EXISTENCE AND STABILITY THEORY OF PANTOGRAPH CONFORMABLE FRACTIONAL DIFFERENTIAL PROBLEM

by

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The purpose of this paper is to investigate the existence and uniqueness (EU) of solutions to a class of conformable fractional differential equations (DE) with delay term using Krasnoselskii's fixed point theorem. The proposed problem is devoted to non-local initial value problems. Such problems are increasingly occurred in applications like in the filed of quantum mechanics and electrodynamics. The theoretical analysis is further enriched by establishing stability theory due to Ulam and its different kinds including "Ulam-Hyers (UH), generalized Ulam-Hyers (GUH), Ulam-Hyers-Rassias (UHR), and generalized Ulam-Hyers-Rassias (GUHR)" stability for the considered class. The obtain analysis is then testified by an example.

Key words: *conformable fractional derivative, EU of results, stability, Krasnoselskii's fixed point theorem*

Introduction

The arbitrary order derivative is the generalization of the classical order derivative. Fractional calculus started at the same time as ordinary calculus. This subject has been applied in various areas of engineering and sciences in recent past [1-14], respectively. In the available literature, there are different definitions to define fractional derivatives, each definition has its own assumptions. Riemann-Liouville and Caputo fractional derivatives among these definitions are most well known definitions. But these definitions of fractional derivative do not satisfies the chain rule like ordinary order derivative.

To define fractional derivative in such a way that, it enjoy the chain rule, in this regard Khalil [15] introduced a new definition, known is conformable fractional order derivative (CFD), which enjoy the chain rule. Most of the properties of this definition coincide with integer order operator and it can be applied to handel fractional DE more easily Abdeljawad [16]. For more properties of CFD, one may see [15-17]. Also for some applications of CFD, see these articles [18-22], respectively. Although the qualitative theory of fractional DE involving CFD is started quite recently.

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There are large numbers of articles related to the existence of solution results for the DE involving the Riemann-Liouville and Caputo fractional derivatives. For instance using fixed point results to develop sufficient conditions for existence and uniqueness, reader should see [23-26], respectively. While using CFD the existence and stability of fractional DE have not been studied in the existing literature. As we know that the existence theory is important aspect of DE before its analytical study.

Zhong and Wang [27] discussed the EU of the following differential equation under CFD:

$$\mathbf{T}_{\lambda}^{\alpha}\mathcal{F}(\nu)=\phi(\nu,\mathcal{F}(\nu)),\ \ \nu\in\Delta=[\alpha,\beta],\ \ \beta<\infty,\ 0<\lambda\leq 1$$
$$\mathcal{F}(0)=\mathcal{F}_0+\psi(\mathcal{F}) \tag{1}$$

where $\psi \in C[R, R]$, $\phi \in C[\Delta \times R, R]$.

Pantograph type DE are special class of delay DE which involve proportional delay. For the first time this class of DE was studied to improve the speed of electric train. An important construction was made by Ockendon and Tayler [28]. Nowadays such type of DE used in many real world problems. To the best of our knowledge no one consider pantograph equations under CFD. Therefore, motivated by aforementioned work in [27], in the current note, our aim is to study the pantograph equation for EU under CFD. To achieve the desired results, we used Krasnoselskii's fixed point theorem for the EU of results which is a well known theorem of functional analysis. The suggested theorem easily deals with an operator problem that can be splited into two sub operators.

Consider a general class of pantograph equation under CFD:

$$\mathbf{T}_{\lambda}^{\alpha}\mathcal{F}(\nu)=\phi(\nu,(\mathcal{F}\nu),\mathcal{F}(\gamma\nu)),\ \ \nu\in\Delta,\ \ 0<\lambda\leq 1$$
$$\mathcal{F}(0)=\mathcal{F}_0+\psi(\mathcal{F}) \tag{2}$$

where $0 < \gamma < 1$, $\phi \in C[\Delta \times R^2, R]$, and $\psi \in C[\Delta, R]$.

