OPTICAL MODELLING OF THE SPACE-TIME FRACTIONAL ECKHAUS EQUATION

by

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In this paper, the space-time fractional Eckhaus equation is considered and solved using the a direct method (Khater method) to obtain exact solutions. This method produces more solutions when compared to other known methods. The real solutions of this equation are classified as travelling wave, kink, periodic and solitary wave solutions. These solutions are searched with the help of the fractional conformable derivative sense. Some graphs and tables are drawn to interpret the solutions and method. With the interpretation of the results, it is explained that the method used is a reliable, effective, powerful and easily applicable technique for obtaining the solutions of fractional differential equations classes in many fields.

Key words: *conformable derivative, the space-time fractional Eckhaus equation, Khater method*

Introduction

Recently, many scientists have been interested in fractional calculations to better model and analyze physical phenomena. Due to the advantage of fractional calculus in modelling, many mathematical events in many different sciences have been reformulated using fractional calculus [1-5]. There are many PDE that have been reformulated using the fractional derivatives concept, such as the time-fractional Cahn-Allen equation [6], space-time fractional Cahn-Hilliard equation [7], Diffusive predator-prey model [8], Space-time fractional Eckhaus equation [9].

The space-time fractional Eckhaus equation [9-11] is expressed:

$$i\mathrm{D}_{\tau}^{\theta}\Psi + \mathrm{D}_{\kappa}^{2\theta}\Psi + 2\Psi\mathrm{D}_{\kappa}^{\theta}\left|\Psi\right|^{2} + \left|\Psi\right|^{4}\Psi = 0 \tag{1}$$

where

$$\Psi = \Psi(\kappa,\tau),\ \ \Psi : \mathbb{R}^{2} \to \mathbb{C}$$

This equation, put forward by Wiktor Eckhaus, describes how waves propagate in some medium [10]. Many properties of this equation are laid out in reference [11].

Where the fractional derivative $\mathrm{D}_{\tau}^{\theta}$ is of conformable-type and defined [12]:

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$$\mathrm{D}_{\tau}^{\theta} f(\tau)=\lim_{\vartheta \to 0}\frac{f\left(\tau+\vartheta\tau^{1-\theta}\right)-f(\tau)}{\vartheta},\quad f:(0,\infty)\to\mathbb{R},\quad \theta\in(0,1)$$

Some properties of the conformable derivative can be expressed in [12-14].

While studying fractional differential equations, solutions are often obtained with the help of this derivative operator [13-16].

In recent years, fractional differential equations have been studied more than integer order differential equations due to their many advantages in practice in many fields. Some of these fields are biomechanics, engineering, fluid-flow, signal processing, acoustic waves, optical fibers, systems identification and control theory, *etc.* In addition, many researchers have used fractional differential operators when searching for exact solutions of fractional differential equations in space and time. While finding solutions of fractional differential equations in the literature, many methods such as the functional variable method, the Kudryashov method, the trial solution method, the Jacobi elliptic function expansion method, the sine-cosine method, the exp-function method, G'/G-expansion method and others [17-20] have been used.

In this study, the exact solutions of the space-time fractional Eckhaus equation are investigated using the Khater method [6, 17] and with the help of derivative operators. This method is very powerful, effective, reliable and applicable method to get exact and solitary wave solution of non-linear differential models. In addition, this method has many advantages over other known methods [6].

Exact solution of the space-time fractional Eckhaus equation by using conformable derivative

The aim of this section is to investigate exact solutions for the space-time fractional Eckhaus equation with the aid of Khater method.

Suppose that traveling wave transformation:

$$\Psi\left(\kappa,\tau\right)=u\left(\phi\right)\mathrm{e}^{i\left(\kappa-q\tau\right)},\ \phi=\frac{\kappa^{\theta}}{\theta}-p\frac{\tau^{\theta}}{\theta} \tag{2}$$

Putting eq. (2) into eq. (1), we get real and imaginary part of eq. (1):

$$u''(\phi)+4u'(\phi)u(\phi)\left|u(\phi)\right|+\left|u(\phi)\right|^{4}u(\phi)+u(\phi)(q-1)=0,\ \ 2iu'(\phi)-ipu'(\phi)=0 \tag{3}$$

where $p = 2$ is obtained by solving the second equation. Applying $u(\phi)=\sqrt{r(\phi)}$ transform to the first equation gives:

$$2r''(\phi)r(\phi)-r'(\phi)^{2}+8r(\phi)^{2}r'(\phi)+4r(\phi)^{4}+4(q-1)r(\phi)^{2}=0 \tag{4}$$

if we apply the balancing operation highest order of linear and non-linear derivative, we get $N = 1$. Then, the solution will be in the shape:

$$r(\phi)=A_0+A_1A^{\Psi(\varphi)} \tag{5}$$

where A_0 and A_1 are constants and will be obtained later. Also $\Psi(\phi)$ provides the equation:

$$\Psi'(\phi)=\frac{1}{\ln A}\left(\alpha A^{-\Psi(\phi)}+\beta+\sigma A^{\Psi(\phi)}\right) \tag{6}$$

