STUDY OF A COUPLED SYSTEM WITH ANTI-PERIODIC BOUNDARY CONDITIONS UNDER PIECEWISE CAPUTO-FABRIZIO DERIVATIVE

by

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A coupled system under Caputo-Fabrizio fractional order derivative (CFFOD) with antiperiodic boundary condition is considered. We use piecewise version of CFFOD. Sufficient conditions for the existence and uniqueness of solution by applying the Banach, Krasnoselskii's fixed point theorems. Also some appropriate results for Hyers-Ulam (H-U) stability analysis is established. Proper example is given to verify the results.

Key words: *piecewise CFFOD, stability results, fixed point technique*

Introduction

During 19th and 20th century fractional calculus has got considerable attention from researchers. The aforesaid area has an old history like traditional calculus. Due to various applications of fractional calculus, many researchers including Lacroix, Riemann, Laplace, Fourier, Green, Letnikov, Sonsini, Grunwald, Laurent, Krug, Nekrasov, and Weyl have done great contribution in this area. The fundamental concepts have been introduced by aforementioned authors. They give various definitions of fractional order derivatives including various kinds kernels. The most notable definition has been given by Caputo in 1967. The aforementioned definition has been used in very large number of articles describing various real world problems. Some historical points have been given about the said branch in [1, 2]. For past few decades, the theory of differential equations including fractional order derivatives has played an eminent work in the solution of many mathematical model, dynamical system, fluid-flow and real life problems. Fundamental concepts and applications have been described in [3-5], respectively. This process is still in progress, but new concepts and strategies have emerged with in the framework of fractional calculus allowing us to gain new and difficult insights. Fractional

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derivative and integral are defined by Riemann, Liouville and Caputo are used to solve various kinds of process in rheology and cosmology also. A detailed survey on similar problems of the said are have been discussed in [6]. Basic theory was given in [7]. Alshabanat *et al*. [8] have studied a random problems for existence theory. Dokuyucu *et al.* [9] used the fractional derivative of non-singular type operator to model cancer treatment by radiotherapy.

Here we remark that aforementioned operators involve power law kernel often known as singular kernel. But recently a different type of derivative involving non-singular kernel has attracted the interest of many researchers. Caputo and Fabrizio [10] has introduced non-singular exponential type fractional operator also name as $\mathcal{CF}$ operator. Many researchers have currently published valuable works on non-singular kernels type derivatives. Quereshi *et al.* [11] authors have established some new numerical results for various problems by using CFFOD. Verma and Kumar [12] and Algahtani [13] have studied biological process by using aforesaid derivatives. Fundamental concepts and stability results have been established in Abdeljavad [14] and Liu *et al*. [15]. For instance Eiman *et al*. [16] established qualitative theory by using Krassnoselsskii fixed point theorem to a class of implicit differential equations of fractional order with non-singular kernel involving $\mathcal{CF}$ fractional derivative.

Various problems under CFFOD have been investigated very well for numerical as well as theoretical results. Some applications of said derivatives have been studied in [17-19], respectively. Also some theatrical, numerical and stability results have studied in [20-22], respectively. Qualitative analysis for coupled systems has been considered in many papers. As such systems have many applications in various filed like synchronization, we refer chaotic systems [23-25], respectively. Further boundary value problems have significant applications in modelling various engineering process. Different kinds of boundary value problems have given proper attention in last several years. Aqlan *et al.* [26] investigated a boundary value problems (BVP) with anti-periodic boundary conditions. Such type of problems play significant roles in modelling periodic process of physical and dynamical systems. Therefore, Atangana *et al*. [27] have introduced new concepts of fractional calculus in piece wise form. Piecewise version of derivatives have also the ability to describe many problems with more precise way.

Inspired from the mentioned work, we consider the following class of coupled systems under CFFOD:

$$\begin{aligned} {}^{PCF}\mathcal{D}^{\kappa}\phi(t) &= \omega\big(t,\phi(t),\varphi(t)\big),\ 0<\kappa\le 1 \\ {}^{PCF}\mathcal{D}^{\kappa}\varphi(t) &= \psi\big(t,\phi(t),\varphi(t)\big),\ t\in[0,\tau] \\ \phi(0) &= -\phi(\tau) \\ \varphi(0) &= -\varphi(\tau) \end{aligned} \tag{1}$$

where ${}^{PCF}\mathcal{D}^{\kappa}$ indicates the PCF derivative of order $\kappa > 0$ and ω, ψ: $[0, \tau]\times\mathcal{R}\times\mathcal{R}\to\mathcal{R}$ given functions are continuous. We establish existence results by using Schuader *et al*., theorems which play a prominent role in this regard. The approach of H-U stability provides an approximate answer for the exact solution for differential equation. Some articles in which H-U stability has been discussed for FDE under CFFOD. The considered problem is more general than that studied in [28]. The said stability has also studied for Singular type problems also [23, 24]. Therefore, we also investigate H-U stability to the considered problem. An example is also evaluated for the validity of the present work.

Basic results

We recollect some needful results from [27].

Definition 1.. Let ϕ be a continuous function then piece wise version of Caputo-Fabrizio integral, with $\kappa(0, 1]$ is defined:

$$ {}^{PCF}\mathrm{I}^{\kappa}\phi(t)=\begin{cases}\displaystyle\int_0^{t_1}\phi(s)\mathrm{d}s, & \text{if} \quad t\in[0,t_1]\\ \displaystyle\frac{1-\kappa}{\mathcal{M}(\kappa)}\phi(t)+\frac{\kappa}{\mathcal{M}(\kappa)}\int_a^t\phi(s)\mathrm{d}s, & \text{if} \quad t\in[t_1,\tau]\end{cases} \tag{2} $$

Definition 2. Let ϕ be a continuous function, then PCF derivative is defined:

$$ {}^{PCF}\mathcal{D}^{\kappa}\phi(t)=\begin{cases}\dfrac{\mathrm{d}\phi}{\mathrm{d}t}, & \text{if} \quad t\in[0,t_1]\\ {}^{CF}\mathcal{D}^{\kappa}\phi(t), & \text{if} \quad t\in[t_1,\tau]\end{cases} \tag{3} $$

For FDE under PCF derivative the given results.

