S\int_0^\varepsilon\left(E\varepsilon_x + E\sum_{i=2}^{n}\alpha_i\varepsilon_x^i\right)\mathrm{d}\varepsilon_x = \frac{1}{2}SE\left(\frac{\partial u}{\partial x}\right)^2 + SE\sum_{i=2}^{n}\frac{1}{i+1}\alpha_i\left(\frac{\partial u}{\partial x}\right)^{(i+1)} \quad (2)$$

Utilizing Hamilton variational principle [14, 15], $\delta\int_{t_1}^{t_2}\int_{x_2}^{x_2}(T-W)\mathrm{d}x\mathrm{d}t = 0$, we can make $F = T - W$, and obtain:

$$F = \frac{1}{2}\rho_l S\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{4}\rho_l SR^2\nu^2\left(\frac{\partial^2 u}{\partial t\partial x}\right) - \frac{1}{2}SE\left(\frac{\partial u}{\partial x}\right)^2 - SE\sum_{i=2}^{n}\frac{1}{i+1}\alpha_i\left(\frac{\partial u}{\partial x}\right)^{(i+1)} \quad (3)$$

The Euler equation is:

$$F_u - \frac{\partial F_{u_x}}{\partial x} - \frac{\partial F_{u_y}}{\partial y} + \frac{\partial^2 F_{u_{xx}}}{\partial x^2} + \frac{\partial^2 F_{u_{yy}}}{\partial y^2} + \frac{\partial^2 F_{u_{xy}}}{\partial x \partial y} - ... = 0 \tag{4}$$

The longitudinal wave motion equation of the non-linear elastic rod considering the lateral effect is obtained:

$$\frac{\partial^2 u}{\partial t^2} - \frac{E}{\rho_l}\left[1 + \sum_{i=2}^{n} i\alpha_i \left(\frac{\partial u}{\partial x}\right)^{i-1}\right]\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\nu^2 R^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} = 0 \tag{5}$$

where $c_0^2 = E/\rho_l$, is the longitudinal wave velocity. Making n equal to three and substituting it into eq. (5), we can obtain:

$$\frac{\partial^2 u}{\partial t^2} - \frac{E}{\rho_l}(1 + 2\alpha_2 u_x + 3\alpha_3 u_x^2)\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\nu^2 R^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} = 0 \tag{6}$$

Then we make:

$$\alpha' = 3\alpha_3, \quad \beta' = \frac{1}{2}\nu^2 R^2, \quad \mu' = \frac{4\beta'}{\alpha' c_0^2}$$

and perform the following transformation:

$$\xi = x - c_0 t, \ \ \tau = \alpha' t \tag{7}$$

Substituting eq. (7) and α', β', and μ' into eq. (6), we can obtain:

$$2c_0\alpha'\tau u_\xi + c_0^2\left(2\alpha_2 u_\xi + \alpha' u_\xi^2\right)u_{\xi\xi} + \frac{1}{4}\mu'\alpha' c_0^4 u_{\xi\xi\xi\xi} = 0 \tag{8}$$

Then we make the following changes:

$$e = \frac{\partial u}{\partial \xi}, \quad y = \frac{2}{c_0}\xi, \quad \alpha'' = \frac{2\alpha_2}{\alpha'} = \frac{2\alpha_2}{3\alpha_3}, \quad \beta'' = 1, \quad \mu' = \frac{2\nu^2 R^2}{3\alpha_3 c_0^2} \tag{9}$$

Based on the previous transformation, eq. (9) can be simplified:

$$\tau e + \alpha'' e e_y + \beta'' e^2 e_y + \mu' e_{yyy} = 0 \tag{10}$$

When $t = 1/(3\alpha_3)$, eq. (10) can be converted to:

$$e + \alpha'' e e_y + \beta'' e^2 e_y + \mu' e_{yyy} = e - \alpha\frac{D}{Dy}e^2 + \beta\frac{D}{Dy}e^3 - \kappa\frac{D^3}{Dy^3}e = 0 \tag{11}$$

where

$$\alpha = -\frac{\alpha_2}{3\alpha_3}, \quad \beta = \frac{1}{3}\beta'', \quad \kappa = -\frac{2\nu^2 R^2}{3\alpha_3 c_0^2}, \quad y = \frac{2}{c_0}\left(x - \frac{c_0}{3\alpha_3}\right), \quad e = \frac{\partial u}{\partial \xi}$$

is the strain in the rod, eq. (11) is the strain dependent KdV-mKdV equation in a non-linear elastic rod.

