A NEW APPROACH TO FRACTIONAL DIFFERENTIAL EQUATIONS

by

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In this work, we define fractional derivative of order $\varsigma > 0$, with no restrictions on the domain of the function, and give its geometry. We derive some rules and properties for the proposed new approach and show that if fractional order converges to an integer order, then each rule converges to the corresponding rule of derivative under this integer. On applications side we show that it has ability to convert various type of FDE to ODE and vice versa. Finally, we solve several FDE given in literature through the new approach.

Key words: *mean function, fractional derivative, ODE, FDE*

Introduction

Fractional calculus has got much attention in last few decades due to its significant applications in various field of science and technology. Infact the evidentiary properties and memory characteristics of various materials and process can be excellently described via fractional order derivatives and integrals as compared to integer order. Therefore, researchers of science and technology who are using the concept of mathematical modelling increasing used fractional calculus instead of classical calculus. For some real world applications in various disciplines of science including physics, biology, dynamics and engineering field we refer the reader some sources as [1-5]. Keeping in mind the importance of fractional calculus, researchers have given more attention on the area and developed various kinds of results. The area devoted to existence theory of solutions has been increasingly developed, see few articles [6-8]. Also the areas of numerical analysis and optimization of FDE have been developed very well in past few years, see [9, 10]. Historically in 1965, L'Hospital asked a question about the fractional derivative:

$$f^{(1/2)}(x) = \frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} f(x)$$

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Here it is remarkable that fractional derivative has not a unique definition and can be expressed by numbers of ways. The first notable definition was given by Reimann, detail see [11]:

$$f^{(\varsigma)}(x)=\frac{\mathrm{d}^{\varsigma}}{\mathrm{d}x^{\varsigma}}f(x),\ \text{ text for }\ \varsigma\in[n-1,n)$$

and defined

$$f^{(\varsigma)}(x)=\frac{1}{\Gamma(n-\varsigma)}\frac{\mathrm{d}^{n}}{\mathrm{d}x^{n}}\int_{a}^{x}\frac{f(u)\mathrm{d}u}{(x-u)^{\varsigma-n+1}}$$

for any real number a. The said definition was slightly modified by Caputo in 1967 as:

$$f^{(\varsigma)}(x)=\frac{1}{\Gamma(n-\varsigma)}\int_{a}^{x}\frac{f^{(n)}(u)\mathrm{d}u}{(x-u)^{\varsigma-n+1}}$$

Further the Conformable fraction derivative [12] is defined:

$$f^{(\varsigma)}(x)=\lim_{h\to 0}\frac{f\left(x+hx^{1-\varsigma}\right)-f\left(x\right)}{h}$$

for all $x < 0$.

Recently Caputo and Fabrizio [13] defined a new operator involves non-singular kernel which has got much attention. The said operator can be described for any function $f\in H^1(a, b)$, $b > a$, $\varsigma \in [0, 1]$:

$$D_x^{\varsigma}\left[f(x)\right]=\frac{M(\varsigma)}{1-\varsigma}\int_{\varsigma}^{x}f'(u)\mathrm{e}^{-\varsigma\frac{x-u}{1-\varsigma}}\mathrm{d}u$$

where $M(\varsigma)$ is a normalizing function satisfies $M(0) = M(1) = 1$. In same line Atangana and Baleanu [14] generalized the definition of Caputo Fabrizio to the following more general definition:

$${}_{\varsigma}^{ABR}D_x^{\varsigma}\left[f(x)\right]=\frac{B(\varsigma)}{1-\varsigma}\int_{\varsigma}^{x}f(u)E_{\varsigma}\left[-\varsigma\frac{(x-u)^{\varsigma}}{1-\varsigma}\right]\mathrm{d}u$$

