Hristov, J.: Transient Heat Conduction With Variable Thermophysical Properties
THERMAL SCIENCE: Year 2023, Vol. 27, Special Issue 1, pp. S411-S422

S411

TRANSIENT HEAT CONDUCTION WITH VARIABLE THERMOPHYSICAL PROPERTIES
Power-Law Temperature-Depenent Heat Capacity and Thermal Conductivity

by

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Transient heat conduction in semi-infinite medium with a power-law tempera-ture-dependent thermophysical properties has been solved by Double integral-bal-ance method. Correct formulation of the energy equation with temperature-depen-dent heat capacity is discussed and analyzed.

Key words: *transient heat conduction, double integral-balance method, temperature-dependent thermophysical properties*

Introduction

Transient heat transport by conduction with variable thermophysical parameters such as temperature-dependent specific heat capacity and thermal conductivity is among the challenging engineering problems in both experimental and modelling studies [1-5]. The existence of highly non-linear terms invokes specific approaches in solution methodologies either analytical or numerical [6-10].

Approximate analytical solutions (by the integral-balance method) of transient heat conduction in a semi-infinite domain with variable thermophysical properties expressed as power-law temperature-dependent functional relationships, [4, 5], and the references therein) are developed. One of the main tasks is the correct construction of the governing equation when the transport coefficients such as the specific heat capacity and thermal conductivity are temperature-dependent. This point is of primary importance since all equations, correctly stated or not, could be solved mathematically but the main question is about the adequacy of their solution and physical relevance.

This article considers two main problems:
– The correct formulation of the model equation when the thermophysical properties of the medium are temperature-dependent, and
– Solution approach involving two principle steps: transformation of the governing equation, the Double integration method, and solutions of the Dirichlet and Neumann boundary condition problems.

Solutions of the Dirichlet and Neumann boundary condition problems are developed respectively.

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S412

Hristov, J.: Transient Heat Conduction With Variable Thermophysical Properties
THERMAL SCIENCE: Year 2023, Vol. 27, Special Issue 1, pp. S411-S422

Energy transfer by conduction

The 1-D heat conduction

The 1-D heat conduction model can be generally presented in terms of temperature:

$$\frac{\partial(\rho C_p T)}{\partial t} = \frac{\partial}{\partial x}\left(k(T)\frac{\partial T}{\partial x}\right) \tag{1}$$

where the product $\rho C_p T = h$ is the enthalpy (the internal energy when the mass-specific capacity C_v is used instead C_p).

The model (1) comes directly from the Fourier formulation of the flux:

$$q = -k\frac{\partial T}{\partial x}$$

and the continuity equation

$$\frac{\partial(\rho C_p T)}{\partial t} = -\frac{\partial q}{\partial x}$$

For constant thermophysical properties of the medium (1) can be simplified:

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}, \ \ a = \frac{k}{\rho C_p} \tag{2}$$

This can be considered an exception since in many cases all thermophysical properties are temperature-dependent. Therefore, (1) should be presented:

$$\frac{\partial\left[\rho(T)C_p(T)T\right]}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \tag{3}$$

The functional relationships of ρT, $C_p(T)$, and $k(T)$ depend on types of data correlation fittings such as power-law [4, 5], polynomial [3, 11] or exponential [12].

On the correct heat conduction equation and common errors

The most common case discussed and solved is that with constant density ρ and heat capacity C_p but with temperature-dependent heat conductivity $k(T)$ that transforms eq. (3):

$$\rho C_p \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \Rightarrow \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[a(T)\frac{\partial T}{\partial x}\right], \ \ a(T) = \frac{k(T)}{\rho C_p} \tag{4}$$

It is worth noting that these forms of the heat conduction equation are especially presented here since we like to avoid a common error in its presentation, especially when the heat capacity is temperature-dependent $C_p(T)$. Precisely, we have to bear in mind that by the thermodynamic definitions both C_p and C_v are defined:

$$\mathrm{d}h = C_p \mathrm{d}T \Rightarrow \frac{\mathrm{d}h}{\mathrm{d}T} = C_p, \ \ \mathrm{d}e = C_v \mathrm{d}T \Rightarrow \frac{\mathrm{d}e}{\mathrm{d}T} = C_v \tag{5}$$

assuming that they are not temperature dependent.

