ANALYTICAL AND COMPUTATIONAL RESULTS FOR BOUNDARY-LAYER EQUATIONS IN POROUS MEDIUM

by

Iftikhar AHMAD [a,b]***, Aziz KHAN*** [c]***, Kamal SHAH*** [c,d]***,***
Thabet ABDELJAWAD [c,e*]***, and Saeed ISLAM*** [b]

[a] Department of Mathematics, Abdul Wali Khan University, Mardan, Khyber Pukhtoonkhwa, Pakistan
[b] Department of Mathematics and Statistics, University of Swat, Khyber Pukhtoonkhwa, Pakistan
[c] Department of Mathematics and Sciences, Prince Sultan University, Riyadh, Saudi Arabia
[d] Department of Mathematics, University of Malakand, Chakdara Dir(L), Khyber Pakhtunkhwa, Pakistan
[e] Department of Medical Research, China Medical University, Taiwan

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The aim of this paper is to determine analytic solutions to the steady/unsteady 2-D flows of a Newtonian fluid passing through a porous media. Using the stream function, the analytical general solutions of the non-linear equations corresponding to boundary-layer flows are determined by employing the extended approach of variable separation. Some specific flow problems, described by given initial and boundary conditions, are investigated using the obtained general analytic solutions.

Key words: *boundary-layer, exact solution, porous medium, inverse method*

Introduction

In the past few decades numerous numerical methods are developed to the solutions of the differential equations but for correct qualitative understanding of the physical phenomena, the exact solutions are crucial. A solution with a simple explicit form, generally an expression in finite terms of elementary or other well-known special functions, is referred to as an exact solution. Due to their use in many disciplines of the science and technology, the exact solutions play important role. These may be used for approximating the errors and for verification of reliabilities in the approximate, asymptotic, and numerical analytical methods. The fundamental equations for fluid motion are the Navier-Stokes (NS) equations. These equations are non-linear PDE lacking of the general solution. Due to the non-disappearance of the inertial term these equations have only few exact solutions. A vast variety of different mathematical physics problems may be solved through using extended and functional separation of variables [1-4] can be obtained. By the generalized separation of variables we means, to quest for exact solutions of the NS equations in the form of finite sums. In [5-8], the applicability of the generalized and functional separation of variables to the NS equations and boundary-layer equations is established. They have derived a number of exact solutions. The exact solutions of