# STRUCTURE OF THE ANALYTIC SOLUTIONS FOR THE COMPLEX NON-LINEAR (2+1)-DIMENSIONAL CONFORMABLE TIME-FRACTIONAL SCHRÖDINGER EQUATION

by

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*In this article, the non-linear (2+1)-dimensional conformable time-fractional Schrödinger equation of order α where $0 < \alpha \leq 1$, has been studied within introducing an appropriate fractional traveling wave transformation. The reliable and powerful method, namely the Improved Bernoulli sub equation function method, is applied to investigate some solitary wave, traveling wave and periodic solutions to the aforementioned model which is crucial significance because the model is in the fields of quantum mechanics and energy spectrum. The obtained solutions are new and significant in revealing the pertinent features of the physical phenomenon. Moreover, gotten solutions have been plotted in several kinds, such as in 3-D or 2-D. The impacts of the time evolution are offered in 2-D graphs for visual observation of the properties of the solutions.*

Key words: *Schrödinger equation, conformable fractional derivative, improved bernoulli sub equation function method, analytic solutions, traveling wave*

## Introduction

In applied mathematics, it's conspicuous that fractional differential equations are the generalizations of classical differential equations with integer orders. In recent years, fractional non-linear differential equations have played a consequential role in different research areas, such as wave models, which have gained much attention in several other areas like signal processing, medical processes, mechanics, engineering, fluid, control theory, biology systems, and many others [1-3]. The search for different type solutions of the time-fractional non-linear Schrödinger's model has been represented by numerous scientists and researchers [4-7]. Comparison by the different fractional non-linear wave models, the time-fractional Schrödinger differential equation that proposed by Laskin [3], is one of crucial significance because of the model is in the scope of quantum mechanics and its provides the most general framework for understanding the relationship between the statistical properties of the quantum mechanical path and the structure of the fundamental equations of quantum mechanics. Various powerful and reliable methods have been proposed to obtain the exact analytic and solitary wave solutions of the mathematical models and fractional differential equations such as the Adomian

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decomposition method [8], experimenting of the tanh-coth method in solving some cases of space-time fractional derivative [9], the fractional Adams-Bashforth-Moulton method [10], the Fourier transform method [11], the fractional mapping expansion method [12], the reproducing kernel algorithm [13], the Darboux transform method [14, 15], the reproducing kernel Hilbert space method [16], these methods that contains $G'/G$-expansion method [17-19], the exponential function method [20-23], the Laplace transformation method in the sense of Caputo fractional derivative [24], the tanh-function and the extended tanh-function methods [25, 26], the improved sub-equation method [27], new generalized exponential rational function method [28], the Bernoulli sub-equation function method [29, 30], the Improved Bernoulli sub-equation function method [31], and so on.

The complex non-linear (2+1)-D conformable time-fractional Schrödinger equation:

$$iD_t^\alpha u + Au_{xx} - Bu_{yy} + Cu|u|^2 = 0, \text{ where } 0 < \alpha \le 1, \ i = \sqrt{-1} \tag{1}$$

for the parameters suppose $A = \beta_1$, $B = \beta_2$, and $C = \beta_3$ then eq. (1) become:

$$iD_t^\alpha u + \beta_1 u_{xx} - \beta_2 u_{yy} + \beta_3 u|u|^2 = 0 \tag{2}$$

where $D_t^\alpha(\cdot)$ is the conformable fractional derivative of order $\alpha$.

The main goal of this paper is to study the eq. (1) via using an analytic method, namely the Improved Bernoulli sub equation function method.

## Preliminaries of the conformablef fractional derivative

In spite of the fact that the preceding definitions of fractional derivatives are linear and possess some classical properties, but some inconsistencies are proposed to this definitions of fractional derivatives since they do not share all the properties similar to the first classical derivative. Recently, a new fractional derivative has been developed whose properties coincide with the classical derivative [32], this local fractional derivative is well-behaved and meets all the properties of the first classical derivative this derivative is the conformable fractional derivative. This new operation has been used to study Newtonian mechanics [33], electrical circuits described [34], quantum mechanics [35], and so on. The implementation of the conformable fractional derivatives is easier and quite productive. The conformable fractional derivatives of order $\alpha$ is defined in [32, 36-39]

*Definition 1*. Given function $\mathcal{F}$:[0, ∞) → ℝ. Then the conformable fractional derivative of $\mathcal{F}$ of order $\alpha$ is defined by

$$\mathcal{D}_\alpha(\mathcal{F})(t) = \lim_{\varepsilon \to 0} \frac{\mathcal{F}\left(t + \varepsilon t^{1-\alpha}\right) - \mathcal{F}(t)}{\varepsilon} \tag{3}$$

for all $t > 0$, $\alpha \in (0, 1)$. It can be said the function $\mathcal{F}$ is $\alpha$-conformable differentiable at a point $t > 0$, when the limit in the eq. (3) exists.

One can easily show that $\mathcal{D}_\alpha$ satisfies all the properties in the following theorem.

*Theorem 2*. Let $\alpha \in (0, 1]$ and $\mathcal{F}$, $\mathcal{H}$ be $\alpha$-differentiable at a point $t > 0$. Then:

1. $\mathcal{D}_\alpha(a\mathcal{F} + b\mathcal{H}) = a\mathcal{D}_\alpha(\mathcal{F}) + b\mathcal{D}_\alpha(\mathcal{H}), \ a, b \in \mathbb{R}$.
2. $\mathcal{D}_\alpha(t^q) = qt^{q-\alpha}$ for all $q \in \mathbb{R}$.
3. $\mathcal{D}_\alpha(\kappa) = 0$ for all constant function $\mathcal{F}(t) = \kappa$.
4. $\mathcal{D}_\alpha(\mathcal{F}\mathcal{H}) = \mathcal{F}\mathcal{D}_\alpha(\mathcal{H}) + \mathcal{H}\mathcal{D}_\alpha(\mathcal{F})$.
5. $\mathcal{D}_\alpha\left(\frac{\mathcal{F}}{\mathcal{H}}\right) = \frac{\mathcal{F}\mathcal{D}_\alpha(\mathcal{H}) - \mathcal{H}\mathcal{D}_\alpha(\mathcal{F})}{\mathcal{H}^2}$. Where $\mathcal{H}$ is different from zero.