where $B(\varsigma)$ is the normalization function. Keeping in mind the aforementioned definitions researchers have developed large number of ways to handle FDE for numerical, analytical and qualitative results. In this regards plenty of work has been done. Now instead to deal FDE a sophisticated rule is needed to handle the aforesaid problems in more easy and simple ways. Since classical differential equations are less complicated as compared to FDE for exact or analytical and numerical solution. Because in most cases it is time consuming and expensive to memory. Therefore, a formula or tool is demanding to convert FDE to ODE to save time and waste of memory. Therefore, motivated from the aforementioned literature and many applications of fractional derivatives, in current work, we define a fractional derivative $g^{(\varsigma)}(x)$ of $\varsigma \in [n - 1, n]$ order, where n is a natural number, of an n order differentiable function $g(x)$ and give its geometry. We find that there is no restriction on the domain set and the graph of $g^{(\varsigma)}(x)$ lies inside the graphs of $g^{(n-1)}(x)$ and $g^{(n)}(x)$ if $\varsigma \in [n - 1, n]$. Next, we investigate some rules and properties and show that it has ability to convert any type of FDE to ODE and vice versa. In last, we solve several examples of FDE given in literature with the help of new approach of fractional derivatives.

Main definition, its rules and properties

In this section we enrich some properties and rules of the fractional derivative.

Main definition, rules, properties and application

In this subsection we give our main definition. We argue that if $n - 1 \leq \varsigma \leq n$, for all natural numbers n, then the graph of the fractional derivative $y^{(\varsigma)} = f^{(\varsigma)}(x)$ should lie inside the graph of $f^{(n-1)}(x)$ and $f^{(n)}(x)$ for all n.

Using the previous argument, we define the fractional derivative of a function $y = f(x)$ of order $\varsigma \in [n - 1, n]$, where n is any non-negative integer:

$$f^{(\varsigma)}(x) = (\varsigma - n + 1) f^{(n)}(x) + (n - \varsigma) f^{(n-1)}(x), \quad x \in (-\infty, \infty) \tag{1}$$

with convention that $f^{(0)}(x) = f(x)$.

In particular, when $\varsigma \in [0, 1]$, eq. (1) becomes:

$$f^{(\varsigma)}(x) = \varsigma f^{(1)}(x) + (1-\varsigma) f(x) = \lim_{h\to 0} \frac{\varsigma f(x+h) - [\varsigma - (1-\varsigma)h] f(x)}{h} \tag{2}$$

Geometry of the fractional derivative (2) is given in fig. 1.

From eq. (1) and the figure, we found that as $\varsigma \to n$ then $f^{(\varsigma)}(x) \to f^{(n)}(x)$ while as $\varsigma \to n-1$ then $f^{(\varsigma)}(x) \to f^{(n-1)}(x)$. Thus, $f^{(\varsigma)}(x)$ lies inside $f^{(n-1)}(x)$ and $f^{(n)}(x)$ for each non-negative integer n and each x. Our definition shows that if $f(x)$ is non-zero constant function then the fractional derivative $f^{(\varsigma)}(x)$ is not zero for $\varsigma \in (0, 1)$ and zero for all $\varsigma \in (n - 1, n)$ for each natural number $n > 1$.

Definition 1. A function $f(x)$ is differentiable of fraction order ς, $n - 1 \leq \varsigma \leq n$, n is any non-negative integer, if it can be expressed as (1).

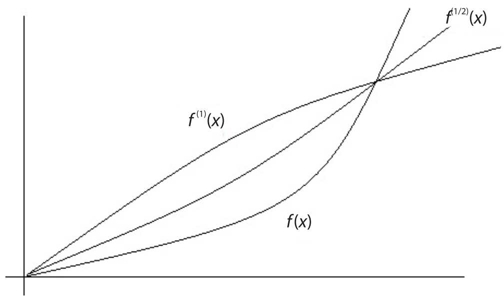

Figure 1. Sketch of the fractional derivative $f^{(\varsigma)}(x)$ for $\varsigma = 1/2$ of the differentiable function $f(x)$ in interval [0, 1]

Rules and properties of the proposed fractional derivative

In this section, we study some rules and properties satisfied by the proposed fractional derivative (1):

– It is linear, that is

$$(f(x) + g(x))^{(\varsigma)} = f^{(\varsigma)}(x) + g^{(\varsigma)}(x)$$

for all real numbers ς.

