REGULARITY FOR A NON-LOCAL DIFFUSION EQUATION WITH RIEMANN-LIOUVILLE DERIVATIVE

by

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Our main goal in this paper is to investigate the regularity of the mild solution fractional diffusion equation which can be used in the modelling of heat transfer with memory effects. Under some various assumptions of the input data, we obtain two main results. We also provide the upper bound and lower bound of the source function. The main tool is to use complex evaluations involving the Wright function.

Key words: *fractional diffusion equation, Riemman-Liouville, regularity*

Introduction

In the last decade, fractional calculus has been seen as a fundamental role in modelling a considerable number of natural phenomena. Fractional calculus with definitions related to integrals and derivatives has described many different types of operators with non-integer degrees [1, 2]. As far as we know, there are currently several definitions for fraction derivatives and fraction integrals, such as Riemann-Liouville, Caputo, Hadamard, Riesz, Griinwald-Letnikov, Marchaud, *etc*. Although most of them have been extensively studied, most mathematicians are interested and studied the two derivative Caputo derivative and Riemann-Liouville.

In this paper, we are interested to study the diffusion equation:

$$\begin{gathered}\partial_t v(t,x) = \partial_t^{1-\alpha}\Delta v(t,x) + \psi(t) f(x),\ (x,t)\in\Omega\times(0,T)\\ u(t,x)=0,\ x\in\partial\Omega, t\in(0,T]\end{gathered} \tag{1}$$

with $v(0, x) = 0$ for $x \in \Omega$ and the condition:

$$\int_0^T v(t,x)\mathrm{d}t = \theta(x) \tag{2}$$

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where $\partial_t^{1-\alpha}$ is called the Riemann-Liouville fractional derivative of order $1-\alpha \in (0, 1)$ which is defined by [2, 3]:

$$\partial_t^{1-\alpha} v(t,x) = \frac{1}{\Gamma(\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t (t-\tau)^{\alpha-1} v(\tau,x)\mathrm{d}\tau,\ t>0 \tag{3}$$

where $\Gamma(\cdot)$ is the Gamma function.

The attraction of the main eq. (1) comes from the presence of the fractional derivative which can help us to model some certain diffusion process efficiently. Particularly, we take the heat transfer process as an example. In the theory of the classical heat equation, the speed of the heat flow is infinite, however, in a material with thermal memory, the propagation speed can be finite. Therefore, the time-fractional diffusion equation has been proposed to provide a different approach for investigating anomalous diffusions with particle sticking and trapping phenomena, such as the aforementioned heat transfer process, for more details about the natural arising of time-fractional diffusion equation or other fractional models, we refer the reader to [4-11] and references therein. In our problem (1), we recall that the main equation has been investigated by several mathematicians. Thach *et al*. [12] studied a non-local problem for semilinear version of eq. (1). The authors applyied Banach fixed point theorem com- bined with some techniques on Mittag-Leffler functions to obtain the existence, uniqueness of the mild solution.

Wanga *et al.* [13], focus on the existence of solutions of fractional differential equations with non-local integral condition:

$$\begin{aligned} &D_{0^+}^{\alpha} u = F(t,u(t)),\ (x,t) \in (0,\pi)\times(0,T) \\ &t^{1-\alpha} u\,|_{t=0} = \lambda \int_0^T u(t)\mathrm{d}t + h,\ h \in \mathbb{R} \end{aligned} \tag{4}$$

where $\lambda \geq 0$. They obtained the existence of extremal solutions by using method of upper and lower solutions combined with monotone iterative technique. A gerenralized version of eq. (4) is investigated in [14] by using a monotone iterative technique by introducing upper and lower solutions.

