PERFORMANCE OF MESHLESS METHOD OF LINES FOR THE SOLUTION OF THE GENERALIZED SEVENTH-ORDER KORTEWEG-DE VRIES EQUATION HAVING APPLICATIONS IN FLUID MECHANICS

by

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In this article, we investigate the execution of a meshless method of line (MMOL) to solve general seventh-order Korteweg-de Vries (KdV7) equations numerically. The suggested meshless technique uses radial basis functions (RBF) for spatial derivatives and the Runge-Kutta (RK) method for time derivatives to solve the governing equation. To produce an efficient numerical solution, three different types of RBF are used. The method's output is successfully compared to the exact solution.

Key words: *meshless method of line, radial basis functions, KdV7*

Introduction

Mathematical models are important tools for understanding the behavior of numerous applications in various areas such as fluid dynamics, engineering or physics. Most mathematical models are integrated into practical applications as non-linear differential equations. Therefore, efforts have been undertaken to develop more reliable and practical methods for finding exact and approximate solutions to non-linear PDE. The phenomenon of the travelling wave occurs in many areas, for example in non-linear optics, fluid dynamics, plasma physics, optical chemical dynamics, *etc.* [1]. In the field of fluid dynamics, the authors in [2, 3] discovered the important interaction properties of a solitary wave solution of KdV, and the non-linear PDE as a travelling wave in shallow water model is entitled as the KdV equation. In [3], the Cauchy problem for the KdV equation is considered by the invention of the inverse spectral transformation is the most profound breakthrough in the development of modern non-linear mathematical sciences. Then the KdV equation was expanded to a higher order, and the seventh degree KdV equation [4] appeared for the first time in the work to investigate the stability of KdV under the singular perturbation.

The general KdV7 equation [5-7]:

$$u_t + au^3u_y + bu_y^3 + cuu_yu_{yy} + du^2u_{yyy} + eu_{2y}u_{3y} + fu_yu_{4y} + guu_{5y} + u_{7y} = 0 \tag{1}$$

with the corresponding initial and boundary conditions.

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Meshless methods are a class of numerical methods that are used to simulate in essentially every field of science, mathematics, and computational biology. It has been one of the hottest topics in computational mathematics in recent years, with an increasing number of scholars dedicating themselves to the study of meshfree methods, which have been suggested to solve various types of ODE and PDE. To solve PDE utilizing meshless methods with freely distributed collocations in the computational domain, and these collocation points participate to the approximation via assumed global or local basis functions. As contrary to most mesh-based methods, the spatial domain is represented by a set of nodes in meshless methods. As a result, there is no need for predetermined connectivity between the nodes. These methods solve the challenges of dimensionality. Meshless methods are efficient and produced better accuracy and can compute the solution in both regular and irregular computational domains. Meshless techniques based on radial basis functions have some limitations, the most significant of which is choosing the optimal shape-parameter value and dense ill-conditioned matrices. To avoid these weaknesses, researchers have introduced several techniques which makes these methods more efficient and accurate. These approaches have recently been tested in a variety of applications [8-19].

In this paper, generalized KdV7 equation is solved by meshless method of lines using radial basis functions. The numerical results are compared with the exact solutions.

Methodology for the general KdV7 equation

A radial basis function is a function that has an independent variable, *e.g.*:

$$\Phi_i\left(y\right)=\left(c^2+\left\|y-y_i\right\|\right)^{1/2}\ \text{(multiquadric)},\ \Phi_i\left(y\right)=\left(c^2+\left\|y-y_i\right\|\right)^{-1/2}\ \text{(inverse multiquadric)},$$

$$\text{and}\ \Phi_i\left(y\right)=\mathrm{e}^{-\alpha\left\|y-y_i\right\|^2}\ \text{(Gaussian)}$$

where c is appositive shape parameter.

