EXACT SOLITARY WAVE SOLUTION FOR THE DRINFELD-SOKOLOV SYSTEM

By

***Bei YANG[a], Guang-Sheng CHEN[b,*]***

[a]Guangxi Transport Vocational and Technical College, Nanning, Guangxi, 530216, China.
[b]College of Mathematics and Computer Science,
Guangxi Science & Technology Normal University, Laibin, Guangxi, 546199, China

*In this work, we mainly investigate the Drinfeld-Sokolov system by employing the functional variable method. Some new solitary wave and periodic solutions are successfully derived. The dynamic characteristics of these obtained solitary wave solutions are elaborated by plotting some 3D and 2D figure.*
Key words: *Drinfeld-Sokolov system, functional variable method, solitary wave solution*

## Introduction

In recent years, nonlinear evolution equations have been widely used to explain complex natural phenomena stemmed from engineering and science [1-6]. For instance, the Drinfeld-Sokolov system, proposed by Drinfeld and Sokolov, was first adopted to describe the long waves of small amplitude on the surface of inviscid fluid. The Drinfeld-Sokolov system is given by the following two equations [7]

$$\frac{\partial w}{\partial t}+\frac{\partial u^2}{\partial x}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}-a\frac{\partial u}{\partial x^3}+3bu\frac{\partial w}{\partial x}+3cw\frac{\partial u}{\partial x}=0, \tag{2}$$

where $a$, $b$, $c$ are constant.

The Drinfeld-Sokolov system has been studied by many famous scholars by using a variety of effective analytical methods. For example, in [7], Wazwaz studied the Drinfeld-Sokolov system by using sine-cosine method and tanh function method, and obtained some solition and periodic solutions, which can describe natural phenomena. In [8], Garrido and Bruzon used Lie group method to derive the exact solutions of Drinfeld-Sokolov system, which are called travelling wave solutions. In [9], Yao *et al* employed the ansatz technique to study the Drinfeld-Sokolov system and obtain some new solitary wave solutions, which can formulate the properties of this system. More results related with Drinfeld-Sokolov system can be found in cited References.

The main objective of this manuscript is to study the Drinfeld-Sokolov system by employing the functional variable method, which is very simple and direct analytical scheme. We successfully derive some novel solitary wave and periodic solutions. These

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* Corresponding author. E-mail: guangshengchen1980@163.com

obtained solutions are completely new and have not yet appeared in other literature. Finally, some 3D and 2D graphs are presented with suitable parameters.

**Functional variable method**

Consider the following nonlinear evolution equation

$$H(\frac{\partial w}{\partial x^2}, \frac{\partial w}{\partial t^2}, w, w^3) = 0 . \quad (3)$$

Use the travelling wave transformation

$$w(x,t) = \Phi(\xi) \ , \quad (4)$$

$$\xi = \lambda x + \beta t \ . \quad (5)$$

Substitute eq. (4) and eq. (5) into eq.(3) and have

$$H((\lambda^2, \beta^2)\frac{d\Phi}{d\xi^2}, \Phi, \Phi^3) = 0 . \quad (6)$$

Assume that unknown function $\Phi$ is

$$\frac{d\Phi}{d\xi} = F(\Phi) . \quad (7)$$

Hence, we obtain

$$\frac{d^2\Phi}{d\xi^2} = \frac{1}{2}\frac{d(F(\Phi))^2}{d\Phi} \quad (8)$$

$$\frac{d^3\Phi}{d\xi^3} = \frac{1}{2}\frac{d^2(F(\Phi))^2}{d\Phi^2}\sqrt{F^2} \quad (9)$$

and

$$\frac{d^4\Phi}{d\xi^4} = \frac{1}{2}\left[\Phi^2 \frac{d^3((F(\Phi))^2)}{d\Phi^3} + \frac{d^2(F(\Phi))^2)}{d\Phi^2}\frac{d(F(\Phi))^2)}{d\Phi}\right]. \quad (10)$$

So , eq. (3) can be converted into the following form

$$H(\frac{d^2F(\Phi)}{dW^2}, \Phi, F(\Phi), (F(\Phi))^3) = 0 . \quad (11)$$

By solving eq. (11), the $\Phi$ can be derived.