Zhai and Jiang [15] considered the following non-local problem:

$$\begin{aligned} &D_{0^+}^{\alpha} u + F(t,v(t)) = a, t \in (0,T) \\ &D_{0^+}^{\beta} v + F(t,u(t)) = b, t \in (0,T) \\ &u(0) = 0,\ u(T) = \int_0^T \phi(t)u(t)\mathrm{d}t \\ &v(0) = 0,\ v(T) = \int_0^T \phi(t)v(t)\mathrm{d}t \end{aligned} \tag{5}$$

where $1 < \alpha, \beta < 2$. The authors established the existence and uniqueness of solutions for the new coupled system by using a fixed point theorem.

Our goal of this paper is to derive the regularity of mild solutions and the source funtion f of problems (1) and (2). We also provide the lower bound of the source function f. Our main approach is based on some evaluations on the Wright functions, the Mittag-Leffler functions and the relationship between them. Besides, we applied Parseval's equality to the Fourier series of functions in Hilbert scales spaces and some related techniques to deal with singular integrals.

Preliminaries

Consider the Mittag-Leffler function, which is defined:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)}$$

($z \in \mathbb{C}$), for $\alpha > 0$ and $\beta \in \mathbb{R}$. When $\beta = 1$, it is abbreviated as $E_\alpha(z) = E_{\alpha,1}(z)$. We have the following lemma which useful for next proof.

Lemma 1. Let $0 < \alpha < 1$. Then the function $z \mapsto E_{\alpha,\alpha}(z)$ no negative root. Moreover, there exists a constant $\overline{C}_\alpha$ such that:

$$0\le E_{\alpha,\alpha}(-z)\le\frac{\overline{C}_\alpha}{1+z},\ \ z>0 \tag{6}$$

Lemma 2. Let $0 < \alpha < 1$. Then there exists two constants C^- and C^+ such that:

$$\frac{C^-}{1+z}\le E_{\alpha,1}(-z)\le\frac{C^+}{1+z},\ \ z>0 \tag{7}$$

Definition 1. For positive number $r \geq 0$, we also define the Hilber scale space:

$$H^r(0,\pi)=\left\{w\in L^2(0,\pi):\sum_{j=1}^{\infty}j^{2r}\left\langle w,\sqrt{\frac{2}{\pi}}\sin(jx)\right\rangle^2<+\infty\right\} \tag{8}$$

with the following norm

$$\|u\|_{H^r(0,\pi)}=\left(\sum_{j=1}^{\infty}j^{2r}\left\langle w,\sqrt{\frac{2}{\pi}}\sin(jx)\right\rangle^2\right)^{1/2}$$

Lemma 3. The following equality holds, see [16]:

$$E_{\alpha,1}(-y)=\int_0^{\infty}M_\alpha(r)e^{-yr}dr,\ \text{ for }\ y\in\mathbb{C} \tag{9}$$

and

$$E_{\alpha,\alpha}(-y)=\int_0^{\infty}rM_\alpha(r)e^{-yr}dr,\ \text{ for }\ y\in\mathbb{C} \tag{10}$$

where the Wright function $M_\alpha(r)$ is defined:

$$M_\alpha(r):=\sum_{j=0}^{\infty}\frac{r^j}{j!\Gamma(-\alpha j+1-\alpha)},\ \ 0<\alpha<1 \tag{11}$$

In addition, we get the following fact that:

$$\mathbf{W}_\alpha(\theta)\ge 0,\ \text{ for }\theta>0\ \text{ and }\int_0^{\infty}\mathbf{W}_\alpha(\theta)d\theta=1 \tag{12}$$

Lemma 4. [16] For $\alpha \in (0, 1)$ and $b > -1$, the following properties hold:

$$\int_0^\infty \theta^b \mathbf{W}_\alpha(\theta)\mathrm{d}\theta = \frac{\Gamma(b+1)}{\Gamma(b\alpha+1)} \tag{13}$$

Lemma 5. [2] Let $0 < \alpha < 1$. Then for any $z > 0$, we have:

$$\frac{C_1}{1+z} \le E_{\alpha,1}(-z) \le \frac{C_2}{1+z} \tag{14}$$

for C_1, $C_2 > 0$.