However, if

$$h(T) = \rho(T)C_p(T)T \ \ \text{or} \ \ e(T) = \rho(T)C_v(T)T$$

then the time derivatives of the enthalpy (internal energy) should be:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\rho(T)C_p(T)T\right] = \rho(T)C_p(T)\frac{\mathrm{d}T}{\mathrm{d}t} + C_p(T)T\frac{\mathrm{d}\rho(T)}{\mathrm{d}t} + T\rho(T)\frac{\mathrm{d}C_P(T)}{\mathrm{d}t} \tag{6}$$

Hristov, J.: Transient Heat Conduction With Variable Thermophysical Properties
THERMAL SCIENCE: Year 2023, Vol. 27, Special Issue 1, pp. S411-S422

S413

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big[\rho(T)C_v(T)T\Big] = \rho(T)C_v(T)\frac{\mathrm{d}T}{\mathrm{d}t} + C_v(T)T\frac{\mathrm{d}\rho(T)}{\mathrm{d}t} + T\rho(T)\frac{\mathrm{d}C_v(T)}{\mathrm{d}t} \tag{7}$$

Then, for instance, if we consider the common case [3, 5] of temperature-dependent heat capacity $C_p(T)$, but with constant density ρ, the time derivative of $h(T)$:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \rho\left[C_p(T)\frac{\mathrm{d}T}{\mathrm{d}t} + T\frac{\mathrm{d}C_p(T)}{\mathrm{d}t}\right] \tag{8}$$

which reduces to

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \rho C_p \frac{\mathrm{d}T}{\mathrm{d}t}$$

only when C_p = const. We especially stressed the attention on this point because heat conduction models such as, see [5, 13-20] and many others:

$$\rho C_p(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left[k(T)\frac{\partial T}{\partial x}\right] \tag{9}$$

are physically and mathematically incorrect. Some particular examples are presented in the *Appendix*.

Functional relationship correlating $C_p(T)$ and $k(T)$

This work addresses power-law functional relationships of $C_p(T)$ and $k(T)$ namely:

$$C_p(T) = C_{p0}T^\alpha, \ k(T) = k_0 T^\beta, \ \alpha > 0, \ \beta > 0, \ \beta < 0 \tag{10}$$

wher $\beta > 0 <$ and $\beta < 0$ correspond to the thermal conductivity of composites [4] and pure metals [5], respectively, see comments in [5, 21]. Further, experimental data of $C_p(T)$, for instance, are scaled concerning a reference temperature T_{ref} or a reference temperature range such as:

$$\Delta T_{\mathrm{ref}} = T_{\mathrm{ref}(\max)} - T_{\mathrm{ref}(\min)}$$

that is:

$$C_p(T) = C_{p0}\left(\frac{T}{T_{\mathrm{ref}}}\right)^\alpha = C_{p0}\left(T^*\right)^\alpha, \ C_p(T) = \left(\frac{T}{\Delta T_{\mathrm{ref}}}\right)^\alpha = C_{p0}\left(T^{**}\right)^\alpha \tag{11}$$

If such non-dimensalization procedures are applied to all terms of the governing model (1), then its dimensionless form:

$$\frac{\partial}{\partial t}\left(\rho C_{p0}\left(T^*\right)^{\alpha+1}\right) = \frac{\partial}{\partial x}\left(k_0\left(T^*\right)^\beta \frac{\partial T^*}{\partial x}\right) \tag{12}$$

In the sequel we will avoid the asterisks assuming T as a dimensionless temperature-see eq. (13).

