NEW ANALYTICAL METHOD FOR CUBIC KLEIN-GORDON EQUATION

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In this paper, the (2+1)-dimensional cubic Klein-Gordon model is investigated, which is used to described the propagation of dislocation in crystals. A simple and efficient analytical technology is successfully employed to seek some new periodic and solitary wave solutions, which is called sine-cosine method . The physics properties of these obtained periodic and solitary wave solutions are illustrated by corresponding graphs.

Key words:*cubic Klein-Gordon equation, sine-cosine method, solitary wave solution*

Introduction

Consider the (2+1)-dimensional cubic Klein-Gordon equation which yields as [1]

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial t^2}+au+bu^3=0\,. \tag{1}$$

Eq.(1) is a very important mathematical model in physics and engineering, which is used to described the propagation of fluxions in Josephson junctions and the propagation of dislocation in crystals. In [1], Khan and Akbar studied the (2+1)-dimensional cubic Klein-Gordon equation by using the modified simple equation method, and successfully obtained some soliton solutions.

In recent decades, it is a very popular topic to study the solitary wave and periodic solutions of evolution equations. So far, a number of excellent mathematical methods have been proposed, such as (G'/G)-expansion method [2], rational sinh-cosh method [3], Lie group method [4], Yang-Machado-Baleanu-Cattain wave method [5-7], Lie symmetry scheme [8], Variational method [9,10], and so on [11-16].

In this work, the sine-cosine method is adopted to study the (2+1)-dimensional cubic Klein-Gordon equation. Some new periodic and solitary wave solutions are derived. The proposed method is a powerful mathematical technique to solve different types of evolution equations. Finally, these new solutions are elaborated by sketching some graphs.

The Sine-Cosine Method

Consider the following evolution equation

$$H\left\{\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2},\frac{\partial^2 u}{\partial t^2},u,u^3\right\}=0\,. \tag{2}$$

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Apply the following travelling wave transformation

$$\begin{cases} u(x,y,t) = U(\xi) \\ \xi = cx + dy + ft \end{cases}. \tag{3}$$

Substitute eq.(3) into eq.(2), and have

$$H\left\{\left(c^2, d^2, f^2\right)\frac{d^2U}{d\xi^2}, U, U^3\right\} = 0. \tag{4}$$

Assume that solution of eq.(4) is the following form

$$U(\xi) = h\cos^{\alpha}\left(\lambda\xi\right), |\xi| < \frac{\pi}{2\lambda}, \tag{5}$$

or

$$U(\xi) = h\sin^{\alpha}\left(\lambda\xi\right), |\xi| < \frac{\pi}{2\lambda}. \tag{6}$$

where h, λ, α are parameters.

When we select eq.(5), we obtain

$$U(\xi) = h\cos^{\alpha}\left(\lambda\xi\right), \tag{7}$$

$$U^3(\xi) = h^3\cos^{3\alpha}\left(\lambda\xi\right), \tag{8}$$

$$\frac{d^2U}{d\xi^2} = \frac{h\alpha^2\lambda^2\cos^{\alpha}\left(\lambda\xi\right)\sin^2\left(\lambda\xi\right)}{\cos^2\left(\lambda\xi\right)} - h\alpha\lambda^2\cos^{\alpha}\left(\lambda\xi\right) - \frac{h\alpha\lambda^2\cos^{\alpha}\left(\lambda\xi\right)\sin^2\left(\lambda\xi\right)}{\cos^2\left(\lambda\xi\right)} \tag{9}$$

We substitute eqs.(7)-(9) into eq.(4), and obtain a trigonometric equation on $\cos^n\left(\lambda\xi\right)$. Then, we collect the coefficients of the same power in $\cos^n\left(\lambda\xi\right)$, and set them is zero. By solving the obtained system of algebraic equations, these parameters are derived.

Applications

Consider the (2+1)-dimensional cubic Klein-Gordon equation as follows

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} + au + bu^3 = 0. \tag{10}$$

Use the travelling wave transformation

$$\begin{cases} u(x,y,t) = U(\xi) \\ \xi = cx + dy + ft \end{cases}. \tag{11}$$

Substituting eq.(11) into eq.(10), we have

$$\frac{d^2U}{d\xi^2} + AU + BU^3 = 0 \tag{12}$$

where

$$A = \frac{a}{c^2 + d^2 - f^2}, B = \frac{b}{c^2 + d^2 - f^2}. \tag{13}$$

Assume that solution of eq.(12) is the following form

$$U(\xi)=h\cos^{\alpha}\left(\lambda\xi\right). \tag{14}$$

So, we have

$$\frac{d^2U}{d\xi^2}=-h\alpha^2\lambda^2\cos^{\alpha}\left(\lambda\xi\right)+h\alpha^2\lambda^2\cos^{\alpha-2}\left(\lambda\xi\right)-h\alpha\lambda^2\cos^{\alpha-2}\left(\lambda\xi\right). \tag{15}$$

Put eq.(14) and eq.(15) into eq.(12), the following result is obtained

$$-\alpha^2\lambda^2\cos^{\alpha}\left(\lambda\xi\right)+Bh^2\cos^{3\alpha}\left(\lambda\xi\right)+\alpha^2\lambda^2\cos^{\alpha-2}\left(\lambda\xi\right)-\alpha\lambda^2\cos^{\alpha-2}\left(\lambda\xi\right)+A\cos^{\alpha}\left(\lambda\xi\right)=0. \tag{16}$$