Integrate both sides of eq. (11) at the same time, and take the constant of integration as zero:

$$\left(\frac{1}{2}-\alpha\right)e^2+\beta e^3-\kappa\frac{D^2}{Dy^2}e=0 \tag{12}$$

Equation (12) is a non-linear equation with quadratic non-linear term [16], and its periodic solution can be obtained *via* various methods [17-23]. This paper focuses itself on prediction of a new type of solutions in a fractal with great precision.

Fractal KdV-MKdV equation and solitary wave solution

The previous derivation is obtained by assuming a continuum rod or a smooth boundary. During the machining process, a rod moves actually along an unsmooth boundary, and the traditional vibration theory can not take into account the unsmooth boundary. Kou, *et al.* [24, 25] used the fractional convolutional neural network to predict the chatter vibration accurately. When air is considered as a fractal medium, by suitable controlling the fractal dimensions, the pull-in instability of a micro-electromechanical system can be eliminated [26-29]. Now the fractal vibration theory [30-33] and fractal solitons [34-38] become hot topics in both mathematics and engineering due to the amazing properties of the fractal vibration systems, for example, the low frequency property at the initial stage and the asymptotic periodicity.

Hinted by the cited literature, we extend eq. (12) into the fractal-fractional one:

$$\frac{1-2\alpha}{2}e^2+\beta e^3-\kappa\frac{D^2}{{}^H Dy^{2\mu}}e=0 \tag{13}$$

$$y^\mu=\frac{2}{c_0}\left(x^\mu-\frac{c_0}{3\alpha_3}\right) \tag{14}$$

where μ represents the two-scale fractal dimensions [39, 40], $(D/{}^H Dy^\mu)e$ is He's the fractal derivative with respect to y^μ [41-43].

Here we want to establish a variational formulation for eq. (13), so that the basic properties of the fractal soliton can be clearly elucidated. In 2021, He, *et al.* [44] suggested a variational approach to the fractal solitons with great success, following [44], Sun [45] studied the fractal solitary waves for Ablowitz-Kaup-Newell-Segur water wave and Klein-Gordon equation [46] with great success.

The variational principle of eq. (13) can be established by the semi-inverse method [47]:

$$J(u)=\int_0^\infty\left[\frac{1}{2}\left(\frac{D}{{}^H Dy^\mu}e\right)^2+\left(\frac{1-2\alpha}{6\kappa}\right)e^3+\frac{\beta}{4\kappa}e^4\right]\mathrm{d}\xi^\mu \tag{15}$$

The Euler-Lagrange equation of eq. (15) can be described:

$$\frac{D^2 e}{{}^H Dy^{2\mu}}+\left(\frac{1-2\alpha}{2\kappa}\right)e^2+\frac{\beta}{\kappa}e^3=0 \tag{16}$$

The semi-inverse method [47] has its own unique approach to search for a variational formulation from the governing differential equations [48-52].

From eq. (16), we can obtain He-Weierstrass function [53]:

$$H=\frac{1}{2}z^2+\left(\frac{1-2\alpha}{2\kappa}\right)e^2+\frac{\beta}{\kappa}e^3-\left[\frac{1}{2}\left(\frac{De}{{}^{H}Dy^{\mu}}\right)^2+\left(\frac{1-2\alpha}{2\kappa}\right)e^2+\frac{\beta}{\kappa}e^3\right]-$$

$$-\left(z-\frac{De}{{}^{H}Dy^{\mu}}\right)\frac{De}{{}^{H}Dy^{\mu}}=\frac{1}{2}z^2-\frac{1}{2}\left(\frac{De}{{}^{H}Dy^{\mu}}\right)^2-\left(z-\frac{De}{{}^{H}Dy^{\mu}}\right)\frac{De}{{}^{H}Dy^{\mu}} \tag{17}$$

where the variable z is defined:

$$z=\frac{De}{{}^{H}Dy^{\mu}}e \tag{18}$$

From eq. (17), it is obvious that:

$$H(y,e,e^{\mu},z)=0,\qquad \frac{\partial^2 H}{\partial z^2}>0 \tag{19}$$

Equation (19) indicates that eq. (15) is a minimal variational principle.