On the values of the exponents α and β

Here, concerning numerical simulations, we have to see if are there real experimentally determined values of exponents α and β. This information is rare in the literature since the common data fittings use polynomial expressions facilitating computer simulation but making hard analytical solutions. In all cases encompassed here, as reported in [5], the data fitting reveals $\alpha = 0.65$ and $\beta = -0.55$ for steel AISI 1008 (satisfies the condition $|\beta| < 1$). Otherwise,

S414

Hristov, J.: Transient Heat Conduction With Variable Thermophysical Properties
THERMAL SCIENCE: Year 2023, Vol. 27, Special Issue 1, pp. S411-S422

Lin [4] reports $\beta = 1$ and $\beta = 3$ with $\alpha = 0$, for a composite slab. In the former case, we have $\lambda_1 = 3.666$, while in the second case λ_2 ($\beta = 1$) $= 0.55$ and λ_2 ($\beta = 33$) $= 0.25$. To some extent, these values will define the ranges of variations in the following numerical experiments with the developed approximate solutions.

Solutions techniques

Governing equation transformation

For $\beta > 0$ we have:

$$\frac{\partial}{\partial t}\left(\rho C_{p0}T^{\alpha+1}\right) = \frac{\partial}{\partial x}\left(k_0 T^{\beta}\frac{\partial T}{\partial x}\right) \tag{13}$$

Changing the variable $T^{\alpha+1} = \theta$ and consequently, we have: $T = \theta^{1/(\alpha+1)}$ and $T^{\beta} = \theta^{\beta/(\alpha+1)} = \theta^{\gamma}$, where $\gamma = \beta/(\alpha+1)$. Then, we may rewrite eq. (13):

$$\frac{\partial}{\partial t}\left(\rho C_{p0}\theta\right) = \frac{\partial}{\partial x}\left(k_0\theta^{\gamma}\frac{\partial \theta^{\frac{1}{\alpha+1}}}{\partial x}\right) \tag{14}$$

Now, we focus on a transformation of the diffusion term in eq. (14) as a step towards the solutions developed next. In general, we have:

$$y^a\frac{\partial y^b}{\partial x} = \frac{b}{a+b}\frac{\partial y^{a+b}}{\partial x} \tag{15}$$

With $a = \gamma$ and $b = \beta/(\alpha+1)$, related to eq. (14), we have its new form, namely:

$$\frac{\partial \theta}{\partial t} = \frac{a_0}{1+\beta}\frac{\partial^2\theta^{\lambda}}{\partial x^2}, \quad a_0 = \frac{k_0}{\rho C_{p0}}, \quad \lambda = \frac{1+\beta}{1+\alpha} \tag{16}$$

In the case with $\beta < 0$ in all relationships developed in the preceding point β should be taken with a negative sign. It is important to mention that in such a case the denominator in the right-hand side of eq. (16) is $(1-\beta)$, and therefore, this imposes the condition $|\beta| < 1$. Then, the parameter λ becomes $\lambda = (1-\beta)/(1+\alpha)$. This does not change the logic of the solutions developed in the sequel.

Double-integration approach

The integral-balance method [22] is based on integration of the linear diffusion model with constant thermophysical properties eq. (17), as an example:

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \tag{17}$$

with respect to x up to the front δ of the diffusing thermal wave. Therefore, the medium is divided into a disturbed zone $0 \leq x \leq \delta(t)$ where $u(x, t) > 0$ and a undisturbed zone with $u(x, t) = 0$ for $\delta(t) \leq x < \infty$. This concept changes the common boundary condition (for semi-infinite media) $u(x \to \infty) = 0$ by two new conditions at the front (known as Goodman's conditions [22, 23]), namely:

$$u(\delta) = 0, \quad \frac{\partial u}{\partial x}(\delta) = 0 \tag{18}$$

where for the sake of demonstration of the integration approach, the variable u denotes temperature or concentration, thus avoiding misunderstanding with the following solutions of the model (2.1).