Lemma 1. If the right hand side of the given PCF differential equation vanishes at $t=0$, then the solution of:

$$ {}^{PCF}D^{\kappa}\phi(t)=x(t), \text{ with } \kappa\in(0,1] $$
$$ \phi(0)=\phi_0 $$

is given by:

$$ \phi(t)=\begin{cases}\displaystyle\phi_0+\int_0^{t_1}x(s)\mathrm{d}s, & \text{if} \quad t\in[0,t_1]\\ \displaystyle\phi(t_1)+\frac{1-\kappa}{\mathcal{M}(\kappa)}x(t)+\frac{\kappa}{\mathcal{M}(\kappa)}\int_{t_1}^t x(s)\mathrm{d}s, & \text{if} \quad t\in[t_1,\tau]\end{cases} \tag{4} $$

Existence theory

Here we define the Banach spaces:

$$ \mathcal{Z}_1=\{\phi:[0,\tau]\to\mathcal{R},\ \phi\in C([0,\tau])\} \text{ and } \mathcal{Z}_2=\{\varphi:[0,\tau]\to\mathcal{R},\ \varphi\in C([0,\tau])\} $$

subject to the norms as

$$ \|\phi\|_{\mathcal{Z}}=\sup_{t\in[0,1]}|\phi(t)| \text{ and } \|\varphi\|_{\mathcal{Z}}=\sup_{t\in[0,1]}|\varphi(t)|, $$

respectively. Then the product space $\mathcal{Z}1\times\mathcal{Z}2$ is also a Banach space with norm defined:

$$ \|(\phi,\varphi)\|=\|\phi\|+\|\varphi\| $$

Lemma 2. Using *Lemma 1*, the solution of the given anti-periodic BVP:

$$ {}^{PCF}D^{\kappa}\phi(t)=h(t), \ \kappa\in(0,1] $$
$$ \phi(0)=-\phi(\tau) $$

is given by:

$$ \phi(t)=\begin{cases}\displaystyle-\phi(\tau)+\int_0^{t_1}h(s)\mathrm{d}s, & \text{if} \quad t\in[0,t_1]\\ \displaystyle\phi(t_1)+\frac{1-\kappa}{\mathcal{M}(\kappa)}h(t)+\frac{\kappa}{\mathcal{M}(\kappa)}\int_{t_1}^t h(s)\mathrm{d}s, & t\in[t_1,\tau]\end{cases} \tag{5} $$

Proof.

Proof is easy, so we omit it.

Corollary 1. In view of *Lemma 2*, the solution of the coupled system (1) of anti-periodic BVP is given by:

$$\phi(t)=\begin{cases} -\phi(\tau)+\displaystyle\int_0^{t_1}\omega(s,\phi(s),\varphi(s))\mathrm{d}s, \text{ if } t\in[0,t_1] \\ \phi(t_1)+\dfrac{1-\kappa}{\mathcal{M}(\kappa)}\omega(t,\phi(t),\varphi(t))+\dfrac{\kappa}{\mathcal{M}(\kappa)}\displaystyle\int_{t_1}^{t}\omega(s,\phi(s),\varphi(s))\mathrm{d}s, \ t\in[t_1,\tau] \end{cases}$$
$$\varphi(t)=\begin{cases} -\varphi(\tau)+\displaystyle\int_0^{t_1}\psi(s,\phi(s),\varphi(s))\mathrm{d}s, \text{ if } t\in[0,t_1] \\ \varphi(t_1)+\dfrac{1-\kappa}{\mathcal{M}(\kappa)}\psi(t,\phi(t),\varphi(t))+\dfrac{\kappa}{\mathcal{M}(\kappa)}\displaystyle\int_{t_1}^{t}\psi(s,\phi(s),\varphi(s))\mathrm{d}s, \ t\in[t_1,\tau] \end{cases} \tag{6}$$

For the required analysis, we need the following data dependence results to be hold:

Assumption 1. There exist constants $\mathrm{L}_\omega > 0$, $\mathrm{L}_\psi > 0$, such that for

$$\omega,\psi:[0,\tau]\times\mathcal{R}\times\mathcal{R}\to\mathcal{R} \text{ and for every } (\phi,\varphi), (\bar{\phi},\bar{\varphi})\in\mathcal{Z}_1\times\mathcal{Z}_2, \text{ we have}$$

we have:

$$\left|\omega(t,\phi,\varphi)-\omega(t,\bar{\phi},\bar{\varphi})\right|\le \mathrm{L}_\omega\left[|\phi-\bar{\phi}|+|\varphi-\bar{\varphi}|\right]$$

and

$$\left|\psi(t,\phi,\varphi)-\omega(t,\bar{\phi},\bar{\varphi})\right|\le \mathrm{L}_\psi\left[|\phi-\bar{\phi}|+|\varphi-\bar{\varphi}|\right]$$

Assumption 2. For constants and C_ω, D_ω, C_ψ, $D_\psi > 0$ and N_ω, $N_\psi > 0$, the given growth conditions hold:

$$\left|\omega(t,\phi(t),\varphi(t))\right|\le C_\omega\left|\phi(t)\right|+D_\omega\left|\varphi(t)\right| \| +N_\omega$$

and

$$\left|\psi(t,\phi(t),\varphi(t))\right|\le C_\psi\left|\phi(t)\right|+D_\psi\left|\varphi(t)\right|+N_\psi$$