When $\beta^2 - \alpha\sigma < 0$ and $\sigma \neq 0$:

$$A^{\Psi(\phi)} = \frac{-\beta}{\sigma} + \frac{\sqrt{-(\beta^2-\alpha)}}{\sigma}\tan\left(\left(\frac{\sqrt{-(\beta^2-\alpha)}}{\sigma}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right)$$
$$A^{\Psi(\phi)} = \frac{-\beta}{\sigma} + \frac{\sqrt{-(\beta^2-\alpha)}}{\sigma}\cot\left(\left(\frac{\sqrt{-(\beta^2-\alpha)}}{\sigma}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right) \tag{7}$$

When $\beta^2 + \alpha\sigma < 0$ and $\sigma \neq 0$:

$$A^{\Psi(\phi)} = \frac{-\beta}{\sigma} + \frac{\sqrt{\beta^2-\alpha\sigma}}{\sigma}\tanh\left(\left(\frac{\sqrt{\beta^2-\alpha\sigma}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right)$$
$$A^{\Psi(\phi)} = \frac{-\beta}{\sigma} - \frac{\sqrt{\beta^2-\alpha\sigma}}{\sigma}\coth\left(\left(\frac{\sqrt{\beta^2-\alpha\sigma}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right) \tag{8}$$

When $\beta^2 + \alpha\sigma > 0$ and $\sigma \neq 0$, and $\sigma \neq -\alpha$:

$$A^{\Psi(\phi)} = \frac{\beta}{\sigma} + \frac{\sqrt{\beta^2+\alpha^2}}{\sigma}\tanh\left(\left(\frac{\sqrt{\beta^2+\alpha^2}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right)$$
$$A^{\Psi(\phi)} = \frac{\beta}{\sigma} + \frac{\sqrt{\beta^2+\alpha^2}}{\sigma}\coth\left(\left(\frac{\sqrt{\beta^2+\alpha^2}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right) \tag{9}$$

When $\beta^2 + \alpha\sigma < 0$ and $\sigma \neq 0$, and $\sigma \neq -\alpha$:

$$A^{\Psi(\phi)} = \frac{\beta}{\sigma} + \frac{\sqrt{-(\beta^2+\alpha^2)}}{\sigma}\tan\left(\left(\frac{\sqrt{-(\beta^2+\alpha^2)}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right)$$
$$A^{\Psi(\phi)} = \frac{\beta}{\sigma} + \frac{\sqrt{-(\beta^2+\alpha^2)}}{\sigma}\cot\left(\left(\frac{\sqrt{-(\beta^2+\alpha^2)}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right) \tag{10}$$

When $\beta^2 - \alpha < 0$ and and $\sigma \neq -\alpha$:

$$A^{\Psi(\phi)} = \frac{-\beta}{\sigma} + \frac{\sqrt{-(\beta^2-\alpha^2)}}{\sigma}\tan\left(\left(\frac{\sqrt{-(\beta^2-\alpha^2)}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right)$$
$$A^{\Psi(\phi)} = \frac{\beta}{\sigma} + \frac{\sqrt{-(\beta^2-\alpha^2)}}{\sigma}\cot\left(\left(\frac{\sqrt{-(\beta^2-\alpha^2)}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right) \tag{11}$$

When $\beta^2 - \alpha^2 > 0$ and and $\sigma \neq -\alpha$:

$$A^{\Psi(\phi)} = \frac{-\beta}{\sigma} + \frac{\sqrt{(\beta^2-\alpha^2)}}{\sigma}\tanh\left(\left(\frac{\sqrt{(\beta^2-\alpha^2)}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right)$$
$$A^{\Psi(\phi)} = \frac{-\beta}{\sigma} + \frac{\sqrt{(\beta^2-\alpha^2)}}{\sigma}\coth\left(\left(\frac{\sqrt{(\beta^2-\alpha^2)}}{2}\right)\left(\frac{\kappa^\theta}{\theta} - 2\frac{\tau^\theta}{\theta}\right)\right) \tag{12}$$