The solutions in this work will apply an advanced version of the method, known as Double-integration technique [21, 24, 25] which differs from that applied in [26]. The first step of DIM is the integration with respect to the space co-ordinate x, from x to δ, taking into account the boundary conditions (18):

$$\int_x^\delta \frac{\mathrm{d}}{\mathrm{d}t} u(x,t)\,\mathrm{d}x = -a\frac{\partial u(x,t)}{\partial x} \qquad (19)$$

The second steps considers integration of eq. (19) from 0 to δ [24]:

$$\int_0^\delta \int_x^\delta \left[\frac{\mathrm{d}}{\mathrm{d}t} u(x,t)\,\mathrm{d}x\right]\mathrm{d}x = au(0,t) \qquad (20)$$

The final result eq. (20) reveals that the right-hand side depends on the boundary condition at 0 in contracts to the eq. (19), where there is the gradient of the temperature profile. Then, replacing n by an approximate profile u_a expressed in terms of x/δ an ODE describing the time-evolution of $\delta(t)$ can be derived (see further in this work). For example only, if the approach used in [26] is applied then the integral relation:

$$\int_0^\delta \int_0^x \left[\frac{\mathrm{d}}{\mathrm{d}t} u(x,t)\,\mathrm{d}x\right]\mathrm{d}x = au(0,t) \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta xu(x,t)\,\mathrm{d}x = au(0,t) \qquad (21)$$

It is noteworthy that the version eq. (20) is more general, even though the right-hand sides of eqs. (20) and (21) are identical, because it allows applications to cases when the Leibniz rule for differentiation under the integral sign is impossible, *i.e.* when time-fractional operators are used in the model equations [25].

Approximate solutions

Unit step of temperature at the boundary

Now, applying the first integration step of DIM we have:

$$\int_x^\delta \frac{\partial \theta}{\partial t}\,\mathrm{d}x = \frac{a_0}{1+\beta}\left(\frac{\partial \theta^\lambda}{\partial x}\right)\Bigg|_x^\delta = \frac{a_0}{1+\beta}\left(-\frac{\partial \theta^\lambda}{\partial x}\right) \qquad (22)$$

The second integration step yields (for Dirichlet problem with $\theta_{x=0} = 1$):

$$\int_0^\delta \int_x^\delta \frac{\partial \theta}{\partial t}\,\mathrm{d}x\mathrm{d}x = \frac{a_0}{1+\beta}\left(\theta^\lambda\right)_{x=0} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\delta \int_x^\delta \theta\,\mathrm{d}x\mathrm{d}x = \frac{a_0}{1+\beta} \qquad (23)$$

Now, let us suggest an assumed profile as a parabolic profile with an unspecified exponent [23]:

$$T_a = T_s\left(1-\frac{x}{\delta}\right)^n,\ T_s = 1 \qquad (24)$$

and having in mind that $T^{\alpha+1} = \theta$, the assumed profile in terms of the variable θ:

$$\theta_a = \left(1-\frac{x}{\delta}\right)^{n(\alpha+1)} \Rightarrow \theta_a^\lambda = \left(1-\frac{x}{\delta}\right)^{\lambda[n(\alpha+1)]} \qquad (25)$$

S416

Hristov, J.: Transient Heat Conduction With Variable Thermophysical Properties
THERMAL SCIENCE: Year 2023, Vol. 27, Special Issue 1, pp. S411-S422

The profile eq. (25) satisfies the conditions:

$$\theta_{x=0} = 1 \text{ and } \theta_{x=\delta} = \left(\frac{\partial \theta}{\partial x}\right)_{x=\delta} = 0$$

for any values of n, and α, β.

The double integration on left-hand-side of eq. (23) yields:

$$\frac{d\delta^2}{dt} = \frac{a_0}{1+\beta} N(n,\alpha), \quad N(n,\alpha) = \left[n(\alpha+1)+1\right]\left[n(\alpha+1)+2\right] \tag{26}$$

The solution of eq. (26) with the condition $\delta(t=0) = 0$ results in:

$$\delta(t)_T = \sqrt{a_0 t} \sqrt{\frac{N(n,\alpha)}{1+\beta}} \tag{27}$$

Then, in terms of $\theta = T^{\alpha+1}$ and T the approximate profile is:

$$\theta_a = \left(1 - \frac{x}{\sqrt{a_0 t}\sqrt{\frac{N(n,\alpha)}{1+\beta}}}\right)^{n(\alpha+1)} \Rightarrow T_a = \left(1 - \frac{x}{\sqrt{a_0 t}\sqrt{\frac{N(n,\alpha)}{1+\beta}}}\right)^{n} \tag{28}$$