For our main results, we define two operators as:

$$\mathcal{F}=(\mathcal{F}_1,\mathcal{F}_2):\mathcal{Z}_1\times\mathcal{F}_2\to\mathcal{Z}_1\times\mathcal{Z}_2$$

by

$$\mathcal{F}(\phi,\varphi)=(\mathcal{F}_1\phi,\mathcal{F}_2\varphi)$$

Further the operators are expressed in more detailed form as:

$$\mathcal{F}_1(\phi,\varphi)=\begin{cases} -\phi(\tau)+\displaystyle\int_0^{t_1}\omega(s,\phi(s),\varphi(s))\mathrm{d}s, \text{ if } t\in[0,t_1] \\ \phi(t_1)+\dfrac{1-\kappa}{\mathcal{M}(\kappa)}\omega(t,\phi(t),\varphi(t))+\dfrac{\kappa}{\mathcal{M}(\kappa)}\displaystyle\int_{t_1}^{t}\omega(s,\phi(s),\varphi(s))\mathrm{d}s, \ t\in[t_1,\tau] \end{cases} \tag{7}$$

$$\mathcal{F}_2(\phi,\varphi)=\begin{cases}-\varphi(\tau)+\int\limits_0^{t_1}\psi(s,\phi(s),\varphi(s))\mathrm{d}s, \text{ if } t\in[0,t_1]\\ \varphi(t_1)+\dfrac{1-\kappa}{\mathcal{M}(\kappa)}\psi(t,\phi(t),\varphi(t))+\dfrac{\kappa}{\mathcal{M}(\kappa)}\int\limits_{t_1}^{t}\psi(s,\phi(s),\varphi(s))\mathrm{d}s, \ t\in[t_1,\tau]\end{cases} \tag{8}$$

Theorem 1. Under the *Assumption 1*, the problem (1) has unique solution if $\max\{K_1, K_2\} < 1$ where:

$$K_1=t_1(\mathrm{L}_\omega+\mathrm{L}_\psi), \ K_2=\frac{1-\kappa+\tau\kappa}{\mathcal{M}(\kappa)}(\mathrm{L}_\omega+\mathrm{L}_\psi)$$

Proof.

Consider (ϕ, φ), and $(\bar{\phi}, \bar{\varphi})$ in $\mathcal{Z}_1 \times \mathcal{Z}_2$, the detail is given as:

Case I. At $t \in [0, t_1]$, we have:

$$\left|\mathcal{F}_1(\phi,\varphi)-\mathcal{F}_1(\bar{\phi},\bar{\varphi})\right|\le\int\limits_0^{t_1}\left|\omega(s,\phi(s),\varphi(s))-\omega(s,\bar{\phi}(s),\bar{\varphi}(s))\right|\mathrm{d}s \tag{9}$$

Taking maximum values of both sides, we have:

$$\left\|\mathcal{F}_1(\phi,\varphi)-\mathcal{F}_1(\bar{\phi},\bar{\varphi})\right\|\le t_1\mathrm{L}_\omega\left(\left\|\phi-\bar{\phi}\right\|+\left\|\varphi-\bar{\varphi}\right\|\right) \tag{10}$$

exercising again the same procedure in $[0, t_1]$, one has:

$$\left\|\mathcal{F}_2(\phi,\varphi)-\mathcal{F}_2(\bar{\phi},\bar{\varphi})\right\|\le t_1\mathrm{L}_\psi\left(\left\|\phi-\bar{\phi}\right\|+\left\|\varphi-\bar{\varphi}\right\|\right) \tag{11}$$

From eqs. (10) andd (11), we have over $t \in [0, t_1]$:

$$\begin{aligned}\left\|\mathcal{F}(\phi,\varphi)-\mathcal{F}(\bar{\phi},\bar{\varphi})\right\|&=\left\|\mathcal{F}_1(\phi,\varphi)-\mathcal{F}_1(\bar{\phi},\bar{\varphi})\right\|+\left\|\mathcal{F}_2(\phi,\varphi)-\mathcal{F}_2(\bar{\phi},\bar{\varphi})\right\|\le\\ &\le t_1(\mathrm{L}_\omega+\mathrm{L}_\psi)\left\|\phi-\bar{\phi}\right\|+\left\|\varphi-\bar{\varphi}\right\|\end{aligned} \tag{12}$$

Hence using $K_1 = t_1[\mathrm{L}_\omega + \mathrm{L}_\psi]$, one has from eq. (12) that:

$$\left\|\mathcal{F}(\phi,\varphi)-\mathcal{F}(\bar{\phi},\bar{\varphi})\right\|\le K_1\left\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\right\| \tag{13}$$

Now we exercise the same procedure for $t \in [0, t_1]$ of our solution.