When $\alpha\sigma < 0$, $\sigma \neq 0$, and = 0:

$$A^{\Psi(\phi)} = \sqrt{\frac{-\alpha}{\sigma}}\tanh\left(\left(\frac{\sqrt{-\alpha\sigma}}{2}\right)\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)\right),\quad A^{\Psi(\phi)} = \sqrt{\frac{-\alpha}{\sigma}}\coth\left(\left(\frac{\sqrt{-\alpha\sigma}}{2}\right)\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)\right) \quad (13)$$

When $\beta = 0$ and $\alpha \neq -\sigma$:

$$A^{\Psi(\phi)} = \frac{-\left(1+\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right) \pm \sqrt{2\left(1+\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right)}}{\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}-1}$$

$$A^{\Psi(\phi)} = \frac{-\left(1+\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right)}{2\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}} \pm \sqrt{\frac{\left(\mathrm{e}^{4\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right)+\sqrt{2\left(1+\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right)}}{2\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}} \quad (14)$$

When $\beta^2 = \alpha\sigma$:

$$A^{\Psi(\phi)} = \frac{-\alpha\left(\beta\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)+2\right)}{\beta^2\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)} \quad (15)$$

When $\beta = k$, $\alpha = 2k$, and $\sigma = 0$:

$$A^{\Psi(\phi)} = \mathrm{e}\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)-1 \quad (16)$$

When $\beta = k$, $\sigma = 2k$, and $\alpha = 0$:

$$A^{\Psi(\phi)} = \frac{\mathrm{e}\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}{1-\mathrm{e}^{\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}} \quad (17)$$

When $2\beta = \alpha + \sigma$:

$$A^{\Psi(\phi)} = \frac{1+\alpha\mathrm{e}^{1/2(\alpha-\sigma)\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}{1+\sigma\mathrm{e}^{1/2(\alpha-\sigma)\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}},\quad A^{\Psi(\phi)} = \frac{1+\alpha\mathrm{e}^{1/2(\alpha-\sigma)\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}{-1-\sigma\mathrm{e}^{1/2(\alpha-\sigma)\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}} \quad (18)$$

When $-2\beta = \alpha + \sigma$:

$$A^{\Psi(\phi)} = \frac{\alpha + \alpha \mathrm{e}^{1/2(\alpha-\sigma)\left(\frac{\kappa^{\theta}}{\theta} - 2\frac{\tau^{\theta}}{\theta}\right)}}{\sigma + \sigma \mathrm{e}^{1/2(\alpha-\sigma)\left(\frac{\kappa^{\theta}}{\theta} - 2\frac{\tau^{\theta}}{\theta}\right)}} \tag{19}$$

When $\alpha = 0$:

$$A^{\Psi(\phi)} = \frac{\beta \mathrm{e}^{\beta\left(\frac{\kappa^{\theta}}{\theta} - 2\frac{\tau^{\theta}}{\theta}\right)}}{1 + \frac{\sigma}{2} \mathrm{e}^{\beta\left(\frac{\kappa^{\theta}}{\theta} - 2\frac{\tau^{\theta}}{\theta}\right)}} \tag{20}$$

When $\alpha = \beta = \sigma \neq 0$:

$$A^{\Psi(\phi)} = \frac{-\left(\alpha\left(\frac{\kappa^{\theta}}{\theta} - 2\frac{\tau^{\theta}}{\theta}\right) + 2\right)}{\alpha\left(\frac{\kappa^{\theta}}{\theta} - 2\frac{\tau^{\theta}}{\theta}\right)} \tag{21}$$

When $\alpha = \sigma$, $\beta = 0$:

$$A^{\Psi(\phi)} = \tan\left(\frac{\alpha\left(\frac{\kappa^{\theta}}{\theta} - 2\frac{\tau^{\theta}}{\theta}\right) + c}{2}\right) \tag{22}$$

When $\sigma = 0$:

$$A^{\Psi(\phi)} = \mathrm{e}^{\beta\left(\frac{\kappa^{\theta}}{\theta} - 2\frac{\tau^{\theta}}{\theta}\right)} - \frac{\alpha}{2\beta} \tag{23}$$