6. If, in addition, $\mathcal{F}$ is differentiable, then

$$\mathcal{D}_\alpha(\mathcal{F})(t) = t^{1-\alpha}\frac{d\mathcal{F}}{dt}t$$

*Proof.* The prove was omitted, for the detail one can see [32]

*Theorem 3*. Let $\mathcal{F}$ and $\mathcal{H}$ be two differentiable function, such that $\mathcal{H}$ is differentiable at any $t$, and $\mathcal{F}$ is differentiable at any $\mathcal{H}(t)$, and then the conformable derivative obeys the Chain rule, meaning:

$$\mathcal{D}_\alpha((\mathcal{F}\circ\mathcal{H})(x)) = x^{1-\alpha}\mathcal{H}(x)^{1-\alpha}\mathcal{H}'(x)\mathcal{D}_\alpha(\mathcal{F}(t))|_{t=\mathcal{H}(x)} \tag{4}$$

*Proof.* The prove was omitted, for the detail one can see [36].

Aforementioned theorems have a main role in transforming conformable (2+1)-D time-fractional Schrödinger equation given in eq. (1) into a non-linear ODE.

## Improved Bernoulli sub-equation function method (IBSEFM)

Improved Bernoulli sub-equation function method (IBSEFM) is outlined in four succeeding steps.

*Step 1*. Consider the following non-linear fractional differential equation (NLPDE) with $u = u(x, y, t)$ and $D_t^\alpha$ is a conformable fractional derivative:

$$\phi\left(D_t^\alpha u, u, u_x, u_t, u_y, u_{yy}, u_{xx}, \cdots\right) = 0 \tag{5}$$

by setting the fractional wave transformation:

$$u(x,y,t) = U(\xi),\ \xi = \delta_1 x + \delta_2 y + \frac{\delta_3}{\alpha}t^\alpha \tag{6}$$

where $\delta_1$, $\delta_2$, $\delta_3$ are arbitrary non-zero constants and $\delta_3/\alpha$ represent a wave number. By installing eq. (6) into eq. (5) the following complex non-linear ordinary differential equation (NLODE) is obtained:

$$\Phi\left(U, iU', U'', U^2, \cdots\right) = 0 \tag{7}$$

where

$$U = U(\xi),\ U' = \frac{\mathrm{d}U}{\mathrm{d}\xi},\ U'' = \frac{\mathrm{d}^2U}{\mathrm{d}\xi^2}, \cdots$$

where ($'$) is the classical derivative of $U$ with respect to $\xi$.

*Step 2*. Assuming that trail solutions of eq. (7) are in the form:

$$U(\xi) = \frac{\sum_{i=0}^{n} a_i F^i}{\sum_{j=0}^{m} b_j F^j} = \frac{a_0 + a_1F + a_2F^2 + \cdots + a_nF^n}{b_0 + b_1F + b_2F^2 + \cdots + b_mF^m} \tag{8}$$

where

$$F' = bF + dF^M,\ b \neq 0,\ d \neq 0,\ M \in \mathbb{R} - \{0,1,2\} \tag{9}$$

where $F(\xi)$ is the well known Bernoulli differential equation, also $b$, $d$, and $a_i$, $b_i$ with $a_n$, $b_m \neq 0$, should be determining later. Subbing eq. (8), with eq. (9), into eq. (7), one immediately gets:

$$\varphi(F(\xi)) = \mu_s F(\xi)^s + \cdots + \mu_1 F(\xi) + \mu_0 = 0 \tag{10}$$

Applying balancing principles, one start getting a formula for $n$, $m$ and $M$ by comparing the highest order derivatives with the higher degree in the non-linear term.

*Step 3*. Let the coefficients of $\varphi(F(\xi))$ equal to zero, then one gets the algebraic system of equations:

$$\mu_i = 0, \;\; i = 0, \cdots, s \tag{11}$$

By solving system (11), one gets the values of $a_0 ..., a_n, b_0, ..., b_m$, and hence solutions of eq. (7) are found.

*Step 4*. Manually solving eq. (9), gives these two forms of solutions:

$$F(\xi) = \left[ -\frac{d}{b} + \frac{\delta}{\mathrm{e}^{b(M-1)\xi}} \right]^{\frac{1}{1-M}}, \;\; b \neq d \tag{12}$$

and

$$F(\xi) = \left\{ \frac{(\delta - 1) + (\delta + 1)\tanh\left[\dfrac{b(1-M)\xi}{2}\right]}{1 - \tanh\left[\dfrac{b(1-M)\xi}{2}\right]} \right\}^{\frac{1}{1-M}}, \;\; b = d, \;\; \delta \in \mathbb{R}$$

combining founded parameters in the previous step commonly with eq. (12), solutions of eq. (1) are obtaining.