Here, we can define, in a classical way, the Boltzmann similarity variable:

$$\eta = \frac{x}{\sqrt{a_0 t}}$$

or its modified version

$$\eta_\beta = \frac{x}{\sqrt{a_\beta t}}$$

using a reduced thermal diffusivity defined as

$$a_\beta = \frac{a_0}{1+\beta}$$

The profiles produced by the approximations are shown in fig. 1. In the case of $\beta < 1$, which represents the conductivity of pure metals, we obtain solutions with strong parabolic behavior (concave profiles), fig. 1(a), where $|\alpha| < 1 \rightarrow 1$, increases the penetration depth. The situations with $\beta > 1$, however, have convex solutions, fig. 1(b), that correspond to conduction in composites. The profiles in fig. 1(b) require some justification: the heat diffusion model strongly degenerates when $\beta > 0$ [24], and references therein. In eq. (16), the right-hand side can be stated:

$$\frac{\partial^2 \theta^\lambda}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\theta^{\lambda-1}\frac{\partial \theta}{\partial x}\right) \tag{29}$$

In such cases the exponent $n(1+\alpha)$ of the assumed profile, see eq. (28), should be [24]:

$$n(\alpha+1) = \frac{1}{\lambda-1} = \frac{1+\alpha}{\beta-\alpha} \Rightarrow n = \frac{1}{\beta-\alpha} < 1 \tag{30}$$

resulting in convex profiles, with increasing steep fronts, and reduced penetration depths, when β is increased.

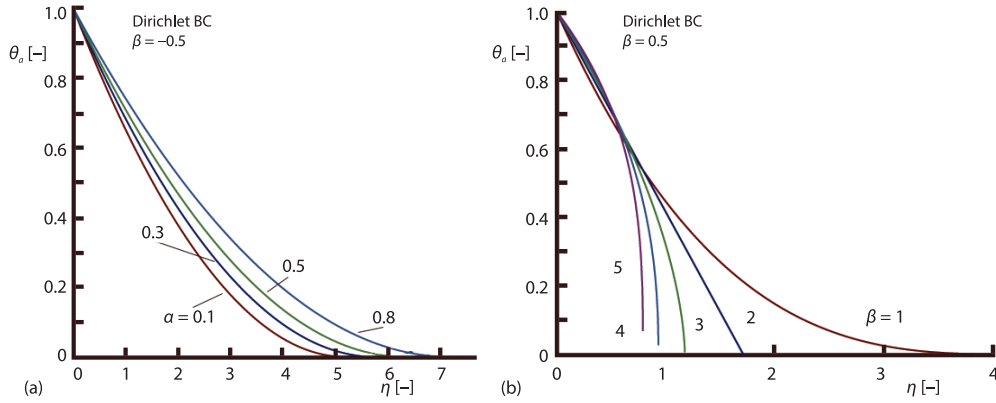**Figure 1. Approximate solutions of the Dirichlet problem; (a) $n = 2$ and $\beta = -0.5 < 0$ (pure metals) and (b) with $\beta > 1$ (composites) and $n = 1/(\beta - \alpha)$;** *Note: the penetration depths are marked at the spots where the profiles cross the abscissa*

Unit step of heat flux at the boundary

With a prescribed flux q_0 at $x = 0$ the boundary condition, see eq. (13):

$$-\frac{k_0}{1+\beta}\frac{\partial}{\partial x}\left(T^{1+\beta}\right) = q_0 \tag{31}$$

With the assumed profile eq. (24) and the relationship eq. (31) we get:

$$q_0 = \frac{k_0 n}{\delta}T^{1+\beta} \Rightarrow T_s = \left(\frac{q}{k_0 n}\right)^{\frac{1}{1+\beta}} \Rightarrow T_s^{1+\alpha} = \left(\frac{q}{k_0 n}\right)^{\lambda}, \quad \lambda = \frac{1+\beta}{1+\alpha} \tag{32}$$