First, we substitute eq. (6) and its derivatives in the eq. (4) and then equalize the coefficients of the different powers of $A^{\Psi(\phi)}$ to zero to obtain algebraic equations. Finally, if these equations are solved with the help of the MATHEMATICA program, four solution sets are obtained:

$$\begin{gathered}
\text{Set 1. } q = \frac{1}{4}(4 - \beta^2 + 4\alpha\sigma),\ A_0 = \frac{1}{4}\left(-\beta + \sqrt{\beta^2 - 4\alpha\sigma}\right),\ A_1 = -\frac{\sigma}{2} \\
\text{Set 2. } q = \frac{1}{4}(4 - \beta^2 + 4\alpha\sigma),\ A_0 = \frac{1}{4}\left(-\beta - \sqrt{\beta^2 - 4\alpha\sigma}\right),\ A_1 = -\frac{\sigma}{2} \\
\text{Set 3. } q = \frac{1}{4}\left(4 - \beta^2 + 2i\sqrt{2}\beta\sqrt{\alpha\sigma} + 2\alpha\sigma\right),\ A_0 = \frac{1}{2}\left(-\beta + i\sqrt{2}\sqrt{\alpha\sigma}\right),\ A_1 = 0 \\
\text{Set 4. } q = \frac{1}{4}\left(4 - \beta^2 - 2i\sqrt{2}\beta\sqrt{\alpha\sigma} + 2\alpha\sigma\right),\ A_0 = \frac{1}{2}\left(-\beta - i\sqrt{2}\sqrt{\alpha\sigma}\right),\ A_1 = 0 \\
\text{Set 5. } q = \frac{1}{16}(16 - 9\beta^2),\ A_0 = -\frac{3\beta}{4},\ A_1 = 0
\end{gathered} \tag{24}$$

Now by substituting the values in Set 1 into eq. (5) the exact traveling wave solution of SFCHE for the Set 1 is obtained:

$$r(\phi)=\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}A^{\Psi(\phi)} \tag{25}$$

If this expression is substituted in eq. (2) after the necessary transformations are made:

$$\Psi(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}A^{\Psi(\phi)}}\,\mathrm{e}^{i\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)} \tag{26}$$

is obtained. There are many different solutions of $\Psi(\kappa,\tau)$ as $A^{\Psi(\phi)}$ takes different values from (7) to (23). We can express some of these solutions:

$$\Psi_{1}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{-\beta}{\sigma}+\frac{\sqrt{-(\beta^{2}-\alpha)}}{\sigma}\tan\left(\frac{\sqrt{-(\beta^{2}-\alpha)}}{\sigma}\right)\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)\right)}\cdot\mathrm{e}^{i\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)}$$

$$\Psi_{2}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{-\beta}{\sigma}+\frac{\sqrt{-(\beta^{2}-\alpha)}}{\sigma}\cot\left(\frac{\sqrt{-(\beta^{2}-\alpha)}}{\sigma}\right)\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)\right)}\cdot\mathrm{e}^{i\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)}$$

$$\Psi_{3}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{-\beta}{\sigma}+\frac{\sqrt{\beta^{2}-\alpha\sigma}}{\sigma}\tanh\left(\frac{\sqrt{\beta^{2}-\alpha\sigma}}{2}\right)\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)\right)}\cdot\mathrm{e}^{i\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)}$$

$$\Psi_{4}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{\beta}{\sigma}+\frac{\sqrt{\beta^{2}+\alpha^{2}}}{\sigma}\tanh\left(\frac{\sqrt{\beta^{2}+\alpha^{2}}}{2}\right)\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)\right)}\cdot\mathrm{e}^{i\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)}$$

$$\Psi_{5}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{\beta}{\sigma}+\frac{\sqrt{\beta^{2}+\alpha^{2}}}{\sigma}\coth\left(\frac{\sqrt{\beta^{2}+\alpha^{2}}}{2}\right)\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)\right)}\mathrm{e}^{i\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)}$$

$$\Psi_{6}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{-\beta}{\sigma}+\frac{\sqrt{-(\beta^{2}-\alpha^{2})}}{\sigma}\tan\left(\frac{\sqrt{-(\beta^{2}-\alpha^{2})}}{2}\right)\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)\right)}\cdot\mathrm{e}^{i\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)}$$

$$\Psi_{7}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{\beta}{\sigma}+\frac{\sqrt{-(\beta^{2}-\alpha^{2})}}{\sigma}\cot\left(\frac{\sqrt{-(\beta^{2}-\alpha^{2})}}{2}\right)\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)\right)}\cdot\mathrm{e}^{i\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)}$$

$$\Psi_{8}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^{2}-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\sqrt{\frac{-\alpha}{\sigma}}\tanh\left(\frac{\sqrt{-\alpha\sigma}}{2}\right)\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)\right)}\cdot\mathrm{e}^{i\left(\frac{\kappa^{\theta}}{\theta}-2\frac{\tau^{\theta}}{\theta}\right)}$$