Stability analysis is necessary for the FO differential equations. Among the several types of stability, UH type stability is one of the most fascinating. Ulam [29] presented asserted stability in 1940, and Hyers [30] investigated it further. Rassiass [31] went on generalize UH stability, UHR stability is a more broad framework. Many researchers have examined asserted stability in the recent few years, for example, see [32-35], respectively. So the proposed problem is also under consideration for UH stability. Finally an example will be provide to verify our establish results.

Fundamental materials

Below are some useful of results of fractional calculus, CFD and UH type stability.

Definition 1. [27] The CFD from α of a function $F \in C(\Delta)$ of order λ is given:

$$\mathbf{T}_{\lambda}^{\alpha}\mathcal{F}(\nu)=\lim_{\delta\to 0}\mathcal{F}\frac{\left(\nu+\delta(\nu-a)^{1-\lambda}\right)-\mathcal{F}(\nu)}{\delta} \tag{3}$$

Definition 2. [27] The fractional integral operator of order α of a function $\mathcal{F} \in C(\Delta)$ of order λ is given:

$$I_{\lambda}^{\alpha}\mathcal{F}(\nu)=\int_{a}^{\nu}(\nu-\alpha)^{\lambda-1}\mathcal{F}(\theta)\mathrm{d}\theta \tag{4}$$

Lemma 1. [27] A function $\mathcal{F} \in C[\alpha, \beta]$ is the solution:

$$\mathbf{T}_\lambda^\alpha \mathcal{F}(\nu) = \phi(\nu, \mathcal{F}(\nu)), \ 0 < \lambda \le 1 \qquad \mathcal{F}(0) = \mathcal{F}_0 \tag{5}$$

$\Leftarrow$ it satisfies the integral equation:

$$\mathcal{F}(\nu) = \mathcal{F}_0 + I_\lambda^\alpha \phi(\nu, \mathcal{F}(\nu)), \ \nu \in [\alpha, \beta] \tag{6}$$

Let $X = C(\Delta)$ denotes Banach space with the norm:

$$\|\mathcal{F}\| = \sup_{\nu\in\Delta}\{|\mathcal{F}|\}, \text{ for all } \mathcal{F} \in X$$

Theorem 1. [36] Krasnoselskii's fixed theorem: Consider Λ be a closed, convex, non-empty subset of X, and assume two operators A and B such that:

– $A\mathcal{F} + B\mathcal{F} \in \Lambda \ \forall \ \mathcal{F} \in \Lambda$.
– A denotes contraction, whereas B denotes compactness and continuity.

Then at least one $\mathcal{F} \in E$ solution exists:

$$A\mathcal{F} + B\mathcal{F} \in \mathcal{F} \tag{7}$$

Main results

We are attempting to build a qualitative theory for our studied problem in this section *Fundametal materialis*. The functional analysis theorems of Banach and Krasnoselskii are used in our study. The equivalent integral equation of *Problem 2* may be easily obtained using *Lemma 1*:

$$\mathcal{F}(\nu) = \mathcal{F}_0 + \psi(\mathcal{F}) + I_\lambda^\alpha \phi(\nu, \mathcal{F}(\nu), \ \mathcal{F}(\gamma\nu)), \ \nu \in \Delta \tag{8}$$

Suppose $X = C(\Delta)$ is a Banach space and define operator H: $X \longrightarrow X$ such that:

$$H\mathcal{F} = A\mathcal{F} + B\mathcal{F} \tag{9}$$

where

$$A = \mathcal{F}_0 + \psi(\mathcal{F}), \ \nu \in \Delta \tag{10}$$

$$B\mathcal{F} = I_\lambda^\alpha \phi(\nu, (\mathcal{F}\nu), \mathcal{F}(\gamma\nu)), \ \nu \in \Delta \tag{11}$$

Consider that the following outcomes are correct:

– There exist $K_1 \in [0,1) \ni$

$$\| \psi(\mathcal{F}_1) - \psi(\mathcal{F}_2) \| \le K_1 \| \mathcal{F}_1 - \mathcal{F}_2 \| \text{ for every } (\nu, \mathcal{F}_1), (\nu, \mathcal{F}_2) \in \Delta \times R$$