According to the suggested methodology, we approximate the function $u(y, t)$, which is denoted by $u^N(y)$:

$$u^N\left(y\right)=\sum_{i=1}^{N}\lambda_i\xi_i=\mathbf{\Phi}^T\left(y\right)\lambda \tag{2}$$

where

$$\mathbf{\Phi}\left(y\right)=\left[\xi_1\left(y\right),\xi_2\left(y\right),...,\xi_N\left(y\right)\right]^T,\ \lambda=\left[\lambda_1,\lambda_2,...,\lambda_N\right]$$

Let

$$u^N\left(y_i\right)=u_i$$

Then

$$\boldsymbol{A}\lambda=\boldsymbol{u},\ \text{where}\ \boldsymbol{u}=\left[u_1,u_2,...,u_N\right]^T \tag{3}$$

$$\boldsymbol{A}=\begin{bmatrix}\mathbf{\Phi}^T\left(y_1\right)\\ \mathbf{\Phi}^T\left(y_2\right)\\ \cdots\\ \mathbf{\Phi}^T\left(y_N\right)\end{bmatrix}=\begin{bmatrix}\xi_1(y_1) & \xi_2(y_1) & \cdots & \xi_N(y_1)\\ \xi_1(y_2) & \xi_2(y_2) & \cdots & \xi_N(y_2)\\ \cdots & \cdots & \ddots & \cdots\\ \xi_1(y_N) & \xi_2(y_N) & \cdots & \xi_N(y_N)\end{bmatrix}$$

It follows from eqs. (2) and (3) that:

$$u^N\left(y\right)=\mathbf{\Phi}^T\left(y\right)\boldsymbol{A}^{-1}\boldsymbol{u}=\boldsymbol{W}(y)u,\ \text{where}\ \boldsymbol{W}(y)=\mathbf{\Phi}^T\left(y\right)\boldsymbol{A}^{-1} \tag{4}$$

Now consider the governing equation:

$$u_t + au^3u_y + bu_y^3 + cuu_yu_{2y} + du^2u_{3y} + eu_{2y}u_{3y} + fu_yu_{4y} + guu_{5y} + u_{7y} = 0 \tag{5}$$

The $y \in [\alpha, \beta]$ with initial condition:

$$u(y,t_0) = u^0(y) \tag{6}$$

and boundary conditions:

$$u(\alpha,t) = \mathrm{B}_1(t),\ u(\beta,t) = \mathrm{B}_2(t) \tag{7}$$

First, according to the MMOL, we discretize the space derivatives utilizing the RBF interpolation by choosing

$$\alpha = y_1 < y_2 < ... < y_{N-1} < y_N = \beta$$

we have:

$$u(y,t) \approx u^N(y,t) = \sum_{i=1}^{N} \mathbf{\Phi}^T(y)\boldsymbol{A}^{-1}u = \boldsymbol{W}(y)\boldsymbol{u} \tag{8}$$

where

$$\mathbf{\Phi}(y) = \left[\xi_1(y), \xi_2(y), \cdots, \xi_N(y)\right]^T,\ \boldsymbol{u} = \left[u_1(t), u_2(t), ..., u_N(t)\right]^T$$
$$\boldsymbol{W}(y) = \mathbf{\Phi}^T(y)\mathrm{A}^{-1} = \left[W(y_1), W(y_2), \cdots, W(y_N)\right]$$

Now utilizing eqs. (5)-(8), we get the following on collocation node y_i:

$$\begin{aligned}&\frac{\mathrm{d}u_i}{\mathrm{d}t} + au_i^3\left(\boldsymbol{W}_y(y_i)\boldsymbol{u}\right) + b\left(\boldsymbol{W}_y(y_i)\boldsymbol{u}\right)^3 + cu_i\left(\boldsymbol{W}_y(y_i)\boldsymbol{u}\right)\left(\boldsymbol{W}_{2y}(y_i)\boldsymbol{u}\right) + \\ &+ \mathrm{d}u_i^2\left(\boldsymbol{W}_{3y}(y_i)\boldsymbol{u}\right) + e\left(\boldsymbol{W}_{2y}(y_i)\boldsymbol{u}\right)\left(\boldsymbol{W}_{3y}(y_i)\boldsymbol{u}\right) + f\left(\boldsymbol{W}_y(y_i)\boldsymbol{u}\right)\left(\boldsymbol{W}_{4y}(y_i)\boldsymbol{u}\right) + \\ &+ gu_i\left(\boldsymbol{W}_{5y}(y_i)\boldsymbol{u}\right) + \left(\boldsymbol{W}_{7y}(y_i)\boldsymbol{u}\right) = 0,\ i = 1, 2, ..., N\end{aligned} \tag{9}$$

where for simplicity $u_i(t)$ is denoted by u_i.

