THERMAL PERFORMANCE OF FRACTAL METASURFACE AND ITS MATHEMATICAL MODEL

by

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How can we explain the thermal phenomenon by a fractal metasurface? This has been puzzling scientists and engineers for at least ten years, and so far no answer has been found. Now, modern mathematics offers a completely new window to physically understand the magical phenomenon that lies far beyond the Fourier law for heat conduction. A fractal-fractional modification of the Fourier law is elucidated, and its extremely high thermal conductivity is mathematically revealed. This article shows that thermal science is the key to nanotechnology.

Key words: metamaterial, surface science, 2-D material, nano-thermodynamics

Introduction

On May 14, 2021, the most famous journal, Science, put forward *125 questions: exploration and discovery*, which had attracted worldwide attention, and many scientists and engineers have tried to find the answer to each question, but not every question found its answer, this paper focuses on a question in chemistry, that is, *How can we measure interface phenomena at the microscopic level*, and no answer has been found yet. A full list of questions can be downloaded from the web: https://www.science.org/do/10.1126/resource.2499249/full/sjtu-booklet.pdf. Now a few years have passed and the question became *How can we explain the thermal phenomena of metasurfaces at the nanoscale level?*. The metasurfaces are a kind of 2-D materials with special properties and amazing functionalities [1-4], the answer to the question is more important than their most advanced applications, for example, invisible cloaks [5], thermal camouflage [6-8], microwave absorbers [9], sophisticated lenses and mirage devices [10], and thermal energy harvesting devices [11]. Only the thermal mechanism allows for the optimal design, so it must be revealed.

Metasurfaces allow us to design any material with the desired thermal conductivity along with other attractive properties, but the physical understanding of this magical phenomenon has not yet been solved. Metasurfaces enable us to design any materials with desired thermal conductivity together with other attractive properties, but its physical understanding of this magic phenomenon has not been solved yet, this paper offers a mathematical approach to metasurface's thermal performance.

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Mathematical model for metasurface's thermal performance

Many intriguing thermal phenomena have been found for metasurfaces [8, 12, 13], physical laws have suddenly seemed to become invalid, and scientists and engineers have tried to find a mathematical answer to these magical phenomena. The first possible mathematical tool that might come to mind is fractal geometry [14, 15], since the surface cannot be explained by the geometric principles of Euclid. The unsmooth boundary, like that of Mandelbrot's fractal coastline [16], has led to the new concept of *fractal metasurface*. Yes, fractal theory has provided a complete path to the amazing thermal properties, but fractal geometry alone is not enough to unravel the mystery.

When there is no way to the mysterious phenomena, someone might ask a question: *Does God set the boundary?* To answer this question, we quote Nobel laureate Wolfgang Pauli's statement: *God made solids, but surfaces were the work of the devil* [17], too many 2-D surfaces are waiting for a physical explanation, from the cell membrane [18] to graphene [19], here a mathematical explanation is given.

Fractal thermodynamics

The concept of fractal thermodynamics was proposed in [20]. Here we apply it to explain the thermal properties of metasurfaces, and the traditional Fourier's law for heat conduction has to be modified to deal with unsmooth metasurfaces.

The traditional 1-D Fourier law is:

$$J = -k\frac{\mathrm{d}T}{\mathrm{d}x} \tag{1}$$

where J is heat flux per area through the metasurface and k – the thermal conductivity.

We re-write eq. (1) in the form:

$$J = -k\frac{\Delta T}{\Delta x} \tag{2}$$

where Δx is the thickness of the metasurface and $\Delta \mathrm{T}$ – the temperature difference between two surfaces.

Since the metasurface is a 2-D planar material with extremely thin thickness, so $x \ll 1$, according to eq. (2), the metasurface has extremely high heat flux, this phenomenon has been widely explained as high thermal conductivity. However, the traditional Fourier cannot explain many other thermal phenomena that occur in the heat conduction of the metasurface. For example, consider two different metasurfaces of the same material, thickness and temperature difference, but with different porosities. The Fourier law gives the following equations:

$$J_1 = -k_1\frac{\Delta T_1}{\Delta x_1} \tag{3}$$

$$J_2 = -k_2\frac{\Delta T_2}{\Delta x_2} \tag{4}$$

As $k_1 = k_2$, $\Delta x_1 = \Delta x_2$, and $\Delta T_1 = \Delta T_2$, we must have $J_1 = J_2$. This implies that the heat fluxes are the same regardless of the geometric properties of the metasurface, which contradicts many experimental observations that the geometric structure plays a key role in the thermal performance. Fourier's law cannot describe the effect of the geometry of the metasurface on its thermal performance.