– There exist c, d, and $c_\phi \ge 0$, $c_2 \in [0,1) \ni$

$$\left|\phi(\nu, \mathcal{F}(\nu), \mathcal{F}(\gamma\nu))\right| \le c\left[\mathcal{F}\left|(\nu)\right|^{c_2} + \mathcal{F}\left|(\gamma\nu)\right|^{c_2}\right] + d \le c_\phi \mid \mathcal{F}(\nu) \mid^{c_2} + d, \text{ where } c_\phi = 2c$$

$$\forall (\nu, \mathcal{F}(\nu), \ \mathcal{F}(\gamma\nu) \in \Delta \times R \times R$$

– There exist constants $L_\nu > 0$, $L_\phi > 0 \ni$

$$\left|\phi(\nu, \bar{\mathcal{F}}(\nu), \bar{\mathcal{F}}(\gamma\nu)) - \phi(\nu, \mathcal{F}(\nu), \mathcal{F}(\gamma\nu))\right| \le L_\nu \left[\bar{\mathcal{F}}\left|(\nu) - \mathcal{F}(\nu)\right| + \left|\bar{\mathcal{F}}(\gamma\nu) - \mathcal{F}(\gamma\nu)\right|\right] \le$$

$$\le L_\phi \left|\bar{\mathcal{F}}(\nu) - \mathcal{F}(\nu)\right|, \text{ where } L_\phi = 2L_\nu, \ \forall \ \nu \in \Delta, \text{ and } \bar{F}, \ F \in R$$

The previous results are key to our analysis.

Theorem 2. The eq. (2) has at least on solution, under the assumptions (*i*) and (*ii*).

Proof 1. Let, $\bar{\mathcal{F}}$, $\mathcal{F} \in E$ where

$$E = \left\{\mathcal{F} \in E : \|\mathcal{F}\| \le \sigma, \sigma > 0\right\}$$

the set is closed convex, then:

$$\left\|A\bar{\mathcal{F}} - A\mathcal{F}\right\| = \sup_{\nu\in\Delta}\left|(A\bar{\mathcal{F}})(\nu) - (A\mathcal{F})(\nu)\right| \le \sup_{\nu\in\Delta}\left|\psi(\nu,\bar{\mathcal{F}}) - \psi(\nu,\mathcal{F})\right| = K_1\left\|\bar{\mathcal{F}} - \mathcal{F}\right\|$$

Hence, the letter A stands for contraction. Next, for each $\mathcal{F} \in E$, we have to show that B is compact and continuous:

$$\|B\mathcal{F}\| = \sup_{\nu\in\Delta}|B\mathcal{F}| = \left|\int_\alpha^\nu (\nu-\alpha)^{\lambda-1}\phi(\theta,\mathcal{F}(\theta),\mathcal{F}(\gamma\theta))\mathrm{d}\theta\right| \le$$

$$\le \int_\alpha^\nu \left|(\nu-\alpha)^{\lambda-1}\right|\left|\phi\big(\theta,\mathcal{F}(\theta),\mathcal{F}(\gamma\theta)\big)\right|\mathrm{d}\theta \le \frac{\beta^\lambda}{\lambda}\left[c_\phi\|\mathcal{F}\| + d\right]$$

Which shows B is bonded. Further, let ν_1, $\nu_2 \in \Delta$ such that $\nu_1 > \nu_2$, then:

$$\left|(B\mathcal{F})(\nu_1) - (B\mathcal{F})(\nu_2)\right| = \left|\int_\alpha^{\nu_1}(\nu_1-\alpha)^{\lambda-1}\phi(\theta,\mathcal{F}(\theta),\mathcal{F}(\gamma\theta))\mathrm{d}\theta - \int_\alpha^{\nu_2}(\nu_2-\alpha)^{\lambda-1}\phi(\theta,\mathcal{F}(\theta),\mathcal{F}(\gamma\theta))\mathrm{d}\theta\right| \le$$
$$\le \left|\int_\alpha^{\nu_1}(\nu_1-\alpha)^{\lambda-1}\mathrm{d}\theta - \int_\alpha^{\nu_2}(\nu_2-\alpha)^{\lambda-1}\mathrm{d}\theta\right|\left|\phi(\theta,\mathcal{F}(\theta),\mathcal{F}(\gamma\theta))\right| \le \left[\frac{c_\phi\|\mathcal{F}\| + d}{\lambda}\right]\left(\nu_1^\lambda - \nu_2^\lambda\right) \to 0 \text{ as } \nu_1 \to \nu_2 \tag{12}$$

Thus, B is continuous then by Arzel a' Ascoli theorem B is compact. As a result, there is at least one solution the relevant problem.