Lemma 6. [2] Let $0 < \alpha < 1$ and λ, $a > 0$. Then:

$$\partial_t\left[E_\alpha\left(-\lambda t^\alpha\right)\right] = -\lambda t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda t^\alpha\right), \text{ for } t > 0$$

$$\partial_t\left[t^{\alpha-1}E_{\alpha,\alpha}\left(-\lambda t^\alpha\right)\right] = t^{\alpha-2}E_{\alpha,\alpha-1}\left(-\lambda t^\alpha\right), \text{ for } t > 0$$

Main results

Theorem 1. Let us assume that:

$$\overline{B}r^\varepsilon \le \left[\psi(r)\right] \le Br^\delta \tag{15}$$

where $\varepsilon > -1$ and $\delta > -1$. Also, we suppose:

$$\theta \in \mathbb{H}^{q+2m}(\Omega)$$

where $q \ge 0$ and $0 < m < \alpha/(2+\varepsilon)$. Then we get:

$$\|f\|_{\mathbb{H}^q(\Omega)} \le \left(\frac{1+\varepsilon}{\overline{B}\mathcal{D}(m,\alpha,\varepsilon)\int_0^1 M_\alpha(s)\mathrm{d}s}\right)\frac{1}{\left(1-\mathrm{e}^{-mT^{\alpha-1}}\right)^m}\|\theta\|_{\mathbb{H}^{q+2m}(\Omega)} \tag{16}$$

where

$$\mathcal{D}(m,\alpha,\varepsilon) = (\alpha m)^{-m}\left(\frac{m-1}{(2+\varepsilon)m-\alpha}\right)^{\frac{m-1}{m}} T^{\frac{(2+\varepsilon)m-\alpha}{m}}$$

In addition, if $\theta \in \mathbb{H}^{q-2\rho+2m}(\Omega)$ for $0 < \rho < 1$ then we have:

$$\|v(t,.)\|_{\mathbb{H}^q(\Omega)} \lesssim t^{1-\alpha\rho+\delta}\|\theta\|_{\mathbb{H}^{q-2\rho+2m}(\Omega)} \tag{17}$$

where the hidden constant depends on m, α, ε, ρ, δ, T.

Theorem 2. Let us assume that:

$$\overline{B}r^\varepsilon \le \left|\psi(r)\right| \le Br^\delta \tag{18}$$

where $\varepsilon > -1$ and $\delta > -1$. In addition, we suppose that:

$$\theta \in \mathbb{H}^{q+2m}(\Omega)$$

where $q \ge 0$ and $0 < \mathrm{m} < \alpha/(2+\varepsilon)$. Then we have:

$$\|f\|_{\mathbb{H}^q(\Omega)} \ge \frac{(2-\alpha\rho+\delta)}{M_1T^{2-\alpha\rho+\delta}}\|\theta\|_{\mathbb{H}^{q+2\rho}(\Omega)} \tag{19}$$

where

$$M_1 = \frac{BC_\rho \mathbf{B}(1-\alpha\rho, 1+\delta)\Gamma(1-\rho)}{\Gamma(1-\alpha\rho)}$$

Proof of Theorem 1

Let us assume that problem of eq. (1) has a unique solution v. Assume that v is given by Fourier series form:

$$v(t,x) = \sum_{k=1}^{\infty} \left\langle v(t,.),\ \mathrm{e}_k(x)\right\rangle \mathrm{e}_k(x) \tag{20}$$

From [17], we get the estimate:

$$\int_\Omega v(t,x)\mathrm{e}_k(x)\mathrm{d}x = E_{\alpha,1}\left(-k^2 t^\alpha\right)\int_\Omega v(0,x)\mathrm{e}_k(x)\mathrm{d}x + \left(\int_0^t E_{\alpha,1}\left(-k^2(t-r)^\alpha\right)\psi(r)\mathrm{d}r\right)\int_\Omega f(x)\mathrm{e}_k(x)\mathrm{d}x \tag{21}$$