Then, we assume the solution of eq. (13) is [44]:

$$e(y^{\mu})=p\,\mathrm{sech}(y^{\mu}) \tag{20}$$

where p is conversion rate. Substituting eq. (20) into eq. (15), we obtain:

$$J(p)=\int_0^{\infty}\left\{\frac{1}{2}\left[-p\,\mathrm{sech}(y^{\mu})\tanh(y^{\mu})\right]^2+\frac{(1-2\alpha)p^3}{6\kappa}\mathrm{sech}^3(y^{\mu})+\frac{\beta p^4}{4\kappa}\mathrm{sech}^4(y^{\mu})\right\}\mathrm{d}y^{\mu}=$$

$$=\frac{1}{6}p^2+\frac{(1-2\alpha)\pi}{24\kappa}p^3+\frac{\beta}{6\kappa}p^4 \tag{21}$$

Owing to:

$$\frac{DJ}{Dp}=\frac{1}{3}p+\frac{(1-2\alpha)\pi}{8\kappa}p^2+\frac{2\beta}{3\kappa}p^3=0 \tag{22}$$

According to the previous equations, we have:

$$p=-\frac{3\pi(1-2\alpha)}{16\kappa}\pm\frac{\sqrt{9\pi^2(1-2\alpha)^2-512\beta\kappa}}{24k} \tag{23}$$

So the solution of eq. 13 can be obtained:

$$e(y^{\mu})=-\frac{3\pi(1-2\alpha)}{16\kappa}\pm\frac{\sqrt{9\pi^2(1-2\alpha)^2-512\kappa\beta}}{24\kappa}\mathrm{sech}(y^{\mu}) \tag{24}$$

To sum up, we obtain the instantaneous solution of the strain dependent KdV-mKdV equation for non-linear elastic rods in the fractal dimension:

$$e(x^\mu) = -\frac{3\pi(1-2\alpha)}{16\kappa} \pm \frac{\sqrt{9\pi^2(1-2\alpha)^2 - 512\kappa\beta}}{24\kappa} \operatorname{sech}\left[\frac{2}{c_0}\left(x^\mu - \frac{c_0}{3\alpha_3}\right)\right] \tag{25}$$

When $\mu \to 1$, eq. (25) can be transformed into eq. (26):

$$\lim_{\mu\to 1} e(x^\mu) = -\frac{3\pi(1-2\alpha)}{16\kappa} \pm \frac{\sqrt{9\pi^2(1-2\alpha)^2 - 512\kappa\beta}}{24\kappa} \operatorname{sech}\left[\frac{2}{c_0}\left(x - \frac{c_0}{3\alpha_3}\right)\right] \tag{26}$$

Equation (26) is the instantaneous solution of the strain dependent KdV-mKdV equation for non-linear elastic rods in a smooth space.

Effect of the fractal dimensions on the solitary wave

Without losing the generality, we only consider the following situations:

$$e(x^\mu) = -\frac{3\pi(1-2\alpha)}{16\kappa} + \frac{\sqrt{9\pi^2(1-2\alpha)^2 - 512\kappa\beta}}{24\kappa} \operatorname{sech}\left[\frac{2}{c_0}\left(x^\mu - \frac{c_0}{3\alpha_3}\right)\right] \tag{27}$$

Based on the assumptions made earlier, we set α_2 – 1, α_3 – 2, $\alpha = -1/6$, $\beta = 1/3$, $\nu = 0.21$, R = 2.5 mm, c_0 = 21.37 mm/s, $\kappa = 0.792 \times 10^{-4}$, by substituting these parameters into eq. (27), we can obtain:

$$e(x^\mu) = -99.116 + 65.831\operatorname{sech}(0.094x^\mu - 0.667) \tag{28}$$

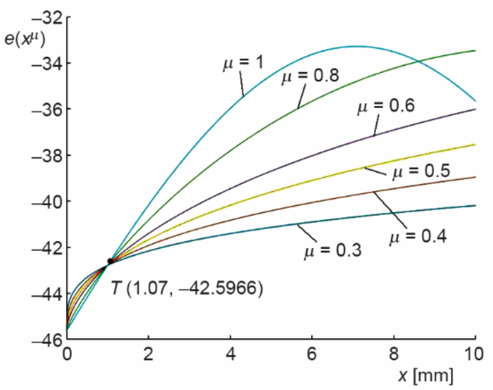

Figure 2. The instantaneous solution of the strain dependent KdV-mKdV equation for non-linear elastic rods with different fractal orders

Figure 2 represents eq. (28) with different fractal orders. By comparing the simulation results, we can obtain that the fractal order only affects the waveform and the position of the peak, and when x = 1.07, the strain calculated by different fractal dimensions is same.

Conclusion

This article presents the derivation of the strain-dependent KdV and mKdV equations in a non-linear elastic rod under specific assumptions. Additionally, we have obtained its fractal form based on He's fractal derivatives. Subsequently, we have proved the minimum variational principle based on the He-Weierstrass function. Using this principle, we are able to obtain the instantaneous solutions under variable fractal dimensions. We find that the fractal order only affects the waveform and the position of the peak, while exact prediction of the peak position is extremely important to guarantee its reliable operation and to monitor its safety.

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