Theorem 3. Consider that there is a constant $r > 0$ that ensures:

$$r = \left(K_1 + \frac{L_\phi\beta^\lambda}{\lambda}\right) < 1 \tag{13}$$

then H has unique fixed point.

Proof 2. Thank to *Banach Theorem* for $\bar{\mathcal{F}}$, $\mathcal{F} \in$ Z, take:

$$\left\|H\bar{\mathcal{F}} - H\mathcal{F}\right\| \le \left\|A\bar{\mathcal{F}} - A\mathcal{F}\right\| + \left\|B\bar{\mathcal{F}} - B\mathcal{F}\right\| \le \left(K_1 + \frac{L_\phi\beta^\lambda}{\lambda}\right)\left\|\bar{\mathcal{F}} - F\right\| = r\left\|\bar{\mathcal{F}} - \mathcal{F}\right\|$$

Hence, H because of the Banach contraction, it has a single fixed point. As a result, the eq. (2) has a unique solution.

Stability analysis

Finally, we examined Ulam type stability as a solution our problem. The definitions of UH stability recall are shown [35].

Let $\mathbf{Z}$: $X \to X$ be an operator satisfying:

$$\mathbf{Z}\mathcal{F} = \mathcal{F}, \text{ for } \mathcal{F} \in \mathrm{X} \tag{14}$$

Definition 3. To show eq. (14) is UH stable if for $\epsilon > 0$ and let $\mathcal{F} \in X$ be any kind of solution of inequality:

$$\left\| \mathcal{F} - \mathbf{Z}\mathcal{F} \right\| \le \epsilon, \text{ for } \nu \in \Delta \tag{15}$$

$\exists$ unique solution $\bar{\mathcal{F}}$ of eq. (14) with $\mathcal{C}_q > 0$ satisfying:

$$\left\| \bar{\mathcal{F}} - \mathcal{F} \right\| \le \mathcal{C}_q \epsilon, \ \nu \in \Delta \tag{16}$$

Definition 4. Let $\mathcal{F}$ of inequality (15) and $\bar{\mathcal{F}}$ be unique solution of eq. (14), if $\exists$ $\varphi \in \mathcal{C}(R, R)$ with $\varphi(0) = 0$ such that:

$$\left\| \bar{\mathcal{F}} - \mathcal{F} \right\| \le \varphi(\epsilon) \tag{17}$$

then eq. (14) is GUH stable.

Remark 1. If $\exists\ \zeta(\nu) \in \mathcal{C}(\Delta, R)$, then $\bar{\mathcal{F}} \in X$ satisfy inequality (15):

$$(i) \quad \zeta(\nu) \le \epsilon, \ \forall \ \nu \in \Delta$$

$$(ii) \quad Z\bar{\mathcal{F}}(\nu) = \bar{\mathcal{F}} + \zeta(\nu), \ \forall \ \nu \in \Delta$$

The following relationship is required for further investigation. Consider the resulting perturbation problem (2) as:

$$\begin{gathered} \mathbf{T}_\lambda^\alpha \mathcal{F}(\nu) = \phi(\nu, \mathcal{F}(\nu), \mathcal{F}(\gamma\nu)) + \zeta(\nu), \nu \in \Delta, \ 0 < \lambda \le 1 \\ \mathcal{F}(0) = \mathcal{F}_0 + \psi(\mathcal{F}) \end{gathered} \tag{18}$$

Lemma 2. For perturbation of eq. (18), the conclusion is true:

$$\left| \mathcal{F}(\nu) - H\mathcal{F}(\nu) \right| \le \frac{\beta^\lambda}{\lambda} \epsilon \tag{19}$$

Proof 3. The result carried out with the help of *Lemma 1*, eq. (8) and *Remark 1*.