Since $v(0, x) = 0$ and the fact that:

$$E_{\alpha,1}\left[-k^2(t-r)^\alpha\right] = \int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\mathrm{d}s$$

we know that

$$\int_\Omega v(t,x)e_k(x)\mathrm{d}x\left(\int_0^t E_{\alpha,1}\left(-k^2(t-r)^\alpha\right)\psi(r)\mathrm{d}r\right)\left(\int_\Omega f(x)e_k(x)\mathrm{d}x\right) =$$

$$= \left(\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\right)\left(\int_\Omega f(x)\mathrm{e}_k(x)\mathrm{d}x\right) \tag{22}$$

In addition, from the condition:

$$\int_0^T v(t,x)\mathrm{d}t = \theta(x)$$

we obtain

$$\int_0^T\int_\Omega v(t,x)e_k(x)\mathrm{d}x\mathrm{d}t = \int_\Omega \theta(x)\mathrm{e}_k(x)\mathrm{d}x$$

This implies that:

$$\left(\int_0^T\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\mathrm{d}t\right)\left(\int_\Omega f(x)\mathrm{e}_k(x)\mathrm{d}x\right) = \int_\Omega \theta(x)\mathrm{e}_k(x)\mathrm{d}x \tag{23}$$

Hence, we arrive:

$$\int_\Omega f(x)\mathrm{e}_k(x)\mathrm{d}x = \left(\int_0^T\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\mathrm{d}t\right)^{-1}\left(\int_\Omega \theta(x)\mathrm{e}_k(x)\ x\right) \tag{24}$$

and

$$\int_\Omega v(t,x)\mathrm{e}_k(x)\mathrm{d}x = \left(\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\right)\cdot$$

$$\cdot\left(\int_0^T\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\mathrm{d}t\right)^{-1}\left(\int_\Omega \theta(x)\mathrm{e}_k(x)\mathrm{d}x\right) \tag{25}$$

Step 1. Evaluation of

$$\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s$$

In view of the inequality

$$\mathrm{e}^{-z} \le C_\rho z^{-\rho}, \quad \rho > 0$$

we find that

$$\mathrm{e}^{-k^2 s(t-r)^\alpha} \le C_\rho k^{-2\rho}s^{-\rho}(t-r)^{-\alpha\rho}$$

Since $\psi(r)$ Br^δ, we know that:

$$\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s \le BC_\rho k^{-2\rho}\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)s^{-\rho}(t-r)^{-\alpha\rho}r^\delta\mathrm{d}r\mathrm{d}s \tag{26}$$

Since $\rho < 1$, it is obvious to see that:

$$\int_0^\infty \mathbf{M}_\alpha(s)s^{-\rho}\mathrm{d}s = \frac{\Gamma(1-\rho)}{\Gamma(1-\alpha\rho)}$$

where we used *Lemma 4*. In addition, since $\delta > -1$ and $\alpha\rho < 1$, we also have:

$$\int_0^t (t-r)^{-\alpha\rho}r^\delta\mathrm{d}r = t^{1-\alpha\rho+\delta}B(1-\alpha\rho, 1+\delta) \tag{27}$$

From some previous observations, we derive that:

$$\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s \le \mathbf{M}_1 k^{-2\rho}t^{1-\alpha\rho+\delta} \tag{28}$$

where we recall $\rho < 1$ and

$$\mathbf{M}_1 = \frac{BC_\rho B(1-\alpha\rho, 1+\delta)\Gamma(1-\rho)}{\Gamma(1-\alpha\rho)}$$

Step 2. Evaluation of

$$\int_0^T\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\mathrm{d}t$$

First, since the fact that $\psi(r) \ge Dr^\varepsilon$ and $\mathbf{M}_\alpha(s) > 0$, for any $s > 0$, we have the observation:

$$\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s \geq \int_0^t\int_0^1 \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s \geq$$

$$\geq \overline{B}\int_0^t\int_0^1 \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2s(t-r)^\alpha}r^\varepsilon\mathrm{d}r\mathrm{d}s \tag{29}$$