Case II: At $t \in [0, t_1]$, if (ϕ, φ), $(\bar{\phi}, \bar{\varphi}) \in \mathcal{Z}_1 \times \mathcal{Z}_2$, then one has:

$$\begin{aligned}\left|\mathcal{F}_1(\phi,\varphi)-\mathcal{F}_1(\bar{\phi},\bar{\varphi})\right|&\le\frac{1-\kappa}{\mathcal{M}(\kappa)}\left|\omega(t,\phi(t),\varphi(t))-\omega(t,\bar{\phi}(t),\bar{\varphi}(t))\right|+\\ &+\frac{\kappa}{\mathcal{M}(\kappa)}\int\limits_{t_1}^{\tau}\left|\omega(s,\phi(s),\varphi(s))-\omega(s,\bar{\phi}(s),\bar{\varphi}(s))\right|\mathrm{d}s\end{aligned} \tag{14}$$

After simplification and applying *Assumption 1*, we have:

$$\begin{aligned}\left\|\mathcal{F}_1(\phi,\varphi)-\mathcal{F}_1(\bar{\phi},\bar{\varphi})\right\|&\le\frac{1-\kappa}{\mathcal{M}(\kappa)}\mathrm{L}_\omega\left(\left\|\phi-\bar{\phi}\right\|+\left\|\varphi-\bar{\varphi}\right\|\right)+\frac{\kappa}{\mathcal{M}(\kappa)}\mathrm{L}_\omega\left(\|\phi-\bar{\phi}\|+\|\varphi-\bar{\varphi}\|\right)(\tau-t_1)\le\\ &\le\left(\frac{1-\kappa}{\mathcal{M}(\kappa)}+\frac{\tau\kappa}{\mathcal{M}(\kappa)}\right)\mathrm{L}_\omega\left(\|\phi-\bar{\phi}\|+\|\varphi-\bar{\varphi}\|\right)\end{aligned} \tag{15}$$

Analogously at $[t_1, \tau]$ incase of second operator, one has:

$$\left\|\mathcal{F}_2(\phi,\varphi)-\mathcal{F}_2(\overline{\phi},\overline{\varphi})\right\| \le \left(\frac{1-\kappa}{\mathcal{M}(\kappa)}+\frac{\tau\kappa}{\mathcal{M}(\kappa)}\right)\mathrm{L}_{\psi}\left[\|\phi-\overline{\phi}\|+\|\varphi-\overline{\varphi}\|\right] \tag{16}$$

From eqs. (15) and (16), we have for $t \in [t_1, \tau]$:

$$\left\|\mathcal{F}(\phi,\varphi)-\mathcal{F}(\overline{\phi},\overline{\varphi})\right\| \le \left(\frac{1-\kappa+\tau\kappa}{\mathcal{M}(\kappa)}\right)(\mathrm{L}_{\omega}+\mathrm{L}_{\psi})\left\|(\phi,\varphi)-(\overline{\phi},\overline{\varphi})\right\| \tag{17}$$

Let

$$K_2=\left(\frac{1-\kappa+\tau\kappa}{\mathcal{M}(\kappa)}\right)[\mathrm{L}_{\omega}+\mathrm{L}_{\psi}]$$

Then eq. (17) yields:

$$\left\|\mathcal{F}(\phi,\varphi)-\mathcal{F}(\overline{\phi},\overline{\varphi})\right\| \le K_2\left\|(\phi,\varphi)-(\overline{\phi},\overline{\varphi})\right\| \tag{18}$$

thus we have:

$$\left\|\mathcal{F}(\phi,\varphi)-\mathcal{F}(\overline{\phi},\overline{\varphi})\right\| \le \begin{cases} K_1\,\|(\phi,\varphi)-(\overline{\phi},\overline{\varphi})\|, & \text{if } t\in[0,t_1] \\ K_2\,\|(\phi,\varphi)-(\overline{\phi},\overline{\varphi})\|, & t\in[t_1,\tau]\end{cases} \tag{19}$$

Finally over $t \in [0, \tau]$, we conclude that:

$$\left\|\mathcal{F}(\phi,\varphi)-\mathcal{F}(\overline{\phi},\overline{\varphi})\right\| \le \max\{K_1,K_2\}\left\|(\phi,\varphi)-(\overline{\phi},\overline{\varphi})\right\| \tag{20}$$

Hence $\mathcal{F}$ is contraction and the given problem has a unique solution.

Theorem 2. Under the hypothesis *Assumption* 2, the proposed coupled system (1) has atleast one solution.

Proof.

Let E be a closed convex subset of $\mathcal{Z}_1 \times \mathcal{Z}_2$, such that:

$$E=\left\{(\phi,\varphi)\in\mathcal{Z}_1\times\mathcal{Z}_2:\left\|(\phi,\varphi)\right\|\le r_{1,2}\right\},\ \text{where } r_{1,2}\ge\frac{\lambda+Mt_1}{1-\theta t_1},\ M=M_{\omega}+M_{\psi},$$

$$\theta=\max\left\{D_{\omega}+D_{\psi},C_{\omega}+C_{\psi}\right\} \text{ and } \lambda=\left|\phi(\tau)\right|+\left|\varphi(\tau)\right|$$

then here we define operator

$$G=(G_1,G_2):E\to E \text{ by } G(\phi,\varphi)=(G_1,G_2)(\phi,\varphi)=(G_1(\phi,\varphi),G_2(\phi,\varphi))$$

where

$$G_1(\phi,\varphi)=\begin{cases} -\phi(\tau)+\displaystyle\int_0^{t_1}\omega(s,\phi(s),\varphi(s))\mathrm{d}s, & \text{if } t\in[0,t_1] \\ \phi(t_1)+\dfrac{1-\kappa}{\mathcal{M}(\kappa)}\omega(t,\phi(t),\varphi(t))+\dfrac{\kappa}{\mathcal{M}(\kappa)}\displaystyle\int_{t_1}^{t}\omega(s,\phi(s),\varphi(s))\mathrm{d}s, & t\in[t_1,\tau]\end{cases} \tag{21}$$

$$G_2(\phi,\varphi)=\begin{cases} -\varphi(\tau)+\displaystyle\int_0^{t_1}\psi(s,\phi(s),\varphi(s))\mathrm{d}s, & \text{if } t\in[0,t_1] \\ \varphi(t_1)+\dfrac{1-\kappa}{\mathcal{M}(\kappa)}\psi(t,\phi(t),\varphi(t))+\dfrac{\kappa}{\mathcal{M}(\kappa)}\displaystyle\int_{t_1}^{t}\psi(s,\phi(s),\varphi(s))\mathrm{d}s, & t\in[t_1,\tau]\end{cases} \tag{22}$$

Here onward analysis is established in numbers of steps:

Case I. For $t \in [0, t_1]$, we have the following step.