**Solitary wave solution of Drinfeld-Sokolov system**

Consider the Drinfeld-Sokolov system as follows

$$\frac{\partial w}{\partial t} + \frac{\partial u^2}{\partial x} = 0 , \quad (12)$$

$$\frac{\partial u}{\partial t} - a\frac{\partial u}{\partial x^3} + 3bu\frac{\partial w}{\partial x} + 3cw\frac{\partial u}{\partial x} = 0. \tag{13}$$

We apply the following transformation

$$w(x,t) = \Phi(\xi)\ , \quad u(x,t) = U(\xi) \tag{14}$$

$$\xi = \lambda x + \beta t\ . \tag{15}$$

Putting eq.(14) and eq.(15) into eq.(12) and eq.(13), we obtain

$$\beta\Phi' + \lambda\left(U^2\right)' = 0, \tag{16}$$

$$\beta U' - a\lambda^3 U''' + 3b\lambda U\Phi' + 3c\lambda\Phi U' = 0, \tag{17}$$

where $\Phi' = d\Phi/d\xi$.

Integrate eq.(16) and neglect the constant, and have

$$\Phi = -\frac{\lambda}{\beta}U^2. \tag{18}$$

Substitute eq.(18) into eq.(17) and integrate once, and obtain the following equation

$$\beta U' - a\lambda^3 U''' + 3b\lambda U\Phi' + 3c\lambda\Phi U' = 0. \tag{19}$$

Put eq.(18) into eq.(19), and get

$$U'' + AU + BU^3 = 0. \tag{20}$$

where

$$A = -\frac{\beta}{a\lambda^3},\ B = \frac{2b+c}{a\lambda\beta}. \tag{21}$$

Then, a functional variable is defined as

$$U' = F(U). \tag{22}$$

Hence, we have

$$U'' = \frac{1}{2}\frac{d(F(U))^2}{dU}. \tag{23}$$

Therefore, the following equation is derived

$$\frac{1}{2}\frac{d(F(U))^2}{dU} + AU + BU^3 = 0. \tag{24}$$

Eq.(24) is calculated as

$$F(U) = \sqrt{-A}U\sqrt{1 + \frac{B}{2A}U^2}. \tag{25}$$

By solving eq.(25), we obtain

$$U(\xi) = \pm\sqrt{\frac{2A}{B}}\csc h(\sqrt{-A}\xi). \tag{26}$$

***Case.1.*** When $-\frac{\beta}{a\lambda^3} < 0$, $\frac{2b+c}{a\lambda\beta} < 0$, the new solitary wave solutions are

$$u_{1,2}(x,t) = \pm\sqrt{-\frac{2\beta^2}{\lambda^2(2b+c)}}\csc h\left(\sqrt{\frac{\beta}{a\lambda^3}}(\lambda x + \beta t)\right). \tag{27}$$

$$w_1(x,t) = \frac{2\beta}{\lambda(2b+c)}\csc h^2\left(\sqrt{\frac{\beta}{a\lambda^3}}(\lambda x + \beta t)\right). \tag{28}$$

***Case.2.*** When $-\frac{\beta}{a\lambda^3} > 0$, $\frac{2b+c}{a\lambda\beta} < 0$, the new periodic solutions are derived as follows

$$u_{3,4}(x,t) = \pm\sqrt{-\frac{2\beta^2}{\lambda^2(2b+c)}}\csc\left(\sqrt{-\frac{\beta}{a\lambda^3}}(\lambda x + \beta t)\right), \tag{29}$$

$$w_2(x,t) = \frac{2\beta}{\lambda(2b+c)}\csc^2\left(\sqrt{-\frac{\beta}{a\lambda^3}}(\lambda x + \beta t)\right). \tag{30}$$

In figure.1(a), we plot the 3D graph of $|u_{1,2}(x,t)|$ with parameters $a = 1$, $b = -2$, $c = 1$, $\lambda = 5$ $\beta = 3$. In figure.1(b), we plot the 2D graph of $|u_{1,2}(x,t)|$ with parameters $a = 2$, $b = -3$, $c = 2$, $\lambda = 4$ $\beta = 2$ at different time $t = 1$ and $t = 5$.