Noting that:

$$\mathrm{e}^{-k^2s(t-r)^\alpha} \geq \mathrm{e}^{-k^2(t-r)^\alpha},\ 0<s<1$$

we get the following estimate

$$\int_0^t\int_0^1 \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2s(t-r)^\alpha}r^\varepsilon\mathrm{d}r\mathrm{d}s \geq \overline{B}\int^t\int_0^1 \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2(t-r)^\alpha}r^\varepsilon\mathrm{d}r\mathrm{d}s = \overline{B}\left(\int_0^1 \mathbf{M}_\alpha(s)\mathrm{d}s\right)\cdot$$

$$\cdot\left(\int_0^t \mathrm{e}^{-k^2(t-r)^\alpha}r^\varepsilon\mathrm{d}r\right) \geq \overline{B}\left(\int_0^1 \mathbf{M}_\alpha(s)\mathrm{d}s\right)\left(\int_0^t \mathrm{e}^{-k^2t^\alpha}r^\varepsilon\mathrm{d}r\int_0^t\right) = \overline{B}\left(\int_0^1 \mathbf{M}_\alpha(s)\mathrm{d}s\right)\frac{\mathrm{e}^{-k^2t^\alpha}t^{\varepsilon+1}}{1+\varepsilon} \tag{30}$$

This implies immediately that:

$$\int_0^T\int_0^t\int_0^1 \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2s(t-r)^\alpha}r^\varepsilon\mathrm{d}r\mathrm{d}s\mathrm{d}t \geq \overline{B}\frac{\int_0^1 \mathbf{M}_\alpha(s)\mathrm{d}s}{1+\varepsilon}\int_0^T \mathrm{e}^{-k^2t^\alpha}t^{\varepsilon+1}\mathrm{d}t \tag{31}$$

Our next purpose is to give the lower bound the integral term on the right above. This is probably no trivial task. For the convenience of the reader, we introduce the Holder-form inequality with another version as follows.

Lemma 7. (Holder's inequality for negative exponents, see [18]. Let $k' < 0$, and $k \in \mathbb{R}$ be such that $(1/k') + (1/k) = 1$. For:

$$f(x) > 0, g(x) \geq 0, \forall x \in \Omega$$

are Lebesgue measurable functions. Then:

$$\int_\Omega fg\mathrm{d}x \geq \left(\int_\Omega |f|^{k'}\mathrm{d}x\right)^{1/k'}\left(\int_\Omega |g|^k\mathrm{d}x\right)^{1/k}$$

Let us choose

$$0 < m < \frac{\alpha}{2+\varepsilon}.\ Let\ b = \frac{\alpha-1}{m}\ \text{and}\ m' = \frac{m}{m-1} < 0$$

We have:

$$\int_0^T \mathrm{e}^{-k^2t^\alpha}t^{\varepsilon+1}\mathrm{d}t = \int_0^T \mathrm{e}^{-k^2t^\alpha}t^b t^{\varepsilon+1-b}\mathrm{d}t \geq \left(\int_0^T \left(\mathrm{e}^{-k^2t^\alpha}t^b\right)^m\mathrm{d}t\right)^{1/m}\left(\int_0^T \left(t^{1+\varepsilon-b}\right)^{m'}\mathrm{d}t\right)^{1/m'} \tag{32}$$

It is obvious to see that:

$$\int_0^T \left(\mathrm{e}^{-k^2t^\alpha}t^b\right)^m\mathrm{d}t = \int_0^T \mathrm{e}^{-mk^2t^\alpha}t^{\alpha-1}\mathrm{d}t \tag{33}$$

Set the variable $z = t^{\alpha-1}$, we have $\mathrm{d}z = \alpha t^{\alpha-1}\,\mathrm{d}t$:

$$\int_0^T \mathrm{e}^{-mk^2 t^\alpha} t^{\alpha-1}\mathrm{d}t = \frac{1}{\alpha}\int_0^{T^{\alpha-1}} \mathrm{e}^{-mk^2 z}\mathrm{d}z = \frac{1-\mathrm{e}^{-mk^2 T^{\alpha-1}}}{\alpha m k^2} \tag{34}$$