Step I. Let $(\phi, \varphi) \in E$. Then at $t \in [0, t_1]$, we have:

$$\begin{gathered}\|G_1(\phi,\varphi)\| \le |\phi(\tau)| + \sup_{t\in[0,t_1]}\int_0^{t_1}|\omega(s,\phi(s),\varphi(s))|\,\mathrm{d}s \le \\ \le |\phi(\tau)| + \left(C_\omega\|\phi\| + D_\omega\|\varphi\| + N_\omega\right)t_1\end{gathered} \tag{23}$$

Similarly at $t \in [0, t_1]$, we also have:

$$\|G_2(\phi,\varphi)\| \le |\varphi(\tau)| + \left(C_\psi\|\phi\| + D_\psi\|\varphi\| + N_\psi\right)t_1 \tag{24}$$

From eqs. (23) and (24), one has:

$$\|G_1(\phi,\varphi)\| + \|G_2(\phi,\varphi)\| \le |\phi(\tau)| + |\varphi(\tau)| + \left[(C_\omega + C_\psi)\|\phi\| + (D_\omega + D_\psi)\|\varphi\| + N_\omega + N_\psi\right]t_1 \tag{25}$$

Let:

$$C_\omega + C_\psi = a, D_\omega + D_\psi = b \ \text{ and } \ |\phi(\tau)| + |\varphi(\tau)| = \lambda, \ \max(a,b) = \theta \ \textit{ and } \ N_\omega + N_\psi = M$$

Then eq. (25) implies:

$$\|G(\phi,\varphi)\| \le \lambda + t_1\theta\|(\phi,\varphi)\| + Nt_1 \le r_{1,2} \tag{26}$$

From which:

$$\|G(\phi,\varphi)\| \le \lambda + t_1\theta\|(\phi,\varphi)\| + Nt_1 \le r_{1,2}$$

where

$$r_{1,2} \ge \frac{\lambda + Mt_1}{1-\theta t_1} \tag{27}$$

Hence $\|G(\phi, \varphi)\| \le r_{1,2}$. Thus G is bounded and $G(\phi, \varphi) \in E$. Which means $G(E) \subseteq E$ in case if $t \in [0, t_1]$.

Case II: At $t \in [t_1, \tau]$.

For $(\phi, \varphi) \in E$ at $t \in [t_1, \tau]$, we have:

$$\begin{gathered}\|G_1(\phi,\varphi)\| \le \left[\sup_{t\in[t_1,\tau]}|\phi(t_1)| + \frac{1-\kappa}{\mathcal{M}(\kappa)}|\omega(t,\phi(t),\varphi(t))| + \frac{\kappa}{\mathcal{M}(\kappa)}\int_{t1}^{t}|\omega(s,\phi(s),\varphi(s))|\,\mathrm{d}s\right] \le \\ \le \left[\sup_{t\in[t_1,\tau]}|\phi(t_1)| + \frac{1-\kappa}{\mathcal{M}(\kappa)}\left(C_\omega|\phi(t)| + D_\omega|\varphi(t)| + N_\omega\right) + \frac{\kappa}{\mathcal{M}(\kappa)}\int_{t_1}^{t}\left(C_\omega|\phi(t)| + D_\omega|\varphi(t)| + N_\omega\right)\mathrm{d}s\right] \le \\ \le |\phi(t_1)| + \frac{1-\kappa}{\mathcal{M}(\kappa)}\left(C_\omega\|\phi\| + D_\omega\|\varphi\| + N_\omega\right) + \frac{\kappa}{\mathcal{M}(\kappa)}\left(C_\omega\|\phi\| + D_\omega\|\varphi\| + N_\omega\right)(\tau - t_1) \le \\ \le |\phi(t_1)| + \frac{1-\kappa}{\mathcal{M}(\kappa)}\left(C_\omega\|\phi\| + D_\omega\|\varphi\| + N_\omega\right) + \frac{\kappa}{\mathcal{M}(\kappa)}\left(C_\omega\|\phi\| + D_\omega\|\varphi\| + N_\omega\right)\tau\end{gathered} \tag{28}$$

In same line one has also:

$$\|G_2(\phi,\varphi)\| \le |\varphi(t_1)| + \frac{1-\kappa}{\mathcal{M}(\kappa)}\left(C_\psi\|\phi\| + D_\psi\|\varphi\| + N_\psi\right) + \frac{\kappa}{\mathcal{M}(\kappa)}\left(C_\psi\|\phi\| + D_\psi\|\varphi\| + N_\psi\right)\tau \tag{29}$$

Now using

$$\Delta_{1,2} = |\phi(t_1)| + |\varphi(t_1)|$$

and the previously defined notation in eq. (27), from eqs. (28) and (29), we have:

$$\|G_1(\phi,\varphi)\| + \|G_2(\phi,\varphi)\| \le \Delta_{1,2} + \frac{(1-\kappa)ar_{1,2}}{\mathcal{M}(\kappa)} + \frac{(1-\kappa)br_{1,2}}{\mathcal{M}(\kappa)} + \left[\frac{(1-\kappa)ar_{1,2}}{\mathcal{M}(\kappa)} + \frac{(1-\kappa)br_{1,2}}{\mathcal{M}(\kappa)}\right]\tau + \\ + \frac{(1-\kappa)N}{\mathcal{M}(\kappa)} + \frac{\kappa N\tau}{\mathcal{M}(\kappa)} \tag{30}$$

