A NUMERICAL STUDY FOR SOLVING MULTI-TERM FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS

by

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In this article, we extended operational matrices using orthonormal Boubaker polynomials of Riemann-Liouville fractional integration and Caputo derivative to find numerical solution of multi-term fractional-order differential equations (FDE). The proposed method is utilized to convert FDE into a system of algebraic equations. The convergence of the method is proved. Examples are given to explain the simplicity, computational time and accuracy of the method.

Key words: *fractional differential equations, orthonormal Boubaker polynomials, Gram-Schmidt process, operational matrix, covergence analysis*

Introduction

In the recent era, fractional calculus (FC) has a vital role in applied mathematics due to its non-local property. Fractional order models have many applications in the areas like biomedical engineering, hydrology, viscoelasticity, electromagnetic, material science, biology, physics, acoustics, electromagnetic, finance, *etc*., to describe the real-world phenomenon [1, 2]. The non-local property is the essential advantage of these equations showing the state of a complex system does not depend only on its current state but also depends upon its all previous conditions. Therefore, fractional derivative (FD) is significant to present long-memory processes and materials, anomalous diffusion, long-range interactions, long-term behaviors, power laws, *etc*. The FD has non-local property, due to which it is hard to find the solution of fractional differential equations (FDE). In today's era, researchers are working on a solution of FDE by developing new analytical and numerical approaches. Widespread applications of FDE motivate the advancement of analytical and numerical techniques to find its solution.

In this paper, we have worked upon finding the numerical solution of multi-order FDE using orthonormal Boubaker polynomials. Multi-order FDE is defined [3]:

$$\mathrm{D}_t^{\mu}u(t)=\sum_{i=1}^{r}a_i\mathrm{D}_t^{\beta_i}u(t)+a_{r+1}u(t)+h(t) \tag{1}$$

$$u^m(0)=c_m,\ \ m=0,1,\cdots n-1 \tag{2}$$

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where $n-1<\mu\leq n$, the coefficients a_i ($i=1, 2, \ldots r+1$) are constants, $0<\beta_1<\beta_2<\ldots<\beta_r<\mu$ and h is known function and $D_t^\mu u(t)$ is the Caputo fractional derivative of order μ.

Polynomial approximation is the best approximation method to find the numerical solution of the FDE. An operational matrix is one of the numerical techniques to solve the FDE. Using different polynomials like Legendre [4, 5], Chebyshev [6], Jacobi [7, 8] and Bernstein polynomials [9], Fermat polynomials [10], Cubic B-spline polynomial [11], *etc.*, operational matrices for fractional derivative and fractional integration constructed by many researchers. Nowadays, researchers are focusing on shifted polynomials like Legendre [12], ψ-shifted [13] to construct the operational matrix for the solution of fractional differential equations of variable order. The Boubaker polynomials were firstly introduced by Boubaker in 2007 to find the solution of the 1-D heat transfer equation [11, 14]. Boubaker polynomials and their applications were discussed by Kobra *et al*. [15]. Using Boubaker polynomials Bolandtalat *et al*. [3] solved multi-term fractional differential equations, Abdelkrim *et al.* solved the Emden-Fowler problem [16]. Spectral method was developed using Boubaker polynomials in [17].The main objective of the proposed method is to convert the multi-term FDE into a set of algebraic equations by expanding the unspecified function within orthonormal Boubaker polynomials using operational matrices of the fractional operators.

Basic definitions and properties

In this section, we recall some of important preliminaries of fractional calculus.

Let $\mu\in\mathbb{R}_+$ and $n=[\mu]$, where [.] is the greatest integer function. The Riemann-Liouville fractional integral $I_t^\mu f(t)$ of order μ is defined [18]:

$$I_t^\mu f(t)=\frac{1}{\Gamma(\mu)}\int_a^t (t-\zeta)^{\mu-1}f(\zeta)\mathrm{d}\zeta,\ \text{where}\ \mu\in[n-1,n) \tag{3}$$

The Caputo fractional derivative of order α is denoted by D_t^μ and is defined [18]:

$$D_t^\mu f(t)=I_t^{(n-\mu)}\left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right)=\frac{1}{\Gamma(n-\mu)}\int_a^t \frac{f^n(\zeta)\mathrm{d}\zeta}{(t-\zeta)^{\mu+1-n}} \tag{4}$$

where, $\mu\in(n-1, n)$. If $\mu=n$,

$$D_t^\mu f(t)=\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$$

The Boubaker polynomials defined [3]:

$$B_n(t)=\sum_{p=0}^{\zeta(n)}\left[\frac{(n-4p)}{(n-p)}\binom{p}{n-p}\right](-1)^p t^{n-2p} \tag{5}$$