## Applications of the described method and results

Applying fractional wave transformation:

$$u = u(x, y, t) = U(\xi) \;\text{ where }\; \xi = \delta_1 x + \delta_2 y + \frac{\delta_3}{\alpha} t^\alpha$$

inserting $u_{xx}$, $u_{yy}$, and $D_t^\alpha u$ into eq. (2) we obtain:

$$i\delta_3 U_\xi + \beta_1 \delta_1^2 U_{\xi\xi} - \beta_2 \delta_2^2 U_{\xi\xi} + \beta_3 U |U|^2 = 0 \tag{13}$$

suppose solution of the eq. (13) take the form

$$U(\xi) = \mathrm{e}^{ik\xi} V(\xi)$$

where $V(\xi)$ is the real function and $k$ – the constant, then one obtain:

$$i\delta_3 \left(ikV + V_\xi\right) + \left(\beta_1 \delta_1^2 - \beta_2 \delta_2^2\right)\left(V_{\xi\xi} + 2ikV_\xi - k^2 V\right) + \beta_3 V^3 = 0 \tag{14}$$

After resetting eq. (14) it can be transformed through dissevering the real and complex parts into the two equations:

$$i\delta_3 V_\xi + 2i\left(\beta_1 \delta_1^2 - \beta_2 \delta_2^2\right) k V_\xi = 0 \tag{15}$$

and

$$\left(\beta_1 \delta_1^2 - \beta_2 \delta_2^2\right) V_{\xi\xi} + \left[\left(\beta_2 \delta_2^2 - \beta_1 \delta_1^2\right) k^2 - \delta_3 k\right] V + \beta_3 V^3 = 0 \tag{16}$$

from eq. (15) one gets:

$$k=\frac{\delta_3}{2\left(\beta_2\delta_2^2-\beta_1\delta_1^2\right)} \tag{17}$$

inserting eq. (17) into eq. (16) after simplifications, we'll obtain a non-linear ODE:

$$4\left(\beta_1\delta_1^2-\beta_2\delta_2^2\right)^2 V_{\xi\xi}+\delta_3^2 V+4\beta_3\left(\beta_1\delta_1^2-\beta_2\delta_2^2\right)V^3=0 \tag{18}$$

take the balance principle between the highest order and a higher degree of the non-linear term in eq. (18) get a vital relation between $n$, $m$, and $M$:

$$n=M+m-1 \tag{19}$$

For the proposal of the values of positive integers in eq. (19) we have the following cases.

*Case 1.* If $m=1$ and $M=3$ then $n=3$ so eqs. (8) and (9) becomes:

$$V(\xi)=\frac{\sum\limits_{i=0}^{3}a_iF^i}{\sum\limits_{j=0}^{1}b_jF^j}=\frac{a_0+a_1F+a_2F^2+a_3F^3}{b_0+b_1F} \tag{20}$$

and

$$F'=bF+dF^3,\ b\neq 0,\ d\neq 0 \tag{21}$$

$$V'=\left(bF+dF^3\right)\left[\frac{a_1+2a_2F+3a_3F^2}{b_0+b_1F}-\frac{b_1\left(a_0+a_1F+a_2F^2+a_3F^3\right)}{\left(b_0+b_1F\right)^2}\right]=\frac{\Psi(F)}{\Omega(F)} \tag{22}$$

notice that there should be at least $a_3\neq 0$, $b_1\neq 0$, one can find:

$$V''=\frac{\Omega(F)\Psi'(F)-\Psi(F)\Omega'(F)}{[\Omega(F)]^2} \tag{23}$$

By installing eqs. (20)-(23) into eq. (18) we obtain an algebraic system of equations to eq. (18), via applying some computer software programs, one able to gets the following sub cases.

*Case 1.1* For $b\neq d$, the following coefficients are obtained:

$$\beta_3=\frac{8b_1^2d^2\left(\beta_2\delta_2^2-\beta_1\delta_1^2\right)}{a_3^2},\ a_0=\frac{a_3bb_0}{2b_1d},\ a_1=\frac{a_3b}{2d},\ a_2=\frac{a_3b_0}{b_1},\ \delta_3=-2\sqrt{2}b\left(\beta_2\delta_2^2-\beta_1\delta_1^2\right) \tag{24}$$

with the mentioned values of parameters in eq. (24) the solution are gained:

$$u_{1,1}(x,y,t)=\frac{a_3b\left(d\mathrm{e}^{2b\xi}+b\delta\right)}{2b_1d\left(b\delta-d\mathrm{e}^{2b\xi}\right)}\exp\left\{-\frac{ib\left[4b\left(\beta_1\delta_1^2-\beta_2\delta_2^2\right)t^\alpha+\sqrt{2}\alpha\left(\delta_1x+\delta_2y\right)\right]}{\alpha}\right\} \tag{25}$$

where

$$\xi=\delta_1x+\delta_2y+\frac{2\sqrt{2}b\left(\beta_1\delta_1^2-\beta_2\delta_2^2\right)}{\alpha}t^\alpha$$

*Case 1.2* If $b \neq d$, then following coefficients are obtained:

$$d = \frac{a_3\sqrt{\beta_3}}{2\sqrt{2}\sqrt{b_1^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}}, \quad a_0 = \frac{\sqrt{2}bb_0\sqrt{b_1^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}}{b_1\sqrt{\beta_3}}$$
$$a_1 = \frac{\sqrt{2}b\sqrt{b_1^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}}{\sqrt{\beta_3}}, \quad a_2 = \frac{a_3 b_0}{b_1}; \delta_3 = 2\sqrt{2}b\left(\beta_1\delta_1^2 - \beta_2\delta_2^2\right) \tag{26}$$

with the referenced values in eq. (26):

$$u_{1,2}(x,y,t) = \frac{1}{b_1}e^{-i\sqrt{2}b\xi}\left[\frac{a_3}{\delta e^{-2b\xi} - \dfrac{a_3\sqrt{\beta_3}}{2\sqrt{2}b\sqrt{b_1^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}}} + \frac{\sqrt{2}b\sqrt{b_1^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}}{\sqrt{\beta_3}}\right] \tag{27}$$

where

$$\xi = \delta_1 x + \delta_2 y + \frac{2\sqrt{2b}\left(\beta_1\delta_1^2 - \beta_2\delta_2^2\right)}{\alpha}$$

profile of the obtained solution in eq. (27) where
$\delta = -\frac{4}{3}$, $\delta_1 = \frac{2}{3}$, $\delta_2 = \frac{3}{4}$, $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{1}{3}$, $\beta_3 = \frac{2}{5}$, $b = \frac{2}{5}$, $b_1 = \frac{3}{7}$, $a_3 = \frac{5}{2}$, $y = \frac{3}{2}$ and $\alpha = \frac{1}{2}$

are graphed:

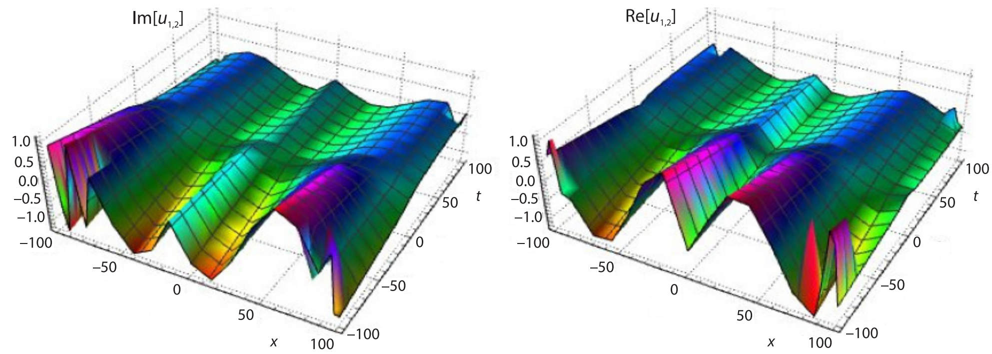


**Figure 1. The 3-D figures for eq. (27) where the values $-100 \leq x \leq 100$, $-100 \leq t \leq 100$**

For the following 2-D graphs values of $t$ are given in the legend:

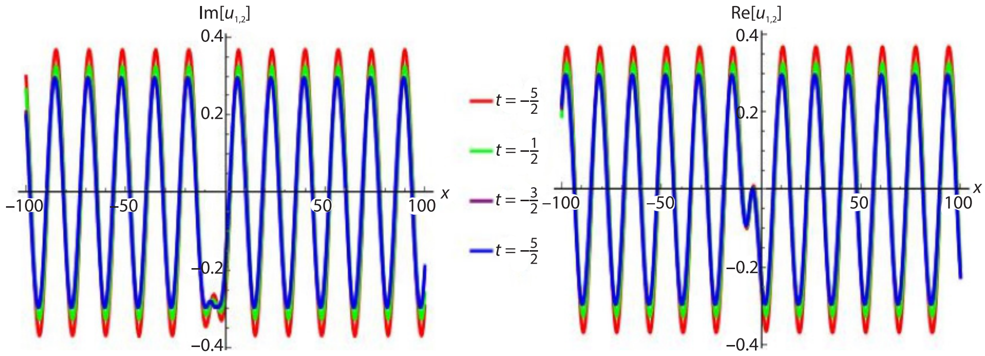


**Figure 2. The 2-D figures for eq. (27) where the values –100 ≤ *x* ≤ 100**

*Case 1.3* If $b \neq d$, then we'll get the coefficients:

$$\beta_2 = \frac{a_0^2\beta_3 + 2b^2b_0^2\beta_1\delta_1^2}{2b^2b_0^2\delta_2^2},\ a_1 = \frac{a_0a_3}{a_2},\ \delta_3 = -\frac{\sqrt{2}a_0^2\beta_3}{bb_0^2},\ d = \frac{a_2b}{2a_0},\ b_1 = \frac{a_3b_0}{a_2} \tag{28}$$

the considered parameters in eq. (28) are generate the solution:

$$u_{1,3}(x,y,t) = \frac{a_0\left(a_2\mathrm{e}^{2b\delta_1x+2b\delta_2y+\frac{2\delta_3t^\alpha}{\alpha}} + 2a_0\delta\right)\exp\left[\frac{2ia_0^2\beta_3t^\alpha}{\alpha b_0^2} - i\sqrt{2}b(\delta_1x+\delta_2y)\right]}{b_0\left(2a_0\delta - a_2\mathrm{e}^{2b\delta_1x+2b\delta_2y+\frac{2\delta_3t^\alpha}{\alpha}}\right)} \tag{29}$$

*Case 2*. If $m = 1$ and $M = 4$ then $n = 4$ so eqs. (8) and (9) are:

$$V(\xi) = \frac{\sum_{i=0}^{4}a_iF^i}{\sum_{j=0}^{1}b_jF^j} = \frac{a_0 + a_1F + a_2F^2 + a_3F^3 + a_4F^4}{b_0 + b_1F} \tag{30}$$

and

$$F' = bF + dF^4,\ b \neq 0,\ d \neq 0 \tag{31}$$

$$V' = \left(bF + dF^4\right)\left[\frac{a_1 + 2a_2F + 3a_3F^2 + 4a_4F^3}{b_0 + b_1F} - \frac{b_1\left(a_0 + a_1F + a_2F^2 + a_3F^3 + a_4F^4\right)}{(b_0 + b_1F)^2}\right] = \frac{\Psi(F)}{\Omega(F)} \tag{32}$$

there should be $a_4 \neq 0$, $b_1 \neq 0$, one can find:

$$V'' = \frac{\Omega(F)\Psi'(F) - \Psi(F)\Omega'(F)}{[\Omega(F)]^2} \tag{33}$$

By restoring eqs. (30)-(33) into eq. (18) we obtain an algebraic system of equations including coefficients from eq. (18), through the use of some computer software programs, one will get the following sub-cases.