$$\|G(\phi,\varphi)\| \le \Delta_{1,2} + \left[\frac{(1-\kappa)a + (1-\kappa)a\tau}{\mathcal{M}(\kappa)} + \frac{(1-\kappa)b + (1-\kappa)b\tau}{\mathcal{M}(\kappa)}\right]r_{1,2} + \left[\frac{(1-\kappa) + \kappa\tau}{\mathcal{M}(\kappa)}\right]N \le r_{1,2} \tag{31}$$

where we use for simplicity

$$p = \frac{(1-\kappa)a(1+\tau) + (1-\kappa)b(1+\tau)}{\mathcal{M}(\kappa)}$$

$$q = \frac{(1-\kappa) + \kappa\tau}{\mathcal{M}(\kappa)}$$

Then:

$$r_{1,2} \ge \frac{\Delta_{1,2} + Nq}{1-p}$$

Now if:

$$\max\left\{\frac{\lambda + Nt_1}{1-\theta t_1}, \frac{\Delta_{1,2} + Nq}{1-p}\right\} \le r_{1,2}, \text{ then } \|G(\phi,\varphi)\| \le r_{1,2}$$

Hence G is bounded in this case also. Also $G(E) \subseteq E$ fo $t \in [t_1, \tau]$.

Step II: Since ω, $\psi \in C$ $[0, \tau]$. Hence G_1, G_2 are continuous in same domains. Now to show that $G = (G_1, G_2)$ is equi-continuous, we discuss two cases. Let $t \in [0, t_1]$, then $t_m < t_n \in [0, t_1]$ we have:

$$|G_1(\phi,\varphi)(t_n) - G_1(\phi,\varphi)(t_m)| \le (t_n - t_m)[C_\omega r_{1,2} + D_\omega r_{1,2} + N_\omega] \to 0 \text{ as } t_m \to t_n \tag{32}$$

and

$$|G_2(\phi,\varphi)(t_n) - G_2(\phi,\varphi)(t_m)| \le (t_n - t_m)[C_\psi r_{1,2} + D_\psi r_{1,2} + N_\psi] \to 0 \text{ as } t_m \to t_n \tag{33}$$

Since G_1, G_2 are bounded over $[0, t_1]$ so G is also bounded. Thus it is uniformly continuous there in also and we have:

$$\|G_2(\phi,\varphi)(t_n) - G_2(\phi,\varphi)(t_m)\| \le (t_n - t_m)[C_\psi r_{1,2} + D_\psi r_{1,2} + N_\psi] \to 0 \text{ as } t_m \to t_n \tag{34}$$

Now if:

$$t_m < t_n \in [t_1, \tau]$$

we have

$$\left|G_1(\phi,\varphi)(t_n)-G_1(\phi,\varphi)(t_m)\right| \le \frac{1-\kappa}{\mathcal{M}(\kappa)}\left|\omega(t_n,\phi(t_n),\varphi(t_n))-\omega(t_m,\phi(t_m),\varphi(t_m))\right|+$$

$$+\frac{\kappa}{\mathcal{M}(\kappa)}\int_{t_m}^{t_n}\left|\omega(s,\phi(s),\varphi(s))\right|\mathrm{d}s \le$$

$$\le \frac{1-\kappa}{\mathcal{M}(\kappa)}\left|\omega(t_n,\phi(t_n),\varphi(t_n))-\omega(t_m,\phi(t_m),\varphi(t_m))\right|+\frac{\kappa}{\mathcal{M}(\kappa)}\Big[(C_\omega+D_\omega)r_{1,2}+N_\omega\Big](t_n-t_m) \tag{35}$$

Hence in eq. (36), we have if $t_m \to t_n$ then right hand side goes to zero. Thus:

$$\left|G_1(\phi,\varphi)(t_n)-G_1(\phi,\varphi)(t_m)\right| \to 0 \text{ with } t_m \to t_n$$

Also G_1 bounded so is uniformly continuous:

$$\left\|G_1(\phi,\varphi)(t_n)-G_1(\phi,\varphi)(t_m)\right\| \to 0 \text{ with } t_m \to t_n$$

In same line, we have:

$$\left\|G_2(\phi,\varphi)(t_n)-G_2(\phi,\varphi)(t_m)\right\| \to 0 \text{ with } t_m \to t_n$$

Therefore

$$\left\|G(\phi,\varphi)(t_n)-G(\phi,\varphi)(t_m)\right\| \to 0 \text{ with } t_m \to t_n$$

Thus after completing all the requirement, we claims that the problem (1) has atleast one solution.

The H-U stability analysis

To derive results regarding H-U stability, let there be a function f which is independent of ϕ and φ, such that $f(0) = 0$,then:

Remark 1. (*i*) $|f(t)| \le \epsilon$, $t \in [0, \tau]$ and (*ii*):

$$\begin{aligned} {}^{PCF}D^{\kappa}\phi(t) &= \omega(t,\phi(t),\varphi(t))+f(t) \\ {}^{PCF}D^{\kappa}\phi(t) &= \psi(t,\phi(t),\varphi(t))+f(t) \\ \phi(0) = -\phi(\tau),\varphi(0) &= -\varphi(\tau),\ t\in[0,\tau] \end{aligned} \tag{36}$$