Then, the assumed profile in terms of both variables T and θ:

$$T_a = \left(\frac{q_0\delta}{k_0 n}\right)^{\frac{1}{1+\beta}}\left(1-\frac{x}{\delta}\right)^{n} \Rightarrow T_a^{1+\alpha} = \theta_a = \left(\frac{q_0\delta}{k_0 n}\right)^{\lambda}\left(1-\frac{x}{\delta}\right)^{n(1+\alpha)} \tag{33}$$

The double integration of eq. (33) (and applying the Leibniz rule) yields:

$$\frac{d}{dt}\left[\int_0^\delta\int_x^\delta\left(\frac{q_0\delta}{k_0 n}\right)^{1/\lambda}\left(1-\frac{x}{\delta}\right)^{n(1+\alpha)}dxdx\right] = \frac{d}{dt}\left[\frac{q_0\delta^{1/\lambda+2}}{N(\alpha,n)}\right] \tag{34}$$

For the right-hand side of eq. (13) with T_s (or θ_s) from eq. (32) we get:

$$\int_0^\delta\int_x^\delta a_0\left(\frac{q_0\delta}{k_0 n}\right)^{1/\lambda}\left(1-\frac{x}{\delta}\right)^{n(1+\alpha)}dxdx = a_0\left(\frac{q_0}{k_0 n}\right)^{1/\lambda}\delta^{1/\lambda} \tag{35}$$

Therefore, the integral-balance relation of DIM yields an equation about δ, namely:

$$\frac{d\delta^{1/\lambda+2}}{dt} = a_0 N(\alpha,n)\delta^{1/\lambda} \tag{36}$$

Denoting $p = 2 + 1/\lambda$ and $z = \delta^{1/\lambda}$ we get from eq. (36):

$$\frac{dz^p}{dt} = bz \quad \text{where} \quad b = a_0 N(\alpha,n)$$

S418

Hristov, J.: Transient Heat Conduction With Variable Thermophysical Properties
THERMAL SCIENCE: Year 2023, Vol. 27, Special Issue 1, pp. S411-S422

This equation was solved in [24] and the solution in terms of $\delta(t)$:

$$\delta(t) = \left[bt\left(\frac{p-1}{p} \right) \right]^{\frac{1}{p-1}} \tag{37}$$

Then, in terms of α and β we get:

$$\delta_{\alpha,\beta}(t) = (a_0 t)^{\frac{1+\beta}{2+\alpha+\beta}} \left[N(\alpha,n)\left(\frac{2+\alpha+\beta}{3+2\alpha+\beta} \right) \right]^{\frac{1+\beta}{2+\alpha+\beta}} \tag{38}$$

For $\alpha = \beta = 0$ we have $\lambda = 1$ and eq. (36) reduces to

$$\delta_{q,\alpha=\beta=0} = \sqrt{a_0 t}\sqrt{\frac{2(n+1)(n+2)}{3}}$$

a result obtained in [26]. Further, for $\alpha = 0$ and $\beta > 0$, the heat conduction model degenerates and the relationship for δ_q is [24]:

$$\delta_{q,\alpha=0} = (a_0 t)^{\frac{1+\beta}{2+\beta}} \left[n^\beta (n+1)(n+2)\frac{\beta+2}{(\beta+1)(2\beta+3)} \right]^{\frac{1+\beta}{2+\beta}} \tag{39}$$

as eq. (27) developed in [24]. It reduces to $\delta_{q,\alpha=\beta=0}$ for $\beta = 0$. Finally, the approximate solution in terms of T_{a-q} and λ:

$$T_{a-q} = \left(1 - \frac{x}{(a_0 t)^{\frac{\lambda}{1+\lambda}}\left[\left(N(\alpha,n)\left(\frac{1+\lambda}{2\lambda+1} \right) \right) \right]^{\frac{\lambda}{1+\lambda}}} \right)^n \tag{40}$$

Hence, δ_q does not propagate proportional to $(a_0 t)^{1/2}$ due to the strong non-linearity in eq. (32). Moreover, a non-Boltzmann similarity variable:

$$\eta_\lambda = \frac{x}{(a_0 t)^{1/(p-1)}} = \frac{x}{(a_0 t)^{\lambda/(1+\lambda)}}$$

is defined. The surface temperature evolution in time is illustrated by the plots in fig. 2 in two cases depending on the variations of α. The retardation effect of α is obvious, *i.e.* the lower values of α, the slower rise of the surface temperature, irrespective of the value of β. The effects β are distinct for $\beta < 1$, fig. 2(a), and $\beta > 1$, fig. 2(b), wherein the latter case there are fast saturations.