Applying the collection the general KdV7 eq. (1) will take the form:

$$\begin{aligned}&\frac{\mathrm{d}U}{\mathrm{d}t} + a\boldsymbol{U}^3 * \left(\boldsymbol{H}_y * \boldsymbol{U}\right) + b\left(\boldsymbol{H}_y * \boldsymbol{U}\right)^3 + c\boldsymbol{U} * \left(\boldsymbol{H}_y * \boldsymbol{U}\right) * \left(\boldsymbol{H}_{2y} * \boldsymbol{U}\right) + \mathrm{d}\boldsymbol{U}^2 * \left(\boldsymbol{H}_{3y} * \boldsymbol{U}\right) + \\ &+ e\left(\boldsymbol{H}_{2y} * \boldsymbol{U}\right) * \left(\boldsymbol{H}_{3y} * \boldsymbol{U}\right) + f\left(\boldsymbol{H}_y * \boldsymbol{U}\right) * \left(\boldsymbol{H}_{4y} * \boldsymbol{U}\right) + g\boldsymbol{U} * \left(\boldsymbol{H}_{7y} * \boldsymbol{U}\right) + \left(\boldsymbol{H}_{7y} * \boldsymbol{U}\right) = 0\end{aligned} \tag{10}$$

where

$$\boldsymbol{U} = \left[u_1, u_2, ..., u_{N-1}, u_N\right]^T,\ \boldsymbol{H}_y = [W_{jy}(y_i)]_{N\times N},\ \boldsymbol{H}_{2y} = [W_{j2y}(y_i)]_{N\times N},\ \boldsymbol{H}_{3y} = [W_{j3y}(y_i)]_{N\times N}$$
$$\boldsymbol{H}_{4y} = [W_{j\,4y}(y_i)]_{N\times N},\ \boldsymbol{H}_{5y} = [W_{j\,5y}(y_i)]_{N\times N},\ \boldsymbol{H}_{7y} = [W_{j\,7y}(y_i)]_{N\times N}$$

the symbol * denote component-wise multiplication of two vectors. We write eq. (10) as:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = G(\boldsymbol{U}) \tag{11}$$

where

$$G(\boldsymbol{U}) = -\begin{pmatrix} a\boldsymbol{U}^3 * \left(\boldsymbol{H}_y * \boldsymbol{U}\right) + b\left(\boldsymbol{H}_y * \boldsymbol{U}\right)^3 + c\boldsymbol{U} * \left(\boldsymbol{H}_y * \boldsymbol{U}\right) * \left(\boldsymbol{H}_{2y} * \boldsymbol{U}\right) + \mathrm{d}\boldsymbol{U}^2 * \left(\boldsymbol{H}_{3y} * \boldsymbol{U}\right) + \\ + e\left(\boldsymbol{H}_{2y} * \boldsymbol{U}\right) * \left(\boldsymbol{H}_{3y} * \boldsymbol{U}\right) + f\left(\boldsymbol{H}_y * \boldsymbol{U}\right) * \left(\boldsymbol{H}_{4y} * \boldsymbol{U}\right) + g\boldsymbol{U} * \left(\boldsymbol{H}_{7y} * \boldsymbol{U}\right) + \left(\boldsymbol{H}_{7y} * \boldsymbol{U}\right) \end{pmatrix}$$

The given initial and boundary conditions:

$$\boldsymbol{U}(t_0)=\left[u^0(y_1),u^0(y_2),\cdots,u^0(y_{N-1}),u^0(y_N)\right]^T \tag{12}$$

$$u_1(t)=\mathrm{B}_1(t),\; u_N(t)=\mathrm{B}_2(t) \tag{13}$$

So far, the first step of the MMOL has been completed. Next, we will utilize the classical four-order Runge-Kutta scheme (RK4) to solve eqs. (11)-(13).

Numerical applications

The accuracy, efficiency and applicability of the proposed MMOL are verified by approximating the solution of the model eq. (1). Two test problems are considered with uniform nodes in the domain [–100, 100]. Throughout the paper, we have used dt = 0.01, k = 0.001. For accuracy measurement, we used the maximum error norm.

Tables 1 and 2 exhibit the numerical results for *Problems 1 and 2*, respectively. Both the tables revealed the efficiency and accuracy of the proposed MMOL.