Lemma 3. The solution of eq. (36) satisfies the following relations:

$$\left\|\phi(t)-\left[-\phi(\tau)+\int_0^{t_1}\omega\big(s,\phi(s),\varphi(s)\big)\mathrm{d}s\right]\right\| \le t_1\epsilon,\ t\in[0,t_1]$$

$$\left|\phi(t)-\left[\phi(t_1)+\frac{1-\kappa}{\mathcal{M}(\kappa)}\omega\big(t,\phi(t),\varphi(t)\big)+\frac{1-\kappa}{\mathcal{M}(\kappa)}\int_{t_1}^{t}\omega\big(s,\phi(s),\varphi(s)\big)\right]\mathrm{d}s\right| \le$$

$$\le\left[\frac{(1-\kappa)+\kappa\tau}{\mathcal{M}(\kappa)}\right]\epsilon,\ t\in[t_1,\tau] \tag{37}$$

and

$$\left|\varphi(t)-\left[-\varphi(\tau)+\int_0^{t_1}\psi\big(s,\phi(s),\varphi(s)\big)\mathrm{d}s\right]\right|\le t_1\epsilon,\quad t\in[0,t_1]$$

$$\left|\varphi(t)-\left[\varphi(t_1)+\frac{1-\kappa}{\mathcal{M}(\kappa)}\psi\big(t,\phi(t),\varphi(t)\big)+\frac{1-\kappa}{\mathcal{M}(\kappa)}\int_{t_1}^{t}\psi\big(s,\phi(s),\varphi(s)\big)\mathrm{d}s\right]\right|\le$$
$$\le\left[\frac{(1-\kappa)+\kappa\tau}{\mathcal{M}(\kappa)}\right]\epsilon,\quad t\in[t_1,\tau] \tag{38}$$

Proof.
In view of *Lemma 2*, the solution of eq. (36) is computed:

$$\phi(t)=\begin{cases}-\phi(\tau)+\displaystyle\int_0^{t_1}\omega(s,\phi(s),\varphi(s))\mathrm{d}s+\int_0^{t_1}f(s)\mathrm{d}s, & \text{if } t\in[0,t_1]\\[2ex] \phi(t_1)+\dfrac{1-\kappa}{\mathcal{M}(\kappa)}\omega(t,\phi(t),\varphi(t))+\dfrac{\kappa}{\mathcal{M}(\kappa)}\displaystyle\int_{t_1}^{t}\omega(s,\phi(s),\varphi(s))\mathrm{d}s & \\[2ex] +\dfrac{\kappa}{\mathcal{M}(\kappa)}f(s)+\dfrac{\kappa}{\mathcal{M}(\kappa)}\displaystyle\int_{t_1}^{t}f(s)\mathrm{d}s, & t\in[t_1,\tau]\end{cases} \tag*{39)}$$

and

$$\varphi(t)=\begin{cases}-\varphi(\tau)+\displaystyle\int_0^{t_1}\psi(s,\phi(s),\varphi(s))\mathrm{d}s+\int_0^{t_1}f(s)\mathrm{d}s, & \text{if } t\in[0,t_1]\\[2ex] \varphi(t_1)+\dfrac{1-\kappa}{\mathcal{M}(\kappa)}\psi(t,\phi(t),\varphi(t))+\dfrac{\kappa}{\mathcal{M}(\kappa)}\displaystyle\int_{t_1}^{t}\psi(s,\phi(s),\varphi(s))\mathrm{d}s & \\[2ex] +\dfrac{\kappa}{\mathcal{M}(\kappa)}f(s)+\dfrac{\kappa}{\mathcal{M}(\kappa)}\displaystyle\int_{t_1}^{t}f(s)\mathrm{d}s, & t\in[t_1,\tau]\end{cases} \tag{40}$$

Now from eq. (39), we have:

$$\left|\phi(t)-\left[-\phi(\tau)+\int_0^{t_1}\omega\big(s,\phi(s),\varphi(s)\big)\mathrm{d}s\right]\right|\le t_1\epsilon,\ \ t\in[0,t_1]$$

$$\left|\phi(t)-\left[\phi(t_1)+\frac{1-\kappa}{\mathcal{M}(\kappa)}\omega\big(t,\phi(t),\varphi(t)\big)+\frac{1-\kappa}{\mathcal{M}(\kappa)}\int_0^{t_1}\omega\big(s,\phi(s),\varphi(s)\big)\mathrm{d}s\right]\right|\le$$
$$\le\left[\frac{(1-\kappa)+\kappa\tau}{\mathcal{M}(\kappa)}\right]\epsilon,\ \ t\in[t_1,\tau] \tag{41}$$

And from eq. (40), we have:

$$\left|\varphi(t)-\left[-\varphi(\tau)+\int_0^{t_1}\psi\left(s,\phi(s),\varphi(s)\right)\mathrm{d}s\right]\right|\le t_1\epsilon,\quad t\in[0,t_1]$$

$$\left|\varphi(t)-\left[\varphi(t_1)+\frac{1-\kappa}{\mathcal{M}(\kappa)}\psi(t,\phi(t),\varphi(t))+\frac{1-\kappa}{\mathcal{M}(\kappa)}\int_{t_1}^{t}\psi\left(s,\phi(s),\varphi(s)\right)\mathrm{d}s\right]\right|\le \tag{42}$$

$$\le\left[\frac{(1-\kappa)+\kappa\tau}{\mathcal{M}(\kappa)}\right]\epsilon,\quad t\in[t_1,\tau]$$

Which satisfy eqs. (37) and (38).

Theorem 3. In view of *Assumption 1* and *Lemma 3*, the solution of system (1) is H-U stable if:

$$\Omega_{t_1,\kappa,K_{1,2}}<1,\ \text{where}\ \max\left\{\frac{2t_1}{1-t_1K_{1,2}},\frac{2t_1}{1-\Omega_{\kappa,K_{1,2}}}\right\}=\Omega_{t_1,\kappa,K_{1,2}} \tag{43}$$

such that

$$\mathrm{L}_\omega+\mathrm{L}_\psi=K_{1,2},\ \Omega_{\kappa,K_{1,2}}=\left(\frac{1-\kappa+\kappa\tau}{\mathcal{M}(\kappa)}\right)K_{1,2}$$

Proof.