Normalized temperature profiles (T_{a-q}/T_s) against the non-Boltzmann similarity variable η_λ are shown fig. 3. In both cases, The effect of α is weaker than the effect of β leading to a degeneration of the heat diffusion equation and yielding convex temperature profiles with sharp fronts and reducing penetration depths.
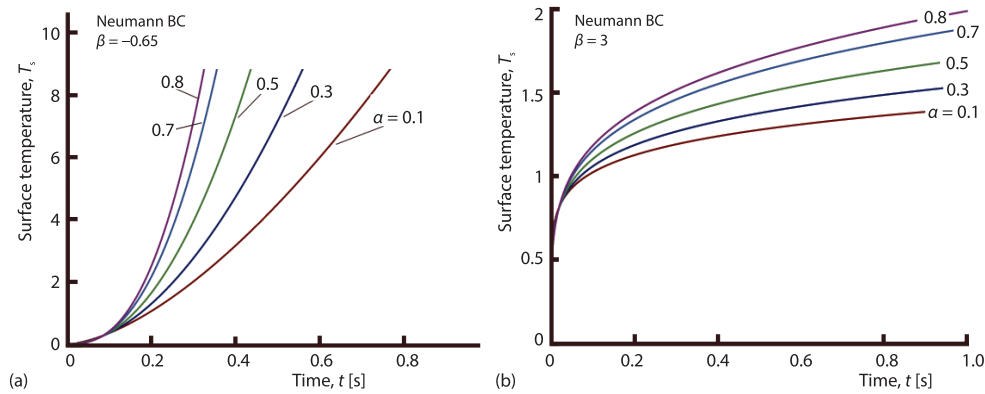
Hristov, J.: Transient Heat Conduction With Variable Thermophysical Properties
THERMAL SCIENCE: Year 2023, Vol. 27, Special Issue 1, pp. S411-S422

S419

**Figure 2. Surface temperature of the Neumann problem with varying α and fixed β;
(a) pure metals with $n = 2$ and $|\beta| < 1$ and (b) composites with $\beta > 1$ and $n = 1/(\beta - \alpha)$**
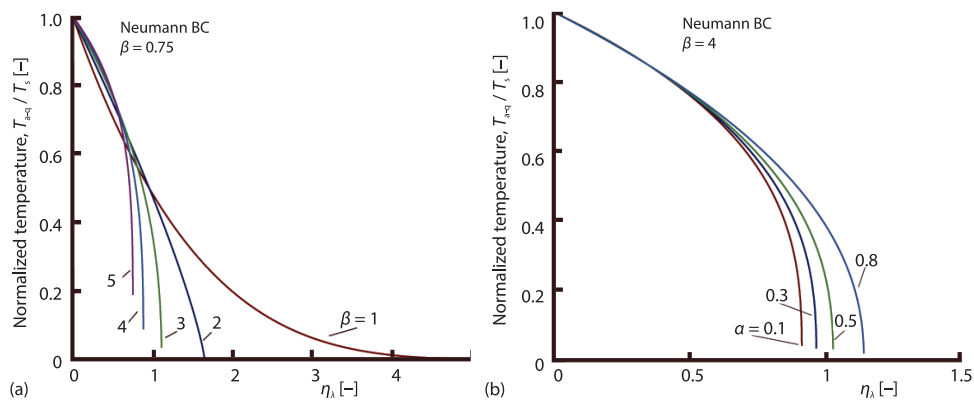

**Figure 3. Normalized temperature profiles of the Neumann problem for composites:
(a) with $\alpha = 0.75$ and varying $\beta > 1$ and (b) with $\beta = 4$ and varying $|\alpha| < 1$ ($\alpha < 0$);
in all cases $n = 1/(\beta - \alpha)$;** *Note: the penetration depths are marked at the spots where
the profiles cross the abscissa*