– When $f(x)$ is non-zero constant and $\varsigma \in [0, 1]$ then $f^{(\varsigma)}(x) = (1 - \varsigma)f(x)$. We see that $f^{(\varsigma)}(x)$ approaches to zero as ς approaches to one. For the case $\varsigma \in (0, 1)$, the fractional derivative $f^{(\varsigma)}(x)$ is not zero, reason is that it lies inside the region bounded by $f^{(1)}(x)$ and $f(x)$. Note that there is no result which shows fractional derivative of a non-zero constant must be zero. In this case, the rule of fraction derivative for a non-zero constant approaches to the rule of non-fractional derivative as the fraction ς approaches to a natural number.

– Let $f(x) = x^r$ for any non-zero real number r and $\varsigma \in [0, 1]$ then $f^{(\varsigma)}(x) = \varsigma r x^{r-1} + (1 - \varsigma)x^r$. Here we find that $f^{(\varsigma)}(x)$ approaches to rx^{r-1} as $\varsigma \to 1$ while $f^{(\varsigma)}(x)$ approaches to

$f^{(0)}(x) = f(x) = x^r$ as $\varsigma \to 0$. In this case, we also find that the rule of fraction derivative for a polynomial approaches to the rule of non-fractional derivative as the fraction ς approaches to a natural number.

– For $\varsigma \in [0, 1]$, the product rule is given as

$$(fg)^{\varsigma}(x) = \varsigma\left(f^{(1)}g + g^{(1)}f\right)(x) + (1-\varsigma)(fg)(x)$$

for all differentiable functions $f(x)$ and $g(x)$. Similar arguments like in *Property 3*, can be given for the product rule of fractional derivative.

– For $\varsigma \in [0, 1]$, the quotient rule is given:

$$\left(\frac{f}{g}\right)^{\varsigma}(x) = \varsigma\left[\frac{\left(f^{(1)}g + g^{(1)}f\right)(x)}{g^2(x)}\right] + (1-\varsigma)\left(\frac{f}{g}\right)(x)$$

with similar arguments like in *Property 4* for quotient rule under fractional derivative.

Now, we are ready to come to the following continuity and differentiability results:

Theorem 1. If the derivative of $g^{(n)}(x)$, for each natural number n exists, then $g(x)$ is differentiable of order ς, for all positive real number $\varsigma < n$.

Proof 1. Existence of the derivative $g^{(n)}(x)$ implies continuity of $g^{(n-1)}(x)$. Using this and definition (1) we get the required result.

In the following result, we discuss the differentiability of the fractional derivative $g^{(\varsigma)}(x)$, for all $\varsigma \in [0, 1]$.

Theorem 2. If $g^{(1)}(x)$ is differentiable, that is $g^{(2)}(x)$ exists, then $g^{(\varsigma)}(x)$ is differentiable.

Proof 2. From eq. (2), we write:

$$\left(g^{(\varsigma)}\right)^{(1)}(x) = \varsigma g^{(2)}(x) + (1-\varsigma)g^{(1)}(x) \tag{3}$$

Result follows from the latter expression.

In the next result, we discuss the continuity of a function and its fractional derivative.

Theorem 3. If $g^{(1)}(x)$ exists then both $g(x)$ and $g^{(\varsigma)}$ are continuous for all $\varsigma \in [0, 1]$.

Proof 3. Existence of $g^{(1)}(x)$ implies continuity of $g(x)$ then, by (2), $g^{(\varsigma)}(x)$ is continuous for all $\varsigma \in [0, 1]$.

Next, we are ready to discuss the convexity under fractional derivative.