*Case 2.1* For $b \neq d$, then we'll get the coefficients:

$$\beta_2 = \frac{6b\beta_1\delta_1^2 + \sqrt{2}\delta_3}{6b\delta_2^2},\ \beta_3 = \frac{3\sqrt{2}b_1^2 d^2\delta_3}{a_4^2 b},\ a_0 = \frac{a_4 b b_0}{2b_1 d},\ a_1 = \frac{a_4 b}{2d},\ a_2 = 0,\ a_3 = \frac{a_4 b_0}{b_1} \tag{34}$$

the declared values of parameters in eq. (34) are gives the solution:

$$u_{2,1}(x,y,t) = \frac{a_4 b \mathrm{e}^{\frac{3ib\left(\delta_3 t^\alpha + \alpha\delta_1 x + \alpha\delta_2 y\right)}{\sqrt{2}\alpha}}\left[d\mathrm{e}^{3b\left(\frac{\delta_3 t^\alpha}{\alpha} + \delta_1 x + \delta_2 y\right)} + b\delta\right]}{2b_1 d\left[b\delta - d\mathrm{e}^{3b\left(\frac{\delta_3 t^\alpha}{\alpha} + \delta_1 x + \delta_2 y\right)}\right]} \tag{35}$$

*Case 2.2*. If $b \neq d$, then we'll obtain the coefficients:

$$a_0 = -\frac{3ibb_0\sqrt{\beta_1\delta_1^2 - \beta_2\delta_2^2}}{\sqrt{2}\sqrt{\beta_3}},\ a_1 = -\frac{3ibb_1\sqrt{\beta_1\delta_1^2 - \beta_2\delta_2^2}}{\sqrt{2}\sqrt{\beta_3}},\ \delta_3 = -3\sqrt{2}b\left(\beta_1\delta_1^2 - \beta_2\delta_2^2\right)$$
$$a_3 = -\frac{3i\sqrt{2}b_0 d\sqrt{\beta_1\delta_1^2 - \beta_2\delta_2^2}}{\sqrt{\beta_3}},\ a_4 = -\frac{3i\sqrt{2}b_1 d\sqrt{\beta_1\delta_1^2 - \beta_2\delta_2^2}}{\sqrt{\beta_3}},\ a_2 = 0 \tag{36}$$

with the specified values in eq. (36) one gets the solution:

$$u_{2,2}(x,y,t) = \mathrm{e}^{\frac{3ib\left(\frac{\delta_3 t^\alpha}{\alpha} + \delta_1 x + \delta_2 y\right)}{\sqrt{2}}}\left\{-\frac{3ib\sqrt{\beta_1\delta_1^2 - \beta_2\delta_2^2}}{\sqrt{2}\sqrt{\beta_3}} - \frac{3i\sqrt{2}d\sqrt{\beta_1\delta_1^2 - \beta_2\delta_2^2}}{\sqrt{\beta_3}\left[\delta\mathrm{e}^{-3b\left(\frac{\delta_3 t^\alpha}{\alpha} + \delta_1 x + \delta_2 y\right)} - \frac{d}{b}\right]}\right\} \tag{37}$$

*Case 2.3*. If $b \neq d$, then we'll gets the coefficients:

$$\beta_1 = \frac{6b\beta_2\delta_2^2 + \sqrt{2}\delta_3}{6b\delta_1^2},\ d = -\frac{ia_4\sqrt{b}\sqrt{\beta_3}}{\sqrt[4]{2}\sqrt{3}b_1\sqrt{\delta_3}},\ a_2 = 0$$
$$a_1 = \frac{i\sqrt{3}\sqrt{b}b_1\sqrt{\delta_3}}{2^{3/4}\sqrt{\beta_3}},\ a_3 = \frac{a_4 b_0}{b_1},\ a_0 = \frac{i\sqrt{3}\sqrt{b}b_0\sqrt{\delta_3}}{2^{3/4}\sqrt{\beta_3}} \tag{38}$$

with the known parameters mentioned in eq. (38) we will obtain the solution:

$$u_{2,3}(x,y,t)=$$

$$=\frac{\exp\left[-\frac{3ib\left(\delta_3 t^{\alpha}+\alpha\delta_1 x+\alpha\delta_2 y\right)}{\sqrt{2}\alpha}\right]\left[3a_4\sqrt{b}\sqrt{\beta_3}\sqrt{\delta_3}\,\mathrm{e}^{3b\left(\frac{\delta_3 t^{\alpha}}{\alpha}+\delta_1 x+\delta_2 y\right)}+3\sqrt[4]{2}i\sqrt{3}bb_1\delta\delta_3\right]}{\sqrt{\beta_3}\left[6\sqrt{b}b_1\delta\sqrt{\delta_3}+2^{3/4}i\sqrt{3}a_4\sqrt{\beta_3}e^{3b\left(\frac{\delta_3 t^{\alpha}}{\alpha}+\delta_1 x+\delta_2 y\right)}\right]} \tag{39}$$

profiles of the obtained solution in eq. (39) where

$$\delta=-\frac{1}{2},\ b=\frac{3}{5},\ b_1=\frac{3}{10},\ \delta_1=\frac{3}{4},\ \delta_2=\frac{2}{3},\ \delta_3=\frac{5}{4},\ \beta_3=\frac{1}{2},\ a_4=\frac{2}{3},\ y=-\frac{1}{4}\ \text{and}\ \alpha=\frac{1}{4}$$

are outlined:

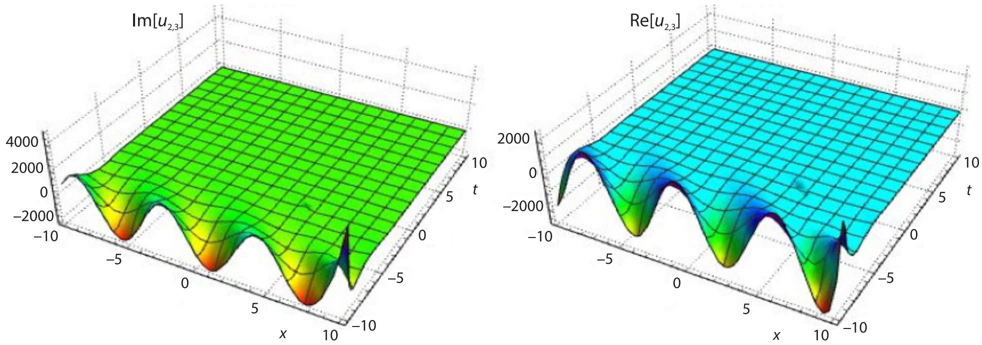


**Figure 3. The 3-D figures for eq. (39) where the values $-10 \le x \le 10$, $-10 \le t \le 10$**

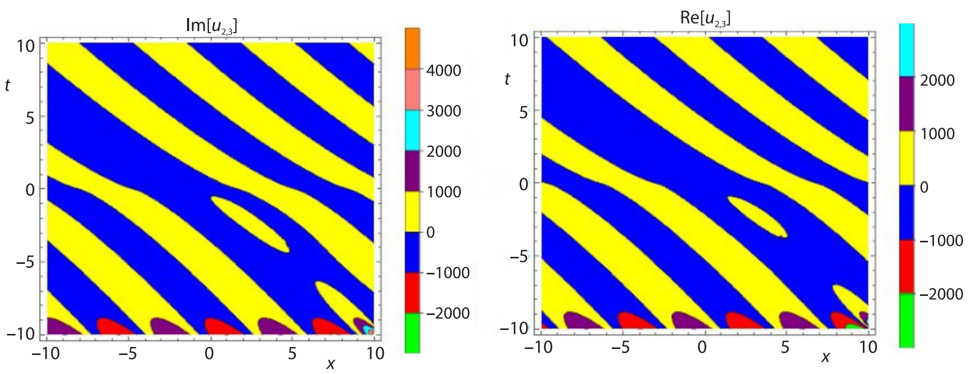


**Figure 4. Contour surfaces for eq. (39) where the values $-10 \le x \le 10$, $-10 \le t \le 10$**

In the following 2-D graphs values of $t$ are given in the legend:

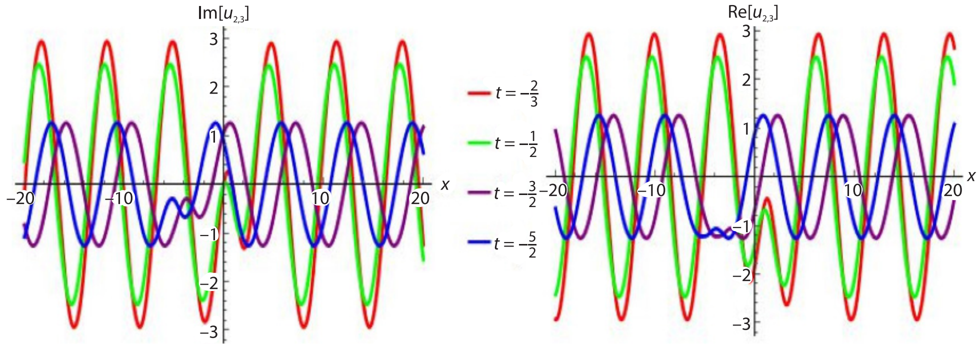


**Figure 5. The 2-D figures for eq. (39) where the values $-20 \le x \le 20$**

*Case 3*. If $m = 2$ and $M = 3$ then $n = 4$ so eq. (8) and eq. (9) are:

$$V(\xi)=\frac{\sum_{i=0}^{4}a_iF^i}{\sum_{j=0}^{2}b_jF^j}=\frac{a_0+a_1F+a_2F^2+a_3F^3+a_4F^4}{b_0+b_1F+b_2F^2} \tag{40}$$

and

$$F'=bF+dF^3,\ b\neq 0,\ d\neq 0 \tag{41}$$

$$V'=-\frac{\left(a_4F^4+a_3F^3+a_2F^2+a_1F+a_0\right)\left(2b_2F+b_1\right)F'}{\left(b_2F^2+b_1F+b_0\right)^2}+\frac{\left(4a_4F^3+3a_3F^2+2a_2F+a_1\right)F'}{b_2F^2+b_1F+b_0}=$$

$$=\frac{\Psi(F)}{\Omega(F)} \tag{42}$$

there should be $a_4 \neq 0$, $b_2 \neq 0$, one can find:

$$V''=\frac{\Omega(F)\Psi'(F)-\Psi(F)\Omega'(F)}{[\Omega(F)]^2} \tag{43}$$

By restoring eqs. (40)-(43) into eq. (18) we obtain an algebraic system of equations to eq. (18), applying some computer software programs, one will gets the following sub cases.

*Case 3.1* If $b \neq d$, then we'll get the coefficients:

$$\beta_3=\frac{8b_2^2d^2\left(\beta_2\delta_2^2-\beta_1\delta_1^2\right)}{a_4^2},\ a_1=\frac{a_4bb_1}{2b_2d},\ a_2=\frac{a_4b}{2d}$$
$$a_3=\frac{a_4b_1}{b_2},\ b_0=0,\ \delta_3=2\sqrt{2}b\left(\beta_2\delta_2^2-\beta_1\delta_1^2\right),\ a_0=0 \tag{44}$$

the declared values of parameters in eq. (44) are gives the solution:

$$u_{3,1}(x,y,t)=$$

$$=\frac{a_4\exp\left\{\dfrac{ib\left[\sqrt{2}\alpha\left(\delta_1 x+\delta_2 y\right)-4b\left(\beta_1\delta_1^2-\beta_2\delta_2^2\right)t^\alpha\right]}{\alpha}\right\}\left\{\dfrac{2d}{\delta\exp\left[-2b\left(\dfrac{\delta_3 t^\alpha}{\alpha}+\delta_1 x+\delta_2 y\right)\right]-\dfrac{d}{b}}+b\right\}}{2b_2 d} \tag{45}$$

profile of the obtained solution in eq. (45) where

$$\delta=-\frac{4}{3},\ y=-\frac{1}{2},\ b=\frac{3}{4},\ \delta_1=\frac{2}{3},\ \delta_2=\frac{1}{2},\ \beta_1=\frac{3}{2},\ \beta_2=\frac{2}{3},\ d=\frac{2}{5},\ a_4=\frac{5}{2},\ b_2=\frac{1}{2},\ \alpha=\frac{9}{10}$$

are figured in the following.