Conclusion

The article presents approximate analytical solutions of transient heat conduction with variable (power-law) thermophysical properties by application of the double-integration technique of the integral-balance method. The solution is preceded by an analysis of the proper construction of the modelling equation when both the specific heat capacity and heat conductivity are temperature-dependent. The solution approach by a special transformation of the non-linear heat conduction models is especially emphasized. The approach is demonstrated by the Dirichlet and Neumann boundary condition problems. The main efforts are on the creation of suitable solutions to correctly formulated problems and demonstrating their feasibility and physical adequacy. Optimization of the approximate temperature profiles concerning the assumed profile exponent n upon various values of α and β draws future study beyond the scope of this work.

Appendix: Correct and incorrect left-hand sides of the model equation

Finally, let consider again, with particular examples the general problem discussed in section *On the correct heat conduction equation and common errors.*

Power-law approximation of $C_p(T)$

With:

$$C_p(T) = Cp_0 T^\alpha$$

when the rules of the energy balance are violated, we get:

$$\rho C_{p0} T^\alpha \frac{\partial T}{\partial t} = \rho C_{p0} \frac{1}{\alpha+1} \frac{\partial T^{\alpha+1}}{\partial t} \tag{41}$$

Then, the model equation becomes:

$$\frac{\partial}{\partial t}\left(T^{\alpha+1}\right) = a_0 \frac{1+\alpha}{1+\beta} \frac{\partial^2}{\partial x^2}\left(\frac{\partial T^{\beta+1}}{\partial x}\right) \Rightarrow \frac{\partial \theta}{\partial t} = \frac{a_0}{\lambda} \frac{\partial^2 \theta^\lambda}{\partial x^2} \tag{42}$$

However, with a correct formulation of the enthalpy

$$h = \left(\rho C_{p0} T^\alpha\right) T \text{ we get } \frac{\partial h}{\partial t} = \rho C_{p0} \frac{\partial T^{\alpha+1}}{\partial t}$$

and the correct model eq. (13).

Polynomial approximation of $C_p(T)$

With a polynomial approximation, for example, $C_{p0} = C_0 + C_1 T + C_2 T^2$ the incorrect formulation:

$$\rho C_p(T) \frac{\partial T}{\partial t} \Rightarrow \rho\left(C_0 + C_1 T + C_2 T^2\right)\frac{\partial T}{\partial t} = \rho \frac{\partial}{\partial t}\left(C_0 T + \frac{C_1}{2} T^2 + \frac{C_2}{3} T^3\right) \tag{43}$$

On the other hand, with the correct enthalpy formulation we get:

$$h = \rho C_p(T) T = \rho\left[\left(C_0 + C_1 T + C_2 T^2\right) T\right] = \rho\left(C_0 T + C_1 T^2 + C_2 T^3\right) \tag{44}$$

Then, the time derivative is:

$$\frac{\partial h}{\partial t} = \rho\left(C_0 + 2C_1 T + 3C_2 T^2\right)$$

Exponential approximation of $C_p(T)$

Let the following approximation is applied:

$$C_p(T) = C_{p0} e^{\alpha T}$$

Then, the correct enthalpy formulation is:

$$h = \rho C_p(T) T = \rho C_{p0} e^{\alpha T} T$$

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial t}\left(\rho C_{p0} e^{\alpha T} T\right) = \rho C_{p0} e^{\alpha T}\left(1+\alpha\right)\frac{\partial T}{\partial t} \tag{45}$$

while the incorrect formulation is

$$\rho C_{p0} e^{\alpha T} \frac{\partial T}{\partial t}$$

At the end of these comments, we have to stress the attention that in all studies, referred to here as incorrect formulation of the heat transfer equation [5, 13-20] and many others, there are numerical solutions, which hiders to see the incorrect results coming from incorrect equations.

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S422

Hristov, J.: Transient Heat Conduction With Variable Thermophysical Properties
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