Problem 1. Consider the general KdV7 equation:

$$u_t + 140u^3u_y + 70u_y^3 + 280uu_yu_{2y} + 70u^2u_{3y} + 70u_{2y}u_{3y} + 42u_yu_{4y} + 14uu_{5y} + u_{7y} = 0$$

which is known as Seventh-order Lax equation [5-7] with the exact solution:

$$u(y,t)=2k^2\sec h^2\left[k(y-64k^6t\right]$$

Table 1. Numerical results for *Problem 1*

t	N	c	RBF	Maximum error
0.05	9	0.4000	MQ	$6.1960\cdot10^{-18}$
		$1\cdot10^{-4}$	GA	$2.0117\cdot10^{-19}$
		$1\cdot10^{-68}$	IMQ	$2.1176\cdot10^{-21}$
	41	0.0310	MQ	$6.6940\cdot10^{-15}$
		1.8000	GA	$2.1176\cdot10^{-21}$
		$1\cdot10^{-69}$	IMQ	$6.3527\cdot10^{-21}$
0.5	9	0.4000	MQ	$6.1960\cdot10^{-17}$
		$1\cdot10^{-4}$	GA	$2.0117\cdot10^{-18}$
		$1\cdot10^{-68}$	IMQ	$2.1176\text{e}\cdot10^{-20}$
	41	0.0310	MQ	$6.6935\cdot10^{-14}$
		1.8000	GA	$2.1176\cdot10^{-20}$
		$1\cdot10^{-69}$	IMQ	$6.3527\cdot10^{-20}$
1.00	9	0.4000	MQ	$1.2392\cdot10^{-16}$
		$1\cdot10^{-4}$	GA	$4.0234\cdot10^{-18}$
		$1\cdot10^{-68}$	IMQ	$4.2352\cdot10^{-20}$
	41	0.0310	MQ	$1.3386\cdot10^{-13}$
		1.8000	GA	$4.2352\cdot10^{-20}$
		$1\cdot10^{-69}$	IMQ	$1.2705\cdot10^{-19}$

Problem 2. Consider the general KdV7 equation

$$u_t + 252u^3u_y + 63u_y^3 + 378uu_yu_{2y} + 126u^2u_{3y} + 63u_{2y}u_{3y} + 42u_yu_{4y} + 21uu_{5y} + u_{7y} = 0$$

which is known as Seventh-order Sawada-Kotera-Ito equation [5-7], and the exact solution is

$$u(y,0) = \frac{4k^2}{3}\left(2 - 3\tanh^2\left(k\left(y - \frac{256k^6}{3}t\right)\right)\right)$$

Table 2. Numerical results for *Problem 2*

t	N	c	RBF	Maximum error
0.05	9	0.71000	MQ	$9.7197 \cdot 10^{-18}$
		$3 \cdot 10^{-5}$	GA	$2.1176 \cdot 10^{-21}$
		$1 \cdot 10^{-68}$	IMQ	$4.2352 \cdot 10^{-21}$
	41	0.0400	MQ	$1.2867 \cdot 10^{-14}$
		1.8000	GA	$4.2352 \cdot 10^{-21}$
		$1 \cdot 10^{-69}$	IMQ	$1.0588 \cdot 10^{-20}$
0.5	9	0.7100	MQ	$9.7197 \cdot 10^{-18}$
		$3 \cdot 10^{-5}$	GA	$2.1176 \cdot 10^{-21}$
		$1 \cdot 10^{-68}$	IMQ	$4.2352 \cdot 10^{-21}$
	41	0.0400	MQ	$1.2866 \cdot 10^{-13}$
		1.8000	GA	$4.2352 \cdot 10^{-20}$
		$1 \cdot 10^{-69}$	IMQ	$1.0588 \cdot 10^{-19}$
1.00	9	0.7100	MQ	$1.0588 \cdot 10^{-19}$
		$3 \cdot 10^{-5}$	GA	$1.9439 \cdot 10^{-16}$
		$1 \cdot 10^{-68}$	IMQ	$4.2352 \cdot 10^{-20}$
	41	0.0400	MQ	$8.4703 \cdot 10^{-20}$
		1.8000	GA	$2.5731 \cdot 10^{-13}$
		$1 \cdot 10^{-69}$	IMQ	$8.4703 \cdot 10^{-20}$

Conclusion

In this study, we investigated the generalized seventh-order KdV7 equation using the meshless method of line based on radial basis functions as a powerful numerical method. The numerical results show that the inverse multiquadric RBF has the best accuracy among the three RBF in this method for the governing equations. Based on these findings, we propose that the proposed technique be used to non-linear partial differential equation models found in optics, fluid dynamics, plasma physics, optical fibers, chemical dynamics, and other fields.

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