By using the balancing theory, we have

$$3\alpha=\alpha-2. \tag{17}$$

Hence, the following system of algebraic equations is derived

$$\begin{cases} A-\lambda^2=0 \\ Bh^2+2\lambda^2=0 \end{cases}. \tag{18}$$

Solve eq.(18), and we have

$$\lambda=\pm\sqrt{A},h=\pm\sqrt{-\frac{2A}{B}}. \tag{19}$$

Case.1. When $A>0$, $B<0$, the periodic solutions of eq.(10) are

$$u_1(x,t)=\sqrt{-\frac{2a}{b}}\sec\left(\sqrt{\frac{a}{c^2+d^2-f^2}}(cx+dy+ft)\right), \tag{20}$$

$$u_2(x,t)=-\sqrt{-\frac{2a}{b}}\sec\left(\sqrt{\frac{a}{c^2+d^2-f^2}}(cx+dy+ft)\right). \tag{21}$$

Case.2. When $A<0$, $B>0$, the solitary wave solutions of eq.(10) are

$$u_3(x,t)=\sqrt{-\frac{2a}{b}}\sec h\left(\sqrt{-\frac{a}{c^2+d^2-f^2}}(cx+dy+ft)\right), \tag{22}$$

$$u_4(x,t)=-\sqrt{-\frac{2a}{b}}\sec h\left(\sqrt{-\frac{a}{c^2+d^2-f^2}}(cx+dy+ft)\right). \tag{23}$$

In figure.1, we plot the 3D graph of u_1 with parameters $a=2$, $b=-1$, $c=3$, $d=0$, $f=2$. In figure.2, the 2D graph of $|u_1|$ with parameters $a=3$, $b=-2$, $c=4$, $d=0$, $f=2$ at different time $t=1$ and $t=6$. In figure.3, we sketch the 3D graph of u_3 with parameters $a=-4$, $b=3$, $c=5$, $d=0$, $f=3$. In figure.4, the 2D graph of $|u_3|$ with parameters $a=-6$, $b=4$, $c=6$, $d=0$, $f=4$ at different time $t=2$ and $t=7$.

Conclusion

In this paper, we successfully employed the sine-cosine method to solve the (2+1)-dimensional cubic Klein-Gordon, and some new periodic and solitary wave solutions are derived. The dynamical behavior of these obtained solutions are elaborated by plotting some 3D and 2D graphs with proper parameters. In future work, the sine-cosine method will be used to study the fractional evolution equations.

Nomenclature

t -time co-ordinate,[second] x, y -space co-ordinate, [m]

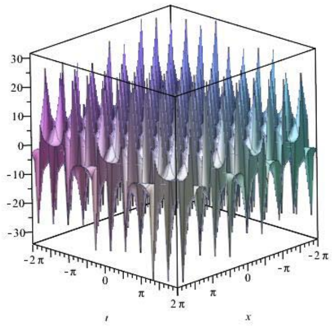

Figure.1. The 3D graph of u_1

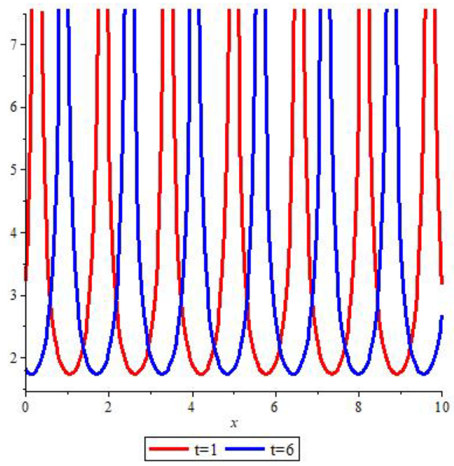

Figure.2. The 2D graph of $|u_1|$ at different time.

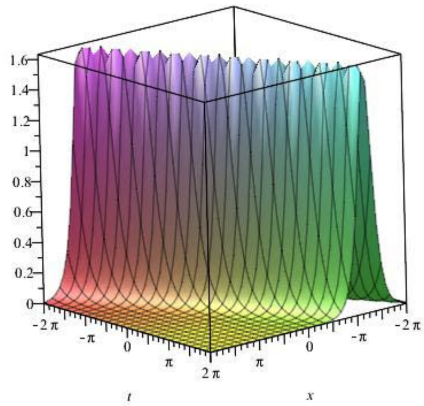

Figure.3. The 3D graph of u_3

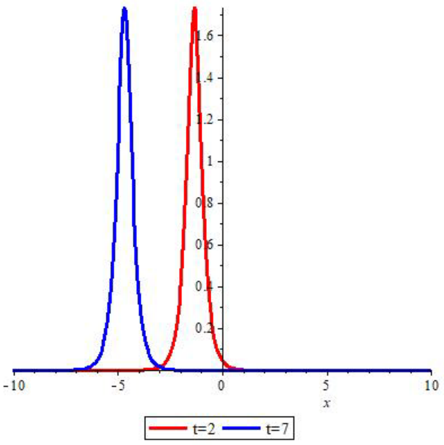

Figure.4. The 2D graph of $|u_3|$ at different time.

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