$$\Psi_9(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^2-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\sqrt{\frac{-\alpha}{\sigma}}\coth\left(\frac{\sqrt{-\alpha\sigma}}{2}\right)\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)\right)}\mathrm{e}^{i\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}$$

$$\Psi_{10,11}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^2-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{-\left(1+\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right)\pm\sqrt{2\left(1+\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right)}}{\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}-1}\right)}\mathrm{e}^{i\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}$$

$$\Psi_{12,13}(\kappa,\tau)=$$

$$=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^2-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{-\left(1+\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right)}{2\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}\pm\sqrt{\frac{\left(\mathrm{e}^{4\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right)+\sqrt{2\left(1+\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}\right)}}{2\mathrm{e}^{2\alpha\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}}\right)}\mathrm{e}^{i\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}$$

$$\Psi_{14}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^2-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{\mathrm{e}^{\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}{1-\mathrm{e}^{\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}\right)}\mathrm{e}^{i\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}$$

$$\Psi_{15}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^2-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{1+\alpha\mathrm{e}^{\frac{1}{2}(\alpha-\sigma)\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}{1+\sigma\mathrm{e}^{\frac{1}{2}(\alpha-\sigma)\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}\right)}\mathrm{e}^{i\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}$$

$$\Psi_{16}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^2-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\frac{\beta\mathrm{e}^{\beta\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}{1+\frac{\sigma}{2}\mathrm{e}^{\beta\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}}\right)}\mathrm{e}^{i\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}$$

$$\Psi_{17}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^2-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\tan\left(\frac{\alpha\left(\dfrac{\kappa^\theta}{\theta}-2\dfrac{\tau^\theta}{\theta}\right)+c}{2}\right)\right)}\,\mathrm{e}^{i\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}$$

$$\Psi_{18}(\kappa,\tau)=\sqrt{\frac{1}{4}\left(-\beta+\sqrt{\beta^2-4\alpha\sigma}\right)-\frac{\sigma}{2}\left(\mathrm{e}^{\beta\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}-\frac{\alpha}{2\beta}\right)}\,\mathrm{e}^{i\left(\frac{\kappa^\theta}{\theta}-2\frac{\tau^\theta}{\theta}\right)}$$

Physical reviews

We prepared some pictures and tables to investigate the physical reviews of the obtained solutions of the space-time fractional Eckhaus equation. In addition, we show the efficiency and accuracy of the Khater method in 3-D and 2-D graphics.

Conclusion

In this study, we applied the introduction of the Khater method, which is more advantageous than many other known methods, for the solution of the space-time fractional Eckhaus equation, and we obtained many solutions for this equation. These solutions are expressed in section 2 *Exact solution of the space-time fractional Eckhaus equation by using conformable derivative*. These solutions can be classified as traveling wave, kink, periodic and solitary wave solutions. The results obtained are given in figs. 1-5 and tabs. 1 and 2 according to the various orders and the different values.

This method is an effective method in terms of providing strong, accurate and reliable solutions in addition the variety of solutions. We examined and interpreted these results in the physical reviews section of our article and showed the effectiveness of the method. It is therefore, applicable to many different differential equations.

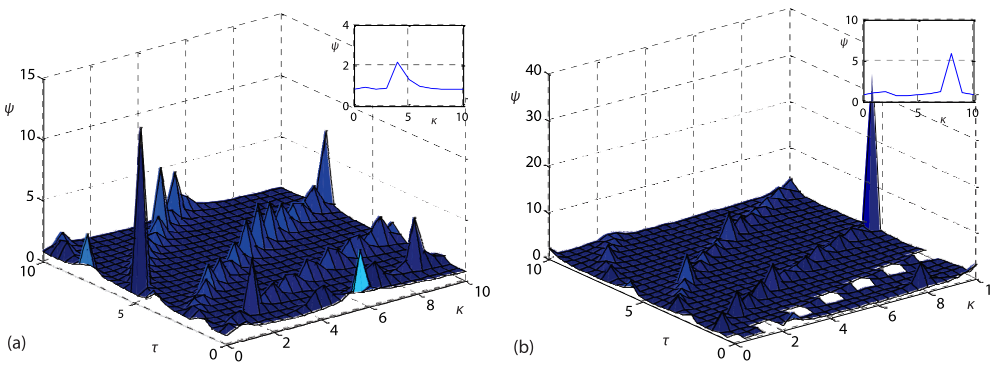

Figure 1. The 3-D graphics of the space-time fractional Eckhaus equation (θ = 0.5); (a) the $\Psi_1(\kappa, \tau)$ exact solution (α = 2, β = 1, σ = 1) and (b) the $\Psi_2(\kappa, \tau)$ exact solution (α = 2, β = 1, σ = 1)