Theorem 4. By choosing *Lemma 2* and *Theorem 3* the solution the problem under consideration (2) is UH stable and GUH stable if $r < 1$.

Proof 4.

$$\left\| \mathcal{F} - \bar{\mathcal{F}} \right\| = \sup_{\nu \in \Delta} \left| \mathcal{F} - H(\bar{\mathcal{F}}) \right| \le \sup_{\nu \in \Delta} \left| \mathcal{F} - H(\mathcal{F}) \right| + \sup_{\nu \in \Delta} \left| H(\mathcal{F}) - H(\bar{\mathcal{F}}) \right| \le \frac{\beta^\lambda}{\lambda} \epsilon + r \left\| \mathcal{F} - \bar{\mathcal{F}} \right\| \le \frac{\beta^\lambda \epsilon}{\lambda(1-r)}$$

Hence, Problem (2) is UH stable. As a result, it is GUH stable.

Definition 5. Consider $\mathcal{F} \in X$, the eq. (14) is UHR stable for $\chi \in \mathcal{C}[\Delta, R]$, if for $\epsilon > 0$ if the inequality holds:

$$\left\| \mathcal{F} - \mathbf{Z}\mathcal{F} \right\| \le \chi(\nu)\epsilon, \text{ for } \nu \in \Delta \tag{20}$$

$\exists$ unique solution $\bar{\mathcal{F}}$ of eq. (14) with $\mathcal{C}_q > 0$ satisfying:

$$\left\| \bar{\mathcal{F}} - \mathcal{F} \right\| \le \mathcal{C}_q \chi(\nu)\epsilon, \ \forall \ \nu \in \Delta \tag{21}$$

Definition 6. Let $\mathcal{F}$ be any solution of inequality (20) and $\bar{\mathcal{F}}$ be unique solution of eq. (14) $\ni$ for $\chi \in \mathcal{C}[\Delta, R]$ if $\exists\ \mathcal{C}_{q\chi}$ and for $\epsilon > 0$:

$$\left\| \bar{\mathcal{F}} - \mathcal{F} \right\| \le \mathcal{C}_{q,\chi} \chi(\nu), \ \forall \nu \in \Delta \tag{22}$$

then eq. (14) is GUHR stable.

Remark 2. If ∃ $\zeta(\nu) \in \mathcal{C}(\Delta, R)$, then $\bar{\mathcal{F}} \in X$ satisfy eq. (20):

$$(i) \quad \left|\zeta(\nu)\right| \le \epsilon\chi(\nu), \quad \forall \quad \nu \in \Delta$$

$$(ii) \quad \mathbf{Z}\bar{\mathcal{F}}(\nu) = \bar{\mathcal{F}} + \zeta(\nu), \quad \forall \quad \nu \in \Delta$$

Lemma 3. For perturbation of eq. (18) the conclusion is true:

$$\left|\mathcal{F}(\nu) - H\mathcal{F}(\nu)\right| \le \frac{\beta^{\lambda}}{\lambda}\chi(\nu)\epsilon \tag{23}$$

Proof 5. Using the *Lemma 1*, eq. (8), and observation (2), one may easily obtain the deired relation.

Theorem 5. Under the *Lemma 3* and *Theorem 3* the solution of the proposed Problem (2) is UHR and GUHR stable if $r < 1$.

Proof 6.

$$\left\|\mathcal{F} - \bar{\mathcal{F}}\right\| = \sup_{\nu\in\Delta} |\mathcal{F} - H(\bar{\mathcal{F}})| \le \sup_{\nu\in\Delta} |\mathcal{F} - H(\mathcal{F})| + \sup_{\nu\in\Delta} |H(\mathcal{F}) - H(\bar{\mathcal{F}})| \le$$
$$\le \frac{\beta^{\lambda}}{\lambda}\epsilon + r \,\|\, \mathcal{F} - \bar{\mathcal{F}} \,\| \le$$
$$\le \frac{\beta^{\lambda}\chi(\nu)\epsilon}{\lambda(1-r)}$$

Hence, Problem (2) is UHR stable. Consequently it is GUHR stable.