Thus we get that:

$$\left(\int_0^T \left(\mathrm{e}^{-k^2 t^\alpha} t^b\right)^m \mathrm{d}t\right)^{1/m} = \left(\frac{1-\mathrm{e}^{-mk^2 T^{\alpha-1}}}{\alpha m k^2}\right)^m = (\alpha m)^{-m} k^{-2m}\left(1-\mathrm{e}^{-mk^2 T^{\alpha-1}}\right)^m \tag{35}$$

Moreover, it is easy to see that:

$$\int_0^T \left(t^{1+\varepsilon-b}\right)^{m'} \mathrm{d}t = \frac{T^{(1+\varepsilon-b)m'+1}}{(1+\varepsilon-b)m'+1} \tag{36}$$

and we can derive that:

$$(1+\varepsilon-b)m'+1 = (1+\varepsilon-b)\frac{m}{m-1}+1 = \frac{(2+\varepsilon)m-\alpha}{m-1} \tag{37}$$

Since the fact that $0 < m < \alpha/(2+\varepsilon)$, we have immediately that $[(2+\varepsilon)m-\alpha]/(m-1) > 0$. Thus, we obtain:

$$\left(\int_0^T \left(t^{1+\varepsilon-b}\right)^{m'} \mathrm{d}t\right)^{\frac{1}{m'}} = \left(\frac{m-1}{(2+\varepsilon)m-\alpha}\right)^{\frac{m-1}{m}} T^{\frac{(2+\varepsilon)m-\alpha}{m}} \tag{38}$$

Combining eqs. (32), (35), and (38), we get that:

$$\int_0^T \mathrm{e}^{-k^2 t^\alpha} t^{\varepsilon+1}\mathrm{d}t \ge \mathcal{D}(m,\alpha,\varepsilon) k^{-2m}\left(1-\mathrm{e}^{-mk^2 T^{\alpha-1}}\right)^m \tag{39}$$

where

$$\mathcal{D}(m,\alpha,\varepsilon) = (\alpha m)^{-m}\left(\frac{m-1}{(2+\varepsilon)m-\alpha}\right)^{\frac{m-1}{m}} T^{\frac{(2+\varepsilon)m-\alpha}{m}}$$

The latter estimate together with eq. (31) allows us to obtain:

$$\int_0^T\int_0^t\int_0^1 \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha} r^\varepsilon \mathrm{d}r\mathrm{d}s\mathrm{d}t \ge \frac{\overline{B}\mathcal{D}(m,\alpha,\varepsilon)\int_0^1 \mathbf{M}_\alpha(s)\mathrm{d}s}{1+\varepsilon} k^{-2m}\left(1-\mathrm{e}^{-mk^2 T^{\alpha-1}}\right)^m \tag{40}$$

Let us claim the upper bound of f. Indeed, from the definition of f as in eq. (24), we use Parseval's equality to obtain:

$$\|f\|^2_{\mathbb{H}^q(\Omega)} = \sum_{k=1}^\infty k^{2q}\left(\int_0^T\int_0^t\int^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2 s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\mathrm{d}t\right)^{-2}\left(\int_\Omega \theta(x)\mathrm{e}_k(x)\mathrm{d}x\right)^2 \le$$

$$\le \left(\frac{1+\varepsilon}{\overline{B}\mathcal{D}(m,\alpha,\varepsilon)\int_0^1 \mathbf{M}_\alpha(s)ds}\right)^2 \frac{k^{2q+4m}}{\left(1-e^{-mk^2 T^{\alpha-1}}\right)^{2m}}\left(\int_\Omega \theta(x)e_k(x)dx\right)^2 \tag{41}$$

Since $k \geq 1$, it is not difficult to verify that the inequality:

$$\frac{1}{\left(1-\mathrm{e}^{-mk^2T^{\alpha-1}}\right)^{2m}} \leq \frac{1}{\left(1-\mathrm{e}^{-mT^{\alpha-1}}\right)^{2m}}$$