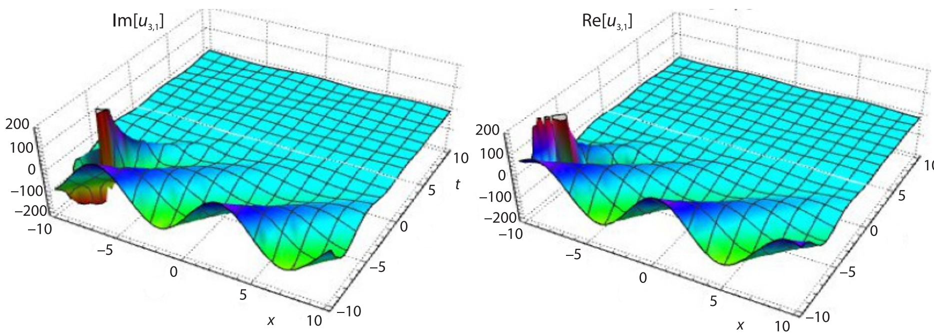


**Figure 6. The 3-D figures for eq. (45) where the values –10 ≤ *x* ≤ 10, –10 ≤ *t* ≤ 10**

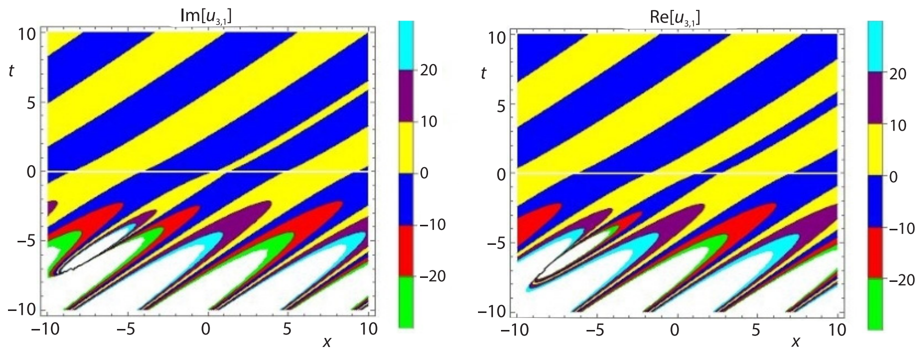


**Figure 7. Contour surfaces for eq. (45) where the values –10 ≤ *x* ≤ 10, –10 ≤ *t* ≤ 10**

For the following 2-D graphs values of $t$ mentioned in the legend.

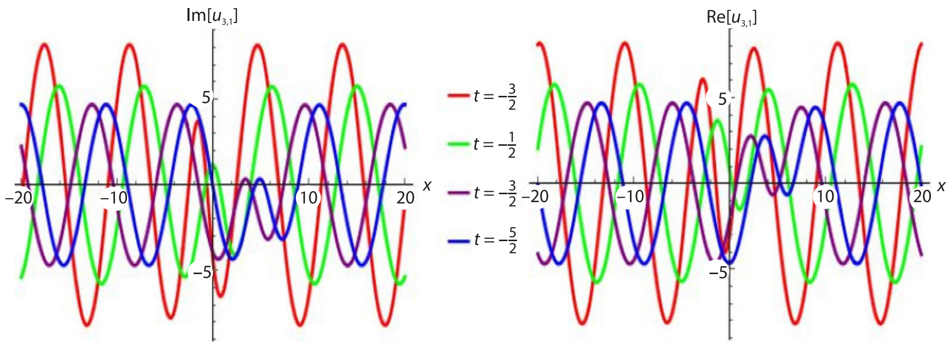


**Figure 8. The 2-D figures for eq. (45) where the values $-20 \le x \le 20$**

*Case 3.2* If $b \neq d$, then we'll get the coefficients:

$$\beta_2 = \frac{4b\beta_1\delta_1^2 - \sqrt{2}\delta_3}{4b\delta_2^2},\ a_1 = -\frac{i\sqrt{b}b_1\sqrt{\delta_3}}{\sqrt[4]{2}\sqrt{\beta_3}},\ a_2 = -\frac{i\sqrt{\delta_3}\left(2b_0 d + bb_2\right)}{\sqrt[4]{2}\sqrt{b}\sqrt{\beta_3}}$$
$$a_0 = -\frac{i\sqrt{b}b_0\sqrt{\delta_3}}{\sqrt[4]{2}\sqrt{\beta_3}},\ a_3 = -\frac{i2^{3/4}b_1 d\sqrt{\delta_3}}{\sqrt{b}\sqrt{\beta_3}},\ a_4 = -\frac{i2^{3/4}b_2 d\sqrt{\delta_3}}{\sqrt{b}\sqrt{\beta_3}} \tag{46}$$

the declared values of parameters in eq. (46) give the solution:

$$u_{3,2}(x,y,t) = \frac{i\sqrt{b}\sqrt{\delta_3}\,\mathrm{e}^{-\frac{i\sqrt{2}b\left(\delta_3 t^{\alpha} + \alpha\delta_1 x + \alpha\delta_2 y\right)}{\alpha}}\left[d\mathrm{e}^{2b\left(\frac{\delta_3 t^{\alpha}}{\alpha} + \delta_1 x + \delta_2 y\right)} + b\delta\right]}{\sqrt[4]{2}\sqrt{\beta_3}\left[d\mathrm{e}^{2b\left(\frac{\delta_3 t^{\alpha}}{\alpha} + \delta_1 x + \delta_2 y\right)} - b\delta\right]} \tag{47}$$

profile of the obtained solution in eq. (45) where

$$\delta = \frac{1}{2},\ b = \frac{3}{5},\ \delta_1 = \frac{1}{2},\ \delta_2 = \frac{3}{4},\ \delta_3 = \frac{2}{3},\ \alpha = \frac{1}{20},\ \beta_3 = \frac{1}{4},\ d = -\frac{2}{3},\ y = \frac{1}{3}$$

are figured in the following.