Theorem 4. If a function $g(x)$ and its first order derivative $g^{(1)}(x)$ increase then so is $g^{(\varsigma)}(x)$ for all $\varsigma \in [0, 1]$. Similarly, if these functions are convex then $g^{(\varsigma)}(x)$ is convex of all $\varsigma \in [0, 1]$.

Proof 4. Since both $g(x)$ and $g^{(1)}(x)$ are increasing so their linear combination with positive coefficients must increases. This gives the first part. Thus using expression (2) we get the $g^{(\varsigma)}(x)$ increases for all ς.

For the second part, as both $g(x)$ and $g^{(1)}(x)$ are convex so is their linear combination with positive coefficients. Thus $g(x)$ is convex of order ς.

Theorem 5. Let $\varsigma \in [0, 1]$ and $g(x)$ is ς differentiable then:

$$\left(g^{(\varsigma)}\right)^{(1)}(x) = g^{(\varsigma+1)}(x) = \left(g^{(1)}\right)^{(\varsigma)}(x) \tag{4}$$

Proof 5. If $\varsigma \in [0, 1]$ then $\varsigma + 1 \in [1, 2]$. Thus, using (2), we can express the right hand side of eq. (4):

$$\begin{aligned} g^{(\varsigma+1)}(x) &= (\varsigma+1-2+1)g^{(2)}(x)+(2-\varsigma-1)g^{(1)}(x)= \\ &= \varsigma g^{(2)}(x)+(1-\varsigma)g^{(1)}(x)= \\ &= \left[\varsigma g^{(1)}+(1-\varsigma)g\right]^{(1)}(x)= \\ &= \left(g^{(\varsigma)}\right)^{(1)}(x) \end{aligned} \tag{5}$$

which is the same as the left hand side of eq. (4).

Let's come to the following commutative property.

Lemma 1. The fractional derivative, defined as (1), satisfies the commutative property as:

$$\left(g^{(\alpha)}\right)^{(\beta)}(x)=\left(g^{(\beta)}\right)^{(\alpha)}(x) \tag{6}$$

for all positive real numbers α, β.

Proof 6. For simplicity, we choose $\alpha, \beta \in [0, 1]$ and express:

$$\begin{aligned} \left(g^{(\alpha)}\right)^{(\beta)}(x) &= \left[\alpha g^{(1)}+(1-\alpha)g\right]^{(\beta)}= \\ &= \left(\alpha g^{(1)}\right)^{(\beta)}+\left[(1-\alpha)g\right]^{(\beta)}= \\ &= \beta\alpha g^{(2)}+\alpha(1-\beta)g^{(1)}+\beta(1-\alpha)g^{(1)}+(1-\alpha)(1-\beta)g \end{aligned} \tag{7}$$

where we have used the linearity rule (*i*) and the product rule *(iv)*.

Similarly, we express the right side of eq. (6):

$$\left(g^{(\beta)}\right)^{(\alpha)}(x)=\alpha\beta g^{(2)}+(1-\alpha)\beta g^{(1)}+\alpha(1-\beta)g^{(1)}+(1-\alpha)(1-\beta)g \tag{8}$$

Comparing (7) and (8) we complete the proof.

Applications of new approach

In this section we show the proposed fractional derivative (1) converts any FDE to an ODE and vice versa and then convert and solve some well-known FDE given in literature.

Theorem 6. The proposed fractional derivative expresses the FDE:

$$a_n(x,y)\frac{d^\alpha y}{dx^\alpha}+a_{n-1}(x,y)\frac{d^\beta y}{dx^\beta}+\ldots+a_1(x,y)\frac{d^\gamma y}{dx^\gamma}+a_0(x,y)=b_0(x,y) \tag{9}$$

where $\alpha \in [n-1, n]$, $\beta \in [n-2, n-1], \ldots, y = \in [0, 1]$ with initial/boundry conditions of the form $y^{(\alpha)}(c) = d$ to ODE:

$$\begin{aligned} &a_n(x,y)(\alpha-n+1)\frac{d^n y}{dx^n}+(a_n(x,y)(n-\alpha)+a_{n-1}(\beta-n+2))\frac{d^{n-1}y}{dx^{n-1}}+\ldots+ \\ &+a_1(x,y)(1-\gamma)y+a_0(x,y)=b_0(x,y) \end{aligned} \tag{10}$$

with initial/boundry conditions of the form

$$(\alpha-n+1)y^{(n)}(c)+(n-\alpha)y^{(n-1)}(c)=d$$

Proof 7. Applying (1) on the fractional derivatives in (9) we express:

$$a_n(x,y)\left[(\alpha-n+1)\frac{d^n y}{dx^n}+(n-\alpha)\frac{d^{n-1}y}{dx^{n-1}}\right]+a_{n-1}(x,y)\left[\beta-(n-1)+1\right]\frac{d^{n-1}y}{dx^{n-1}}+$$

$$+(n-1-\beta)\frac{d^{n-2}y}{dx^{n-2}}]+...+a_1(x,y)\left[(\gamma-1+1)\frac{dy}{dx}+(1-\gamma)y\right]+a_0(x,y)=b_0(x,y) \quad (11)$$

while the initial/boundry condition as

$$(\alpha-n+1)y^{(n)}(c)+(n-\alpha)y^{(n-1)}(c)=d$$

Combining the coefficients we get the result.

Now we are ready to solve and verify the following FDE given in Khalil *et al.* [15].

Example 1. Solve the following fractional differential equation:

$$y^{(1/2)}+y=x^{1/2}+2x^{3/2},\ \ y(0)=0 \quad (12)$$

where $y^{(1/2)}$ is the fractional derivative of y of order 1/2.

Proof 8. Using eq. (2) in (12) we express the given equation:

$$\frac{1}{2}y^{(1)}+\frac{1}{2}y+y=x^{1/2}+2x^{3/2}$$

which further reduces

$$y^{(1)}+3y=2x^{1/2}+4x^{3/2}$$

The latter equation is a non-fractional ODE having integrating factor e^{3x}. Using this, we write:

$$\frac{d}{dx}\left(ye^{3x}\right)=2x^{1/2}e^{3x}+4x^{3/2}e^{3x}$$

Integrating both sides we obtain:

$$y=\frac{4}{3}x^{3/2}+Ae^{-3x}$$

where A is the constant of integration.

Using the initial condition $y(0) = 0$, we get the final solution as $y = (4/3)x^{3/2}$.

To verify our solution, we consider the left hand side of eq. (12) and using eq. (2) we express:

$$y^{(1/2)}+y=\left(\frac{4}{3}x^{3/2}\right)^{(1/2)}+\frac{4}{3}x^{3/2}=$$

$$=\frac{1}{2}\left(\frac{4}{3}x^{3/2}\right)^{(1)}+\frac{1}{2}\left(\frac{4}{3}x^{3/2}\right)+\frac{4}{3}x^{3/2}=$$

$$=x^{1/2}+\frac{2}{3}x^{3/2}+\frac{4}{3}x^{3/2}=$$

$$=x^{1/2}+2x^{3/2}$$

this is the same as the right hand side of (12).

Example 2. Let's convert the following well known FDE given in [14-17] and solve:

$$\frac{d^\alpha}{dx^\alpha}y=\alpha y,\ \ \alpha\in(0,1) \quad (13)$$

to an ODE.

As $\alpha \in (0, 1)$ we choose $n = 1$ in (1), then we express the left hand side of eq. (13):

$$\alpha \frac{d^1}{dx^1} y + (1-\alpha) y = \alpha y$$

which further reduces to a homogeneous linear ODE

$$\alpha \frac{d}{dx} y + (1-2\alpha) y = 0$$

with integrating factor $e^{[(1-2\alpha)/\alpha]x}$. Using the integrating factor, one easily obtain solution of the FDE (13) as $y = Ae^{[(2\alpha-1)/\alpha]x}$, where A is the constant of integration.