Recursive formula for Boubaker polynomials is given:

$$\begin{gathered}B_m(t)=tB_{m-1}(t)-B_{m-2}(t),\ \text{ for }\ m\geq 2\\ B_0(t)=1,\ \ B_1(t)=t\end{gathered} \tag{6}$$

For Gram-Schmidt process(G-S process) let $\phi(t)\in L^1(0, 1)$ be a polynomial and:

$$W=\left\{\phi(t)\,|\,\phi(t)\ \text{ is a polynomialand }\ \phi(t)\in L^1(0,1)\right\} \tag{7}$$

be the inner product vector space with the innner product and norm is defined:

$$< \phi(t),\psi(t) >= \int_0^1 \phi(t)\psi(t)\mathrm{d}t,\ \ \|\phi\|^2 = \int_0^1 \phi(t)^2\,\mathrm{d}t$$

Gram Schmidt orthogonality process is defined to covert any arbitrary basis of the inner product space into orthogonal basis. Now, let us consider $\{v_0, v_1, v_2,...v_n\}$ be an arbitrary basis of V then, orthogonal basis of W is given by the vectors $\{w_0, w_1, w_2,...w_n\}$:

$$w_1(t) = v_1(t)$$

$$w_2(t) = v_1(t) - \frac{< v_1, w_1 > u_1}{\|w_1\|^2}$$

$$w_3(t) = v_3(t) - \frac{< v_3, w_1 > w_1}{\|w_1\|^2} - \frac{< v_2, w_2 > w_2}{\|w_2\|^2}$$

$$w_n(t) = v_n(t) - \sum_{i=1}^{n-1} \frac{< v_n, w_i > w_i}{\|w_i\|^2}$$

For Orthonormal Boubaker polynomials we now apply the process of G-S on (5) to derive orthonormal Boubaker polynomials [19]. Then, we have:

$$\tilde{B}_0(t) = 1,\ \ \tilde{B}_1(t) = \sqrt{3}(-1+2t),\ \ \tilde{B}_2(t) = \sqrt{5}(1-6t+6t^2),\ \ \tilde{B}_3(t) = \sqrt{7}(-1+12t-30t^2+20t^3)$$

Analytical form of orthonormal Boubaker polynomial is given:

$$\tilde{B}_i(t) = \sqrt{2N+1}\sum_{r=0}^{N}(-1)^{N+r}\frac{(N+r)!t^r}{(N-r)!(r!)^2},\ \ N \in \mathbb{N} \tag{7}$$

Properties of fractional operators:

$$I_t^{\mu}\mathrm{D}_t^{\mu}u(t) = u(t) - \sum_{r=0}^{n-1} u^{(r)}(0)\frac{t^r}{r!} \tag{8}$$

$$I_t^{\mu}\mathrm{D}_t^{\beta}u(t) = I_t^{\mu-\beta}u(t) - \sum_{r=0}^{n-1}\frac{u^{(r)}(0)}{\Gamma(\mu-\beta+r+1)}(t-a)^{\mu-\beta+r} \tag{9}$$

where $\beta \in (n-1, n)$.

Function approximation

The function $\phi(t) \in L^1(0, 1)$ is approximated using orthonormal Boubaker polynomials:

$$\phi(t) = \sum_{i=0}^{N} b_i(t)B_i(t) = B^T\tilde{B}(t) \tag{10}$$

where

$$\tilde{B}(t) = \left[\tilde{B}_0(t)\ \ \tilde{B}_1(t)\ldots\tilde{B}_N(t)\right] \text{ and } \tilde{B}_i(t),\quad i = 0,1,\cdots N$$

are orthonormal Boubaker polynomials and $B^T = [b_0\ \ b_1... b_N]$ are unknown orthonormal Boubaker coefficients and N is chosen as any positive integer. The b_i can be derived:

$$b_i = \int_0^1 \phi(t) B_i(t) \mathrm{d}t$$

Let us express, orthonormal Boubaker polynomials into Taylors basis is given:

$$\tilde{B}(t) = \tilde{Z} X(t)$$

where

$$X(t) = [1 \ t \ t^2 \ \ t^3 \cdots t^n]^T$$

and coefficient matrix $\tilde{Z}$ is given:

$$\tilde{Z} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ -\sqrt{3} & 2\sqrt{3} & 0 & 0 & 0 & \cdots & 0 \\ \sqrt{5} & -6\sqrt{5} & 6\sqrt{5} & 0 & 0 & \cdots & 0 \\ -\sqrt{7} & 12\sqrt{7} & -30\sqrt{7} & 20\sqrt{7} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^N \sqrt{2N+1} & (-1)^N \sqrt{2N+1} & & & & & (-1)^N \sqrt{2N+1} \\ & (N^2+N) & & & & & \dfrac{(2N)!}{N!} \end{bmatrix} \tag{11}$$

Operational matrices for fractional operators

Orthonormality plays an important role in the solution of FDE using operational matrix method. In this section, we extend the operational matrices of fractional integration and Caputo derivative using orthonormal Boubaker polynomials.

Operational matrix of the R-L fractional integration

Consider R-L fractional integration of Boubaker vector $\tilde{B}$(t):

$$I_t^\mu \tilde{B}(t) = L^\mu \tilde{B}(t) \tag{12}$$

where L^μ is the operational matrix of R-L fractional integration using orthonormal Boubaker polynomials of order $(N + 1) \times (N + 1)$. Computation of L^μ:

$$\begin{aligned} I_t^\mu \tilde{B}(t) &= \frac{1}{\gamma(\mu)} \int_0^t (t-\zeta)^{\mu-1} \tilde{B}(t) \mathrm{d}\zeta \\ &= \frac{1}{\gamma(\mu)} \int_0^t (t-\zeta)^{\mu-1} \tilde{Z} X(t) \mathrm{d}\zeta \\ &= \tilde{Z} \left[I_t^\mu 1 \ \ I_t^\mu t \ \ I_t^\mu t^2 \ \ I_t^\mu t^3 \cdots I_t^\mu t^N \right]^T \\ &= \tilde{Z} \left[\frac{0!}{\Gamma(\mu+1)} t^\mu \ \ \frac{1!}{\Gamma(\mu+2)} t^{\mu+1} \ \cdots \ \cdots \ \frac{N!}{\Gamma(\mu+1+N)} t^{\mu+N} \right]^T \\ &= \tilde{Z} \tilde{\mathrm{D}} \overline{X}(t) \end{aligned} \tag{13}$$

where

$$\bar{X}(t)=\left[t^{\mu}\;\; t^{\mu+1}\;\; t^{\mu+2}\;\; t^{\mu+3}\cdots t^{\mu+N}\right]^{T}$$

$$\tilde{\mathrm{D}}=\begin{bmatrix} \frac{0!}{\Gamma(\mu+1)} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1!}{\Gamma(\mu+2)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{2!}{\Gamma(\mu+3)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{3!}{\Gamma(\mu+4)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & & & & \frac{N!}{\Gamma(\mu+1+N)} \end{bmatrix}$$

Now, we express the $X(t)$ in terms of orthonormal Boubaker ploynomials:

$$\bar{X}(t)=H\tilde{B}(t)$$

Let us express, $t^{\mu+r}$ by $N+1$ terms of the orthonormal Boubaker basis:

$$t^{\mu+i}\cong h_i\tilde{B}(t)$$

where

$$h_i=[h_{i,0}\;\; h_{i,1}\cdots h_{i,N}],\quad h_{i.j}=\int_0^1 t^{\alpha+i}B_j(t)\mathrm{d}t,\quad H=[h_0\;\; h_1\cdots h_N]^T$$

$$\therefore L^{\mu}\tilde{B}(t)=\tilde{Z}\tilde{\mathrm{D}}H\tilde{B}(t) \tag{14}$$

The matrix $L^{\mu}=\tilde{Z}\tilde{\mathrm{D}}H$ is the required operational matrix of R-L fractional integration operator.

Operational matrix of the Caputo fractional derivative

Consider, Caputo fractional derivative of Boubaker vector $\tilde{B}(t)$:

$$\mathrm{D}^{\mu}\tilde{B}(t)=\mathrm{D}_{\mu}\tilde{B}(t) \tag{15}$$

where D_{μ} is $(N+1)\times(N+1)$ operational matrix for Caputo fractional of derivative using orthonormal Boubaker polynomials. Computation of D_{μ} is given:

$$\begin{aligned} \mathrm{D}^{\mu}\tilde{B}(t)&=\mathrm{D}^{\mu}\tilde{Z}X(t)\\ &=\tilde{Z}\mathrm{D}^{\mu}X(t)\\ &=\tilde{Z}[\mathrm{D}^{\mu}1\;\; \mathrm{D}^{\mu}t\;\; \mathrm{D}^{\mu}t^2\;\; \mathrm{D}^{\mu}t^3\cdots \mathrm{D}^{\mu}t^N]^T\\ &=\tilde{Z}\left[0\;\; \frac{\Gamma(2)}{\Gamma(2-\mu)}t^{1-\mu}\;\;\cdots\frac{\Gamma(N)}{\Gamma(N-\mu)}t^{n-\mu}\right]^T\\ &=\tilde{Z}\mathrm{D}_1X_1(t) \end{aligned} \tag{16}$$

where

$$X_1(t)=\left[t^{-\mu}\ \ t^{1-\mu}\ \ t^{2-\mu}\ \ t^{3-\mu}\cdots t^{N-\mu}\right]^T$$

$$\mathrm{D}_1=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \dfrac{1!}{\Gamma(2-\mu)} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \dfrac{2!}{\Gamma(3-\mu)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \dfrac{3!}{\Gamma(4-\mu)} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & & & & & \dfrac{N!}{\Gamma(N+1-\mu)} \end{bmatrix}$$

Approximate $t^{r-\mu}$ by $N+1$ terms of the orthonormal Boubaker basis:

$$t^{i-\mu}\cong\tilde{E}_i^T\tilde{B}(t)$$

$$\tilde{E}=\left[e_{i,0}\ \ e_{i,1}...e_{i,N}\right]$$

where

$$e_{i,j}=\int_0^1 t^{i-\mu}B_j(t)\mathrm{d}t$$

$$\therefore \mathrm{D}_\mu=\tilde{Z}\mathrm{D}_1\tilde{E}^T\tilde{B}(t) \tag{17}$$

The matrix

$$\mathrm{D}_\mu=\tilde{Z}\mathrm{D}_1\tilde{E}^T$$

is the required operational matrix of Caputo fractional order derivative operator.

Multi-term fractional order differential equation using operational matrix

To solve the multi-term fractional order differential equation, we first operate I_t^μ on the both sides of eq. (1). Therefore, we get:

$$u(t)-\sum_{r=0}^{n-1}u^{(r)}(0)\frac{t^r}{r!}=\sum_{i=0}^{r}a_i\left(I_t^{\mu-\beta_i}u(t)-\sum_{j=0}^{m_i-1}\frac{u^{(k)}(0)}{\Gamma(\mu-\beta_i+j+1)}(t)^{\mu-\beta_i+r}\right)+a_{r+1}I_t^\mu u(t)+I_t^\mu h(t) \tag{18}$$

$$u(t)=\sum_{i=1}^{r}a_iI^{\mu-\beta_i}u(t)+a_{r+1}I^\mu u(t)+w(t) \tag{19}$$

where

$$w(t)=I^\mu h(t)+\sum_{r=0}^{n-1}u^{(r)}(0)\frac{t^r}{r!}-\sum_{j=0}^{m_j-1}\frac{u^{(r)}(0)}{\Gamma(\mu-\beta_i+j+1)}(t)^{\mu-\beta_i+r} \tag{20}$$

Here we assume the polynomial approximation u and w by using given polynomial:

$$u(t) \cong \sum_{i=0}^{N} c_i \tilde{B}_i(t) = C^T \tilde{B}(t), \;\; w(t) \cong \sum_{i=0}^{N} w_i \tilde{g}_i(t) = W^T \tilde{B}(t) \tag{21}$$

Therefore, eq. (19) converted into:

$$C^T - C^T \sum_{i=1}^{r} a_i L^{\mu-\beta_i} - a_{r+1} C^T L^{\mu} - W^T = 0 \tag{22}$$

Solving the system of algebraic eq. (22), we find the value of C^T and hence the required solution.

Here, we discuss convergence of the solution obtained by the proposed scheme in section 5 to the analytical solution of problema (1) and (2).

Theorem 1. Let $u_N(t)$ be the approximate solution of problems (1) and (2) obtained by the proposed scheme in section *Examples*, $u(t)$ is its analytical solution and $R_N(t)$ is the residual error for the approximate solution. Then, $R_N(t)$ tends to zero when $N \to \infty$.

Proof 1. To convert, the given eq. (1) into fractional integral equation we apply I_t^{μ} on both sides of equation. Therefore, we get:

$$u(t) = \sum_{s=0}^{n-1} \frac{t^s}{s!} u^{(s)}(0) + \sum_{i=1}^{r} a_i \left(I_t^{\mu-\beta_i} u(t) - \sum_{j=0}^{m_i-1} \frac{t^{\mu-\beta_i+j}}{\Gamma(\mu-\beta_i+j+1)} u^{(j)}(0) \right) + a_{r+1} I_t^{\mu} u(t) + I_t^{\mu} h(t)$$

so $u_N(t)$ satisfies

$$u_N(t) = \sum_{s=0}^{n-1} \frac{t^s}{s!} u^{(s)}(0) + \sum_{i=1}^{r} a_i \left(I_t^{\mu-\beta_i} u_N(t) - \sum_{j=0}^{m_i-1} \frac{t^{\mu-\beta_i+j}}{\Gamma(\mu-\beta_i+j+1)} u^{(j)}(0) \right) + a_{r+1} I_t^{\mu} y_N(t) + I_t^{\mu} h(t)$$

where the residual function $R_N(t)$ is given

$$R_N(t) = u_N(t) - u(t) + \sum_{i=1}^{r} a_i \left(I_t^{\mu-\beta_i} (u(t) - u_N(t)) \right) + a_{r+1} I_t^{\mu} \left(u(t) - u_N(t) \right)$$

Then we have:

$$|R_N(t)| \le |u_N(t) - u(t)| + \sum_{i=1}^{r} \frac{|a_i|}{\Gamma(\mu-\beta_i+1)} |u(t) - u_N(t)|$$

$$+ \frac{|a_{r+1}|}{\Gamma(\mu+1)} |u(t) - u_N(t)| = \left(1 + \sum_{i=1}^{r} \frac{|a_i|}{\Gamma(\mu-\beta_i+1)} + \frac{|a_{r+1}|}{\Gamma(\mu+1)} \right) |u(t) - u_N(t)| \tag{23}$$

Now, we need to find a bounded for $|u(t) - u_N(t)|$. To do that, let ∈ is the interpolating polynomials to $u(t)$ at points t_ℓ (t_ℓ be the roots of the *shifted Chebyshev polynomials* of degree $N + 1$):

$$u(t) - P_N(t) = \frac{u^{N+1}(\xi)}{(N+1)!} \prod_{i=0}^{N} (t - t_i), \;\; \xi \in [0,1]$$

Using the estimates for Chebyshev interpolation nodes:

$$|u(t) - P_N(t)| \le \frac{\theta}{(N+1)! 2^{2N+1}}, \;\; \forall t \in [0,1]$$

where

$$\left|u^{(N+1)}(t)\right| \le \theta$$

In the finite subspace, the best approximation of any given function $u \in L^2(0, 1)$ is unique. Therefore, we have:

$$\left|u(t)-u_N(t)\right| \le \frac{\theta}{(N+1)!2^{2N+1}}, \quad \forall t \in [0,1] \tag{24}$$

Substituting eq. (25) into eq. (23), we have:

$$\left|R_N(t)\right| \le \frac{\theta}{(N+1)!2^{2N+1}}\left(1+\sum_{i=1}^{r}\frac{|a_i|}{\Gamma(\mu+\beta_i+1)}+\frac{|a_{r+1}|}{\Gamma(\mu+1)}\right)$$

Therefore, it is clear that $R_N(t)$ tends to zero when $N \to \infty$.

Table 1. A comparison between our methods and method in [3] for *Example 1*

t	Exact	[3]	Proposed method for $N = 3$	Proposed method for $N = 4$
	0	0.1252	0.1253	0.1092
0.1	0.3162	0.2993	0.2992	0.3034
0.2	0.4472	0.4396	0.4395	0.4461
0.3	0.5477	0.5517	0.5517	0.5531
0.4	0.6325	0.6414	0.6413	0.6371
0.5	0.7071	0.7140	0.7140	0.7077
0.6	0.7746	0.7753	0.7753	0.7719
0.7	0.8367	0.8309	0.8309	0.8336
0.8	0.8944	0.8864	0.8863	0.8939
0.9	0.9487	0.9472	0.9471	0.9507
1	1	1.0192	1.0190	0.9993

Examples

This section contains, solution of few examples using proposed numerical method.

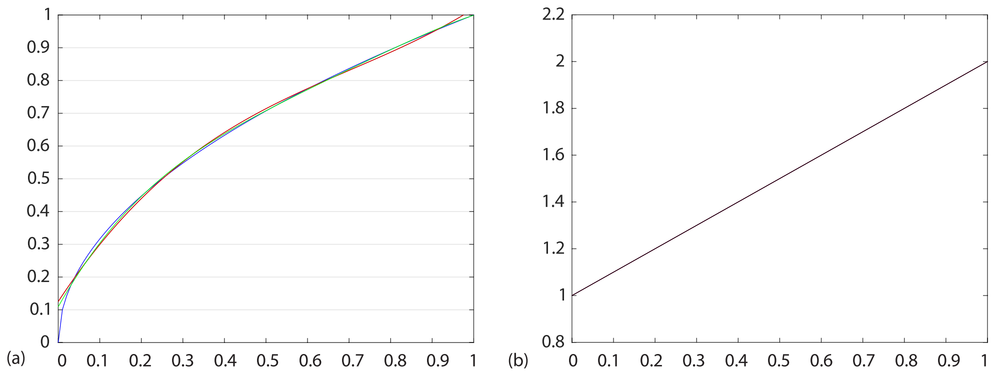
Example 1:

Consider the FDE

$$D^{1/2}u(t)+u(t)=\sqrt{t}+\frac{\sqrt{\pi}}{2}, \quad 0<t<1, \text{with initial condition } u(0)=0 \tag{25}$$

the exact solution in this case: $u(t) = (t)^{1/2}$.

Solution of eq. (25) using orthonormal Boubaker polynomials discussed for $N = 3,4$.



(a) The exact solution and approximate solution of *Example 1* for $N = 3$ and $N = 4$ and (b) exact solution and approximate solution of *Example 2*

Example 2. Multi-order Bagley-Torvik equation FDE:

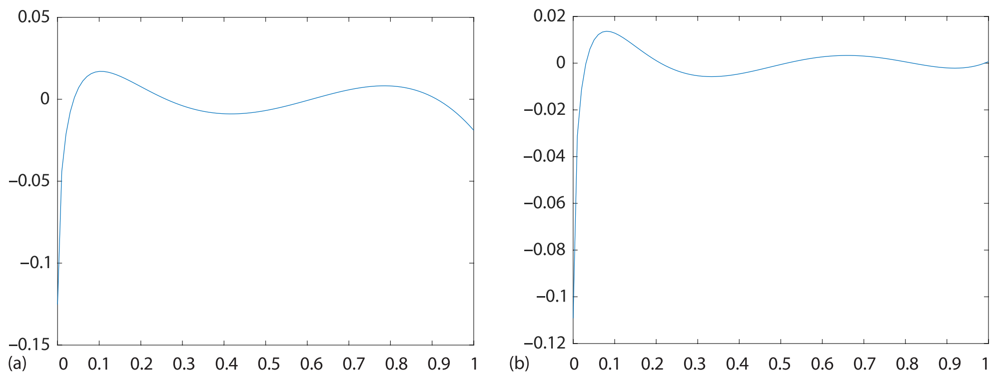
$$D^2u(t) + D^{3/2}u(t) + u(t) = 1 + t, \quad 0 < t < 1 \tag{26}$$

$$u(0) = 1, \quad u'(0) = 1 \tag{27}$$

The exact solution of eq. (26):

$$u(t) = 1 + t \tag{28}$$

Solution of eq. (26) using orthonormal Boubaker polynomials discussed for $N = 3,4$



(a) The absolute error for $N = 3$ and (b) the absolute error for $N = 4$

Conclusion

In this article, we have extended the proposed numerical scheme using orthonormal Boubaker polynomials. By using this method, we derived operational matrix of R-L fractional integration and Caputo derivative. This technique is applied for the solution of multi-term FDE. The convergence analysis is provided. This numerical scheme is applied on few examples to illustrate the accuracy and simplicity of the proposed method. In future, this method can be applied for system of multi-term FDE and variable order FDE. All computational results are obtained by using MATLAB software.

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