Example 1.

$$\mathbf{T}_{1/2}^{0}\mathcal{F}(\nu) = \nu^{3/4} + \frac{1}{b}\left(\frac{\mathcal{F}(\nu)}{1+\left|\mathcal{F}(\nu)\right|^{1/2}} + \frac{\mathcal{F}(\gamma\nu)}{1+\left|\mathcal{F}(\gamma\nu)\right|^{1/2}}\right), \ \nu \in [0,1]$$
$$\mathcal{F}(0) = \mathcal{F}_0 + \frac{1}{a}\left(\frac{\mathcal{F}(\nu)}{1+\left|\mathcal{F}(\nu)\right|^{1/2}}\right), \ \text{where} \ a,b \in R^{+} \tag{24}$$

where

$$\psi(\mathcal{F}) = \frac{1}{a}\left(\frac{\mathcal{F}(\nu)}{1+\left|\mathcal{F}(\nu)\right|^{1/2}}\right), \ \phi\big(\nu, \mathcal{F}(\nu), \mathcal{F}(\gamma\nu)\big) = \nu^{3/4} + \frac{1}{b}\left(\frac{\mathcal{F}(\nu)}{1+\left|\mathcal{F}(\nu)\right|^{1/2}} + \frac{\mathcal{F}(\gamma\nu)}{1+\left|\mathcal{F}(\gamma\nu)\right|^{1/2}}\right)$$

Now:

$$\left|\psi(\mathcal{F}) - \psi(\bar{\mathcal{F}})\right| = \frac{1}{a}\left|\frac{\mathcal{F}(\nu)}{1+\left|\mathcal{F}(\nu)\right|^{1/2}} - \frac{\bar{\mathcal{F}}(\nu)}{1+\left|\bar{\mathcal{F}}(\nu)\right|^{1/2}}\right| \le \frac{1}{a}\left|\mathcal{F} - \bar{\mathcal{F}}\right| \cdot$$

$$\cdot\left|\phi(\nu, \mathcal{F}(\nu), \mathcal{F}(\gamma\nu)) - \phi(\nu, \bar{\mathcal{F}}(\nu), \bar{\mathcal{F}}(\gamma\nu))\right| = \left|\frac{\mathcal{F}(\nu)}{1+|\,\mathcal{F}(\nu)\,|^{1/2}} + \frac{\mathcal{F}(\gamma\nu)}{1+|\,\mathcal{F}(\gamma\nu)\,|^{1/2}}\right| -$$
$$-\left(\frac{\bar{\mathcal{F}}(\nu)}{1+|\,\bar{\mathcal{F}}(\nu)\,|^{1/2}} + \frac{\bar{\mathcal{F}}(\gamma\nu)}{1+|\,\bar{\mathcal{F}}(\gamma\nu)\,|^{1/2}}\right) \le \frac{2}{b}\left|\mathcal{F} - \bar{\mathcal{F}}\right| \cdot \left|\phi(\nu, \mathcal{F}(\nu), \mathcal{F}(\gamma\nu))\right| \le \frac{2}{b}\left|\mathcal{F}\right|^{1/2} + 1$$

Thus ϕ, ψ satisfy condition (*i*)-(*iii*) for

$$\lambda = \frac{1}{2},\ \Delta = [0,1],\ K_1 = \frac{1}{a},\ c_1 = \frac{1}{2},\ L_\phi = c_\phi = \frac{2}{b},\ \text{and}\ d = 1$$

Hence, according to the *Theorem 2* the Problem (24) has at least a solution.

Uniqueness: Since ϕ satisfies condition (*iii*) for $L_\phi = 2/b$ and let $a = 10$, $b = 20$ then:

$$r = 0.3 < 1$$

Hence Problem (24) possess uniqueness solution. The Problem (24) is *UH, GUH, UHR,* and *GUHR* stable, since $r < 1$. The proof is easy, so we left for the reader.

Conflict of interest

No conflict of interest exist.

Autors contribution

Authors have equal contribution.

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