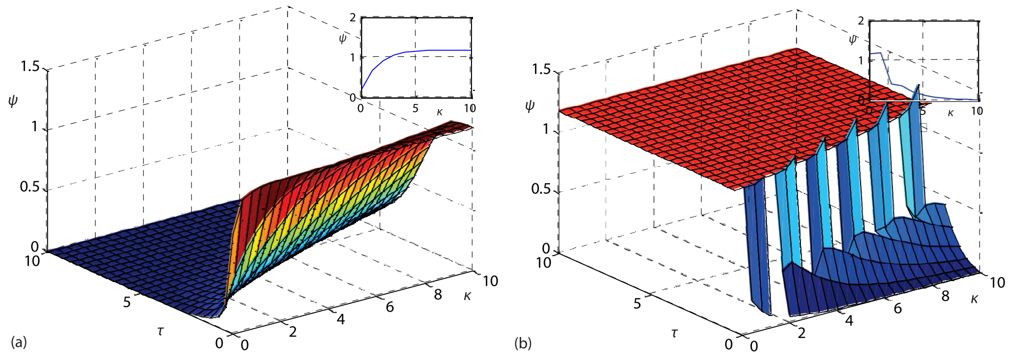

Figure 2. The 3-D graphics of the space-time fractional Eckhaus equation ($\theta = 0.5$); (a) the $\Psi_8(\kappa, \tau)$ exact solution ($\alpha = 2, \beta = 0, \sigma = -1$) and (b) the $\Psi_{10}(\kappa, \tau)$ exact solution ($\alpha = 2, \beta = 0, \sigma = -2$)

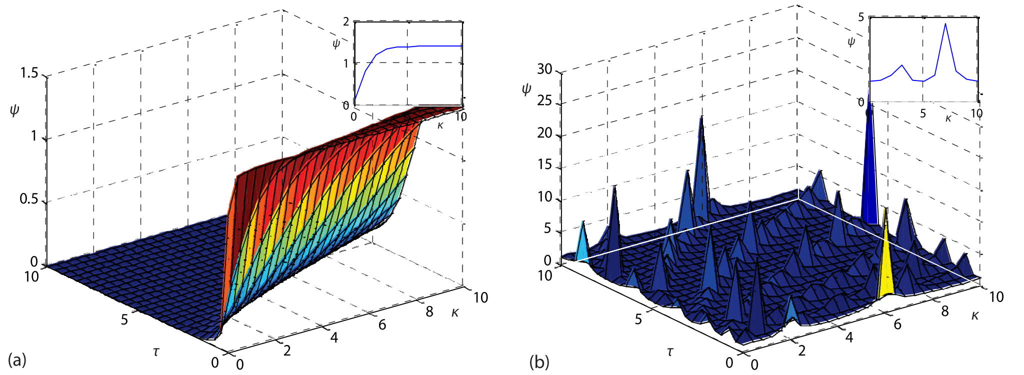

Figure 3. The 3-D graphics of the space-time fractional Eckhaus equation ($\theta = 0.5$); (a) the $\Psi_{16}(\kappa, \tau)$ exact solution ($\alpha = 0, \beta = 2, \sigma = 3$) and (b) the $\Psi_{17}(\kappa, \tau)$ exact solution ($\alpha = 3, \beta = 0, \sigma = 3$)