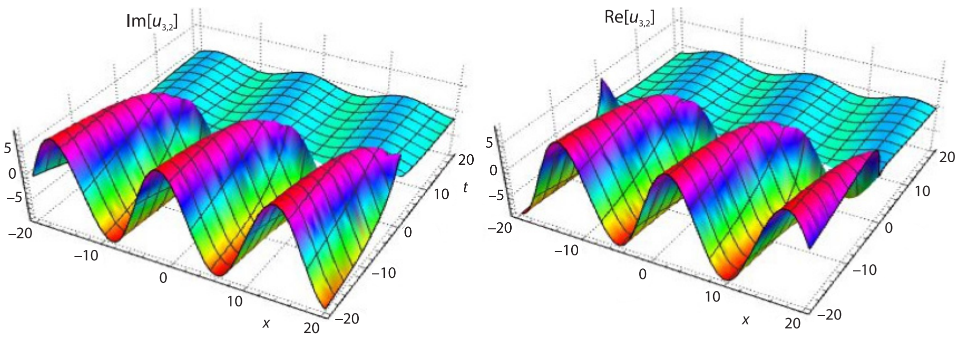


**Figure 9. The 3-D figures for eq. (47) where the values $-20 \le x \le 20$, $-20 \le t \le 20$**

Where the values of $t$ are given in the legend we have:

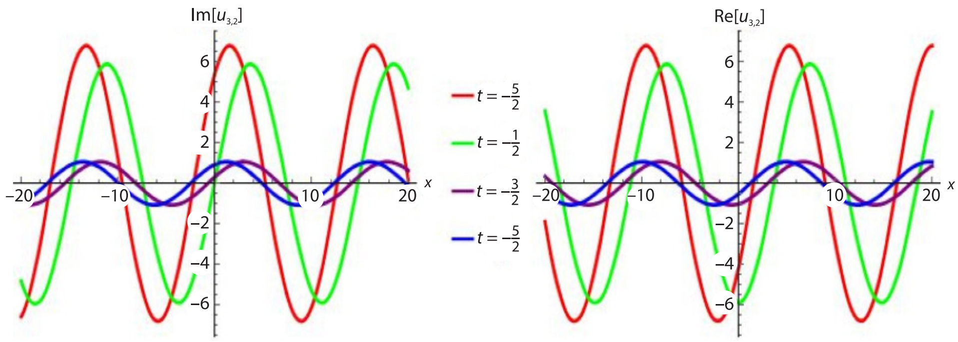


**Figure 10. The 2-D figures for eq. (47) where the values $-20 \le x \le 20$**

*Case 3.3* If $b \neq d$, then we'll get the coefficients:

$$a_0 = \frac{2b^2 b_2^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}{a_4\beta_3}, \quad a_1 = 0, \quad a_2 = -\frac{2\sqrt{2}b\sqrt{b_2^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}}{\sqrt{\beta_3}}, \quad a_3 = 0, \quad b_1 = 0$$
$$b_0 = -\frac{\sqrt{2}bb_2\sqrt{b_2^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}}{a_4\sqrt{\beta_3}}, \quad d = -\frac{a_4\sqrt{\beta_3}}{2\sqrt{2}\sqrt{b_2^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}}, \quad \delta_3 = 2\sqrt{2}b\left(\beta_1\delta_1^2 - \beta_2\delta_2^2\right) \tag{48}$$

the declared values of parameters in eq. (48) gives the solution:

$$u_{3,3}(x,y,t) = \frac{e^{-i\sqrt{2}b\xi}\left[\dfrac{a_4}{\dfrac{a_4\sqrt{\beta_3}}{2\sqrt{2}b\sqrt{b_2^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}} + \delta e^{-2b\xi}} - \dfrac{\sqrt{2}b\sqrt{b_2^2\left(\beta_2\delta_2^2 - \beta_1\delta_1^2\right)}}{\sqrt{\beta_3}}\right]}{b_2} \tag{49}$$

where

$$\xi = \delta_1 x + \delta_2 y + \frac{2\sqrt{2}b\left(\beta_1\delta_1^2 - \beta_2\delta_2^2\right)}{\alpha}t^{\alpha}$$

## Conclusion

In this paper, through the use of the fractional traveling wave transformation, we have smoothly applied an analytic method, namely the improved Bernoulli sub equation function method, to the complex non-linear (2+1)-D conformable time-fractional Schrödinger differential equation of order $\alpha$. As a result, many types of periodic, second order periodic, oscillating travailing waves, and exponential function solutions for this model have been prosperously found. The applied method was efficaciously used to achieve the goal set for this scientific work. It can be optically canvassed that the mentioned method is efficacious, valuable, and vital in determining the exact solutions of fractional differential equations appearing in different branches of the mathematical physics, and engineering sciences. the effects of the time evolution have been presented through the 2-D graphs which will be observed optically. All of the obtained solutions have been verified by substituting them back into their corresponding equation with the aid of symbolic computation software. For the best understanding of the gotten solutions peculiarities, they have been graphed in various types.

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