Let's come to *Examples 4.1, 4.2, 4.3, and 4.4* of Khalil *et al*. [12]. We convert all these fractional order examples to ODE.

Example 3.

$$y^{(0.5)} + y = x^2 + 2x^{3/2}, \ \ y(0) = 0$$

As $0.5 \in [0, 1]$ therefore, applying (1) with $n = 1$, we get $y^{(0.5)} = 0.5y^{(1)} + 0.5y$. Inserting this in the given FDE we get the corresponding ODE:

$$\frac{1}{2} y^{(1)} + \frac{3}{2} y = x^2 + 2x^{3/2}, \ \ y(0) = 0$$

Example 4.

$$y^{(\varsigma)} + y = 0, \quad 0 < \varsigma \le 1$$

As $\varsigma \in [0, 1]$ therefore, applying (1) with $n = 1$, we get:

$$\varsigma y^{(1)} + (2-\varsigma) y = 0$$

Example 5.

$$y^{(0.5)} + \sqrt{x} y = xe^{-x}$$

As $\varsigma \in [0, 1]$ therefore, applying eq. (1) we get:

$$y^{(1)} + \left(1 + 2\sqrt{x}\right) y = x\mathrm{e}^{-x}$$

Example 6.

$$y^{(0.5)} = \frac{x^{3/2} + \sqrt{x} y}{2x + 3y}$$

Applying (1) we easily obtain:

$$(2x+3y) y^{(1)} + \left(2x + 3y - 2\sqrt{x}\right) y = 2x^{3/2}$$

In the next example, we convert the well known Riccati FDE to an ODE given in Syam and Al-Refai [17].

Example 7. Riccati FDE is of the form:

$$T_\varsigma(f)(x) = a_0(x) + f(x) - f^2(x), \ \ x \in (0,1), \varsigma \in (0,1) \tag{14}$$

with $f(0) = 1$ and the fractional term $a_0(x)$ is of the form:

$$a_0(x) = -\frac{\varsigma+1}{\varsigma\Gamma(\varsigma)} x \left[E_{\varsigma,2}\left(-\frac{\upsilon}{1-\varsigma} x^\varsigma - 1 \right) \right] - \left(x^{\varsigma+1} + 1 \right) + \left(x^{\varsigma+1} + 1 \right)^2 \tag{15}$$

As $\varsigma \in (0, 1)$ then $\varsigma + 1 \in (1, 2)$. Using these values, the fractional derivative (1) gives:

$$x^{\varsigma} = \varsigma x^{(1)} + (1-\varsigma)x = \\ = \varsigma + (1-\varsigma)x \tag{16}$$

while

$$x^{\varsigma+1} = \varsigma x^{(2)} + (1-\varsigma)x^{(1)} \tag{17}$$

Inserting eqs. (16) and (17) in the fractional term eq. (15) and simplifying, we get the non-fractional term:

$$a_0(x) = -\frac{\varsigma+1}{\varsigma\Gamma(\varsigma)}x\left[E_{\varsigma,2}\left(-\frac{\upsilon(\varsigma+(1-\varsigma)x)}{1-\varsigma}-1\right)\right]+2+3\varsigma+\varsigma^2 \tag{18}$$

Combining eq. (14) with non-fractional term (18), we get Riccati ODE solution of which gives solution of the given Riccati FDE.

Conclusion

From the applications and geometry of the proposed new definition of fractional derivative showed that it is more applicable in the applied side because it is defined with no restrictions on the domain. Basic properties of the new definition are provided. The definition generalizes the definitions of FD already exist in the literature. Specially, through this definition we can easily convert fractional derivatives to integer order derivatives. All the existence examples solved by other authors in history of fractional derivatives can be easily solved with this approach. In future any kind of FDE which model a real world phenomenon or process can be easily handle by converting to classical order problem.

Competing interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

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