The $(\phi,\varphi)\in\mathcal{Z}_1\times\mathcal{Z}_2$ be the unique solution and $(\bar{\phi},\bar{\varphi})\in\mathcal{Z}_1\times\mathcal{Z}_2$ is any solution of eq. (1). Then for $t\in[0,t_1]$, we have:

$$\left\|\phi-\bar{\phi}\right\|\le t_1\epsilon+t_1\mathrm{L}_\omega\left(\left\|\phi-\bar{\phi}\right\|+\left\|\varphi-\bar{\varphi}\right\|\right) \tag{44}$$

and

$$\left\|\varphi-\bar{\varphi}\right\|\le t_1\epsilon+t_1\mathrm{L}_\psi\left(\left\|\phi-\bar{\phi}\right\|+\left\|\varphi-\bar{\varphi}\right\|\right) \tag{45}$$

Adding eqs. (44), and (45), we have:

$$\left\|\phi-\bar{\phi}\right\|+\left\|\varphi-\bar{\varphi}\right\|\le 2t_1\epsilon+t_1\left(\mathrm{L}_\omega+\mathrm{L}_\psi\right)\left(\left\|\phi-\bar{\phi}\right\|+\left\|\varphi-\bar{\varphi}\right\|\right) \tag{46}$$

Using

$$\mathrm{L}_\omega+\mathrm{L}_\psi=K_{1,2}$$

then we have from eqs. (46) and (47):

$$\left\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\right\|\le\frac{2t_1}{1-t_1K_{1,2}}\epsilon,+t_1(\mathrm{L}_\omega,t\in[0,t_1] \tag{47}$$

In same line for $t\in[0,t_1]$, we have:

$$\left\|\phi-\bar{\phi}\right\|\le t_1\epsilon+\frac{(1-\kappa)\mathrm{L}_\omega}{\mathcal{M}(\kappa)}\left(\left\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\right\|\right)+\frac{\kappa\mathrm{L}_\omega\tau}{\mathcal{M}(\kappa)}\left(\left\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\right\|\right) \tag{48}$$

and

$$\|\varphi-\bar{\varphi}\| \le t_1\epsilon + \frac{(1-\kappa)\mathrm{L}_\psi}{\mathcal{M}(\kappa)}\left(\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\|\right) + \frac{\kappa \mathrm{L}_\psi \tau}{\mathcal{M}(\kappa)}\left(\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\|\right) \tag{49}$$

Adding eqs. (48) and (49), we have:

$$\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\| \le 2t_1\epsilon + \left[\frac{(1-\kappa)}{\mathcal{M}(\kappa)}K_{1,2} + \frac{\kappa\tau}{\mathcal{M}(\kappa)}K_{1,2}\right]\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\| \tag{50}$$

For easiness we put:

$$\Omega_{\kappa,K_{1,2}} = \left(\frac{1-\kappa+\kappa\tau}{\mathcal{M}(\kappa)}\right)K_{1,2}$$

then eq. (50) yields:

$$\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\| \le \frac{2t_1}{1-\Omega_{\kappa,K_{1,2}}}\epsilon, \ \ t\in[t_1,\tau] \tag{51}$$

Hence from (47) and (51), if:

$$\max\left\{\frac{2t_1}{1-t_1K_{1,2}}, \frac{2t_1}{1-\Omega_{\kappa,K_{1,2}}}\right\} = \Omega_{t_1,\kappa,K_{1,2}}$$

then eq. (51) implies:

$$\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\| \le \Omega_{t_1,\kappa,K_{1,2}}\epsilon \tag{52}$$

Hence the solution is H-U stable.

Further, if there exists a non-decreasing function $\Phi:\mathcal{R}^+\to\mathcal{R}^+$ such that $\Phi(\epsilon)=\epsilon$. Then from eq. (52), we have:

$$\|(\phi,\varphi)-(\bar{\phi},\bar{\varphi})\| \le \Phi(\epsilon)\Omega_{t_1,\kappa,K_{1,2}}\epsilon \tag{53}$$

with $\Phi(0) = 0$. Thus the solution is generalized H-U stable also.

Application authenticate our analysis

Example 1.

$$\begin{gathered}
{}^{PCF}\mathcal{D}^{1/2}\phi(t) = \frac{\sin|\phi(t)|+|\varphi(t)|}{t^2+100}, \quad t\in[0,1] \\
{}^{PCF}\mathcal{D}^{1/2}\varphi(t) = \frac{|\phi(t)|+\cos|\varphi(t)|}{t^3+100}, \quad t\in[0,1] \\
\phi(0) = -\phi(1), \quad \varphi(0) = -\varphi(1)
\end{gathered} \tag{54}$$

Then $\mathrm{L}_\omega = 1/100$, $\mathrm{L}_\psi = 1/100$. Let $t_1 = 0.5$, then $K_1 = 1/2(1/50) = 1/100$ and $K_2 = 1/50$. Hence the given problem has a unique solution by *Theorem 2*. Also by calculation

$$\Omega_{t_1,\kappa,K_{1,2}} = 0.789 < 1$$

Hence by *Theorem 3*, the solution is H-U stable. Consequently if we take $\Phi(t) = t$ is a non-decreasing function, then the condition of generalized H-U stability also holds.

Conclusion

By considering a coupled system with anti-periodic boundary conditions under piecewise CFFOD, we have established some adequate results for the existence theory. We have successfully applied Banach and Krassnoselskii fixed point results to establish our required results. Also we have provided sufficient results for H-U stability to the proposed problem. All the results have been verified by taking a pertinent example. In future such analysis can be extended to more general boundary value problems especially those increasing used in modelling boundary-layer problems.

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