From two previous estimates, we deduce that the upper bound for the source term f:

$$\| f \|_{\mathbb{H}^q(\Omega)} \leq \left(\frac{1+\varepsilon}{\overline{B}\mathcal{D}(m,\alpha,\varepsilon)\int_0^1 \mathbf{M}_\alpha(s)ds}\right)\frac{1}{\left(1-e^{-mT^{\alpha-1}}\right)^m}\| \theta \|_{\mathbb{H}^{q+2m}(\Omega)} \tag{42}$$

Let us claim the upper bound of v. Since the definition of eq. (25), we have that:

$$v(t,.)\|^2_{\mathbb{H}^q(\Omega)} = \sum_{k=1}^{\infty} k^{2q}\left(\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\right)^2 \cdot$$

$$\cdot\left(\int_0^T\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\mathrm{d}t\right)^{-2}\left(\int_\Omega \theta(x)\mathrm{e}_k(x)\mathrm{d}x\right)^2 \tag{43}$$

In view of eqs. (28) and (40), we derive that:

$$\| v(t,.) \|^2_{\mathbb{H}^q(\Omega)} \leq | M_1 |^2 \left(\frac{1+\varepsilon}{\overline{B}\mathcal{D}(m,\alpha,\varepsilon)\int_0^1 \mathbf{M}_\alpha(s)ds}\right)^2 \frac{1}{\left(1-e^{-mT^{\alpha-1}}\right)^{2m}} t^{2-2\alpha\rho+2\delta} \cdot$$

$$\cdot\sum_{k=1}^{\infty} k^{2q-4\rho+4m}\left(\int_\Omega \theta(x)\mathrm{e}_k(x)\mathrm{d}x\right)^2 \tag{44}$$

where $0 < \rho < 1$. Hence, from this observation, we can deduce:

$$\| v(t,.) \|_{\mathbb{H}^q(\Omega)} \lesssim t^{1-\alpha\rho+\delta} \| \theta \|_{\mathbb{H}^{q-2\rho+2m}(\Omega)} \tag{45}$$

where the hidden constant depends on m, α, ε, ρ, δ, T.

Proof of Theorem 2

Let us recall:

$$\|f\|^2_{\mathbb{H}^q(\Omega)} = \sum_{k=1}^{\infty} k^{2q}\left(\int_0^T\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\mathrm{d}t\right)^{-2}\left(\int_\Omega \theta(x)\mathrm{e}_k(x)\mathrm{d}x\right)^2 \tag{46}$$

In view of eq. (28), we find that for any $0 < \rho < 1$:

$$\int_0^T\int_0^t\int_0^\infty \mathbf{M}_\alpha(s)\mathrm{e}^{-k^2s(t-r)^\alpha}\psi(r)\mathrm{d}r\mathrm{d}s\mathrm{d}t \leq M_1 k^{-2\rho}\int_0^T t^{1-\alpha\rho+\delta}\mathrm{d}t = \frac{M_1}{2-\alpha\rho+\delta}T^{2-\alpha\rho+\delta}k^{-2\rho} \tag{47}$$

From two previous observations, we can deduce:

$$\|f\|_{\mathbb{H}^q(\Omega)}^2 \geq \frac{(2-\alpha\rho+\delta)^2}{M_1^2 T^{4-2\alpha\rho+2\delta}} \sum_{k=1}^{\infty} k^{2q+4\rho} \left(\int_\Omega \theta(x) \mathrm{e}_k(x) \mathrm{d}x \right)^2 \tag{48}$$

This implies that:

$$\|f\|_{\mathbb{H}^q(\Omega)} \geq \frac{(2-\alpha\rho+\delta)}{M_1 T^{2-\alpha\rho+\delta}} \| \theta \|_{\mathbb{H}^{q+2\rho}(\Omega)} \tag{49}$$

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