According to the two-scale fractal theory, Fourier's law can be modified as [21]:

$$J = -k\frac{\mathrm{d}^\alpha T}{\mathrm{d}x^\alpha} \tag{5}$$

where α is the two-scale fractal dimensions [22],

$$\frac{\mathrm{d}^\alpha T}{\mathrm{d}x^\alpha}$$

is the two-scale fractal derivative [23]:

$$\frac{\mathrm{d}^\alpha T}{\mathrm{d}x^\alpha}(x_0) = \Gamma(1+\alpha)\lim_{x-x_0\to r}\frac{T(x)-T(x_0)}{(x-x_0)^\alpha} \tag{6}$$

where r is the smallest scale to measure the metasurface's thermal performance. When $r \to 0$, it is a metal-like material with α = 1. Equation (6) can be approximately written as:

$$\frac{\partial^\alpha T}{\partial x^\alpha} = (\Delta x)^{1-\alpha}\Gamma(1+\alpha)\lim_{x-x_0\to r}\frac{T(x)-T(x_0)}{x-x_0} \approx (\Delta x)^{1-\alpha}\Gamma(1+\alpha)\frac{\mathrm{d}T}{\mathrm{d}x} \tag{7}$$

So the Fourier law becomes:

$$J = -k(\Delta x)^{1-\alpha}\Gamma(1+\alpha)\frac{\mathrm{d}T}{\mathrm{d}x} \tag{8}$$

It is obvious that the heat flux depends upon not only the temperature difference across the metasurface, but also the fractal dimensions and the measured scale. The thickness Δx scales with the measured scale, r, that is:

$$\Delta x \propto r \tag{9}$$

So the Fourier law for the metasurface's heat conduction can be written as:

$$J = -k_\alpha\frac{\Delta T}{r^\alpha} \tag{10}$$

where k_α is the fractal thermal conductivity and r can be considered as the smallest pore radius of the metasurface. Now we consider two metasurfaces with same thickness, but with different porous structures, according to eq. (10), we have:

$$J_1 = -k_{1\alpha}\lim_{\Delta x\to r_1}\frac{\Delta T}{(r_1)^{\alpha_1}} \tag{11}$$

$$J_2 = -k_{2\alpha}\lim_{\Delta x\to r_2}\frac{\Delta T}{(r_2)^{\alpha_2}} \tag{12}$$

We assume that $\alpha_1 \approx \alpha_2 \approx 1.1$, $r_1 = 10\times10^{-6}$ m and $r_2 = 10\times10^{-9}$ m, so we have:

$$\frac{J_2}{J_1} = \frac{-k_{2\alpha}\lim\limits_{\Delta x\to r_2}\dfrac{\Delta T}{(r_2)^\alpha}}{-k_{1\alpha}\lim\limits_{\Delta x\to r_1}\dfrac{\Delta T}{(r_1)^\alpha}} \approx \left[\frac{10\times10^{-6}}{10\times10^{-9}}\right]^{1.1} = 1995 \tag{13}$$

In the traditional view, the thermal conductivity of the metasurface in nanoscale pores is almost 2000 times higher than that of its microscale partner.

We now return to the traditional heat conduction equation:

$$c\rho\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) \tag{14}$$

where ρ is the density and c – the specific heat capacity. Considering that the porous structure changes with depth profile, eq. (14) can be modified as:

$$c\rho A(x)\frac{\partial T}{\partial t}+\frac{\partial}{\partial x}\left[kA(x)\frac{\partial T}{\partial x}\right]=0 \tag{15}$$

where A is the heat transfer area, while the fractal heat conduction equation is [22]:

$$c\rho A\frac{\partial^{\alpha}\Delta T}{\partial t^{\alpha}}=\frac{\partial^{\alpha}}{\partial x^{\alpha}}\left(k_{\alpha}A\frac{\Delta T}{r^{\alpha}}\right) \tag{16}$$

The time for heat equilibrium $\Delta T = 0$ scales with squared of the smallest measured scale, *i.e.*:

$$\Delta t \propto r^2 \tag{17}$$

For example, if a metasurface with micropores averaging 10 micrometers in diameter takes 0.1 seconds to dissipate a microdevice's wasted heat to its surroundings, a metasurface with nanopores averaging 10 nanometers in diameter takes only 0.1 microseconds. This rapid thermal response is extremely useful for modern applications. The present thermodynamic analysis can also be extended to arbitrary 2-D materials, and some fractal-fractional models in the literature [24-27] can be also greatly simplified for the 2-D cases.

Conclusion

This paper elucidates the fractal thermal performance of a 2-D metasurface, whose geometry (pore size and thickness) plays a key role in heat conduction. A fractal Fourier law can clearly describe the amazing thermal phenomena, the thinner the better for fast heat transfer.

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