In figure.2(a), we sketch the 3D graph of $|w_1(x,t)|$ with parameters $a = 3$, $b = -6$, $c = 2$, $\lambda = 3$ $\beta = 5$. In figure.2(b), we present the 2D graph of $|w_1(x,t)|$ with parameters $a = 4$, $b = -5$, $c = 1$, $\lambda = 2$ $\beta = 7$ at different time $t = 2$ and $t = 7$.

## Conclusion

In the present work, the functional variable method is successfully adopted to study the Drinfeld-Sokolov system, and some new solitary wave and periodic solutions are derived. These solutions are very helpful for understanding the corresponding physical phenomena. In the future work, the functional variable method will be used to solve fractional differential equations.

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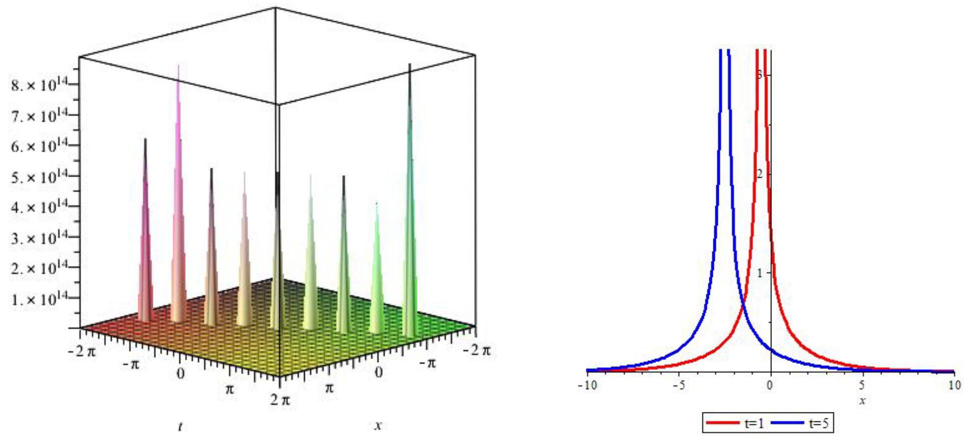
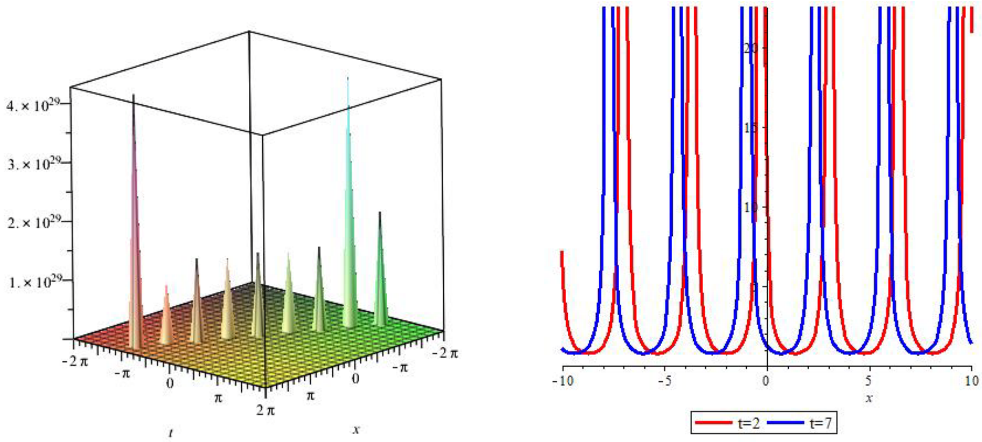
**Nomenclature**

$t$ -time co-ordinate,[s] $x$ -space co-ordinate, [m]




**(a)** **(b)**

**Figure.1. The corresponding 3D and 2D graph of $|u_{1,2}|$**




**(a)** **(b)**

**Figure.2. The corresponding 3D and 2D graph of $|w_1|$**

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