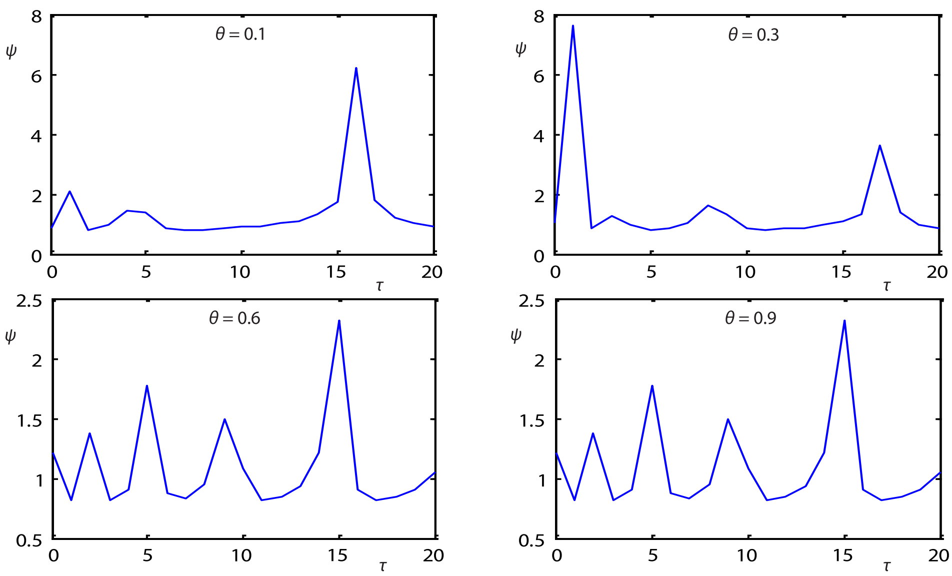

Figure 4. The 2-D graphic of the $\Psi_1(\kappa, \tau)$ exact solution for the space-time fractional Eckhaus equation with distinct θ and τ ($\alpha = 2, \beta = 1, \sigma = 1, \kappa = 0.4$)

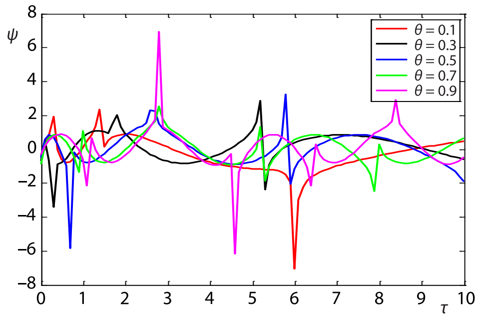

Figure 5. The 2-D graphic of the $\Psi_1(\kappa, \tau)$ exact solution for the space-time fractional Eckhaus equation with distinct θ and τ ($\kappa = 0.8$, $\alpha = 2$, $\beta = 1$, $\sigma = 1$)

Table 1. The some Khater solutions of the space-time fractional Eckhaus equation for different values of τ ($\theta = 0.5$, $\kappa = 0.4$)

τ	$\Psi_1(\kappa,\tau)$	$\Psi_4(\kappa,\tau)$	$\Psi_7(\kappa,\tau)$	$\Psi_{10}(\kappa,\tau)$	$\Psi_{15}(\kappa,\tau)$	$\Psi_{17}(\kappa,\tau)$
0.01	1.43019	0.78941	0.926265	0.207307	1.08855	1.27238
0.02	0.776796	0.266642	0.820317	0.288744	0.860211	1.16545
0.03	0.44447	0.162466	0.70805	0.350386	0.650095	0.30129
0.04	0.209987	0.0867	0.592936	0.398274	0.456245	0.813587
0.05	0.026405	0.78069	0.476272	0.4334796	0.2774	1.9628
0.06	0.124147	0.986279	0.358568	0.461277	0.112564	3.22947
0.07	0.250555	1.13282	0.239878	0.478629	0.0391154	7.27559
0.08	0.357995	1.22711	0.119913	0.487587	0.178397	2.92955
0.09	0.44982	1.27559	0.00193581	0.488812	0.305973	2.60436
0.1	0.528374	1.28431	0.126649	0.482932	0.422484	2.30889

In the previous tab. 1, $\alpha = 2$, $\beta = 1$, $\sigma = 1$ for $\Psi_1(\kappa,\tau)$, $\alpha = 2$, $\beta = 1$, $\sigma = 2$ for $\Psi_4(\kappa,\tau)$, $\alpha = 2$, $\beta = 1$, $\sigma = 1$ for $\Psi_7(\kappa,\tau)$, $\alpha = 2$, $\beta = 0$, $\sigma = -1$ for $\Psi_{10}(\kappa,\tau)$, $\alpha = 1$, $\beta = 2$, $\sigma = 3$ for $\Psi_{15}(\kappa,\tau)$, and $\alpha = 3$, $\beta = 0$, $\sigma = 3$ for $\Psi_{17}(\kappa,\tau)$, are used.

Table 2. The $\Psi_1(\kappa,\tau)$ Khater solution of the space-time fractional Eckhaus equation for different values of τ and θ ($\kappa = 0.8$, $\alpha = 2$, $\beta = 1$, $\sigma = 1$)

τ/θ	0.1	0.3	0.5	0.7	0.9
0.1	0.825091	0.868836	0.813991	0.821221	0.819255
0.2	1.08382	1.10335	0.840896	0.817678	0.814282
0.3	2.20861	3.46163	0.898417	0.841748	0.826528
0.4	0.896632	1.10113	0.984116	0.877759	0.848119
0.5	0.814545	0.85528	1.133331	0.92664	0.877673
0.6	0.82361	0.814423	1.490024	0.99659	0.917073
0.7	0.851672	0.819076	6.21962	1.10844	0.971696
0.8	0.889782	0.838413	1.45352	1.3245	1.05371
0.9	0.938667	0.86545	1.0286	1.9714	1.19398
1	1.00398	0.899465	0.885487	2.7453	1.49933

References

[1] Kilbas, A. A., *et al*., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006

[2] Podlubny, I., *Fractional Differential Equation*, Academic Press, San Diego, USA, 1999

[3] Sabatier, J., *et al*., (Eds.), Advances in Fractional Calculus: *Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, Germany, 2007

[4] Samko, S. G., *et al*., Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Switzerland, 1993

[5] Wu, G. C., *et al*., Fractional Impulsive Differential Equations: Exact Solutions, Integral Equations and Short Memory Case, *Frac. Calc. Appl. Anal*., *2*2 (2019), Mar., pp. 180-192

[6] Khater, M. M. A., *et al*., Elliptic and Solitary Wave Solutions for Bogoyavlenskii Equations System, Couple Boiti-Leon-Pempinelli Equations System and Time-fractional Cahn-Allen Equation, Results in Physics, *7* (2017), July, pp. 2325-2333

[7] Khan, M.M., *et al*., Traveling Wave Solutions for Space-Time Fractional Cahn Hilliard Equation and Space-Time Fractional Symmetric Regularized Long-Wave Equation, *Alexandria Engineering Journal*, *60* (2021), 1, pp. 1317-1324

[8] Ali, M., *et al*., Explicit and Approximate Solutions for the Conformable-Caputo Time-Fractional Diffusive Predator-Prey Model, *Int. J. Appl. Comput. Math*., *7* (2021), 90

[9] Kadkhoda, N., Jafari, H., An Analytical Approach to Obtain Exact Solutions of Some Space-Time Conformable Fractional Differential Equations, *Advances in Difference Equations*, *2019* (2019), 428

[10] Taghizadeh, N., *et al*., The First Integral Method Applied to the Eckhaus Equation, *Appl. Math. Lett*., *25* (2012), 5, pp. 798-802

[11] Calogero, F., Lillo, S. D., The Eckhaus PDE $i\phi t + \boldsymbol{\phi xx} + 2\phi|\phi|^2_x + \phi|\ \phi\ |^4 = 0$, *Inverse Probl*., *3* (1987), 4, pp. 633-681

[12] Khalil, R., *et al.*, A New Definition of Fractional Derivative, *Journal of Computational and Applied Mathematics*, *264* (2014), July, pp. 65-70

[13] Eslami, M., Rezazadeh, H., The First Integral Method for Wu-Zhang System with Conformable Time-Fractional Derivative, *Calcolo*, *53* (2016), Oct., pp. 475-485

[14] Korpinar, Z., *et al*., New Soliton Solutions of the Fractional Regularized Long Wave Burger Equation by Means of Conformable Derivative, *Results in Physics*, *14* (2019), 102395

[15] Z, Korpinar, T., New Optical Hybrid Electromotive of B2-Ferromagnetic Fiber with some Optical Applications, *Optik*, *251* (2022), 168190

[16] Korpinar, T., *et al*., Optical Quasi Flux Density of Heisenberg Ferromagnetic Spin with qHATM Approach, *Optik*, *245* (2021), 16756

[17] Bibi, S., *et al.*, Khater Method for Non-Linear Sharma Tasso-Olever (STO) Equation of Fractional Order, *Results in Physics*, *7* (2017), Nov., pp. 4440-4450

[18] Abdelrahman, M. A. E., *et al*., Exact Solutions of the Cubic Boussinesq and Coupled Higgs System, *Thermal Science*, *24* (2020), Suppl. 1, pp. S333-S342

[19] Aliyu, A.I., *et al*., Adomian-Pade Approximate Solutions to the Conformable Non-Linear Heat Transfer Equation, *Thermal Science*, *23* (2019), Suppl. 1, pp. S235-S242

[20] Aliyu, A. I., *et al*., Approximate Solutions and Conservation Laws of the Periodic Base Temperature of Convective Longitudinal Fins in Thermal Conductivity, *Thermal Science*, *23* (2019), Suppl. 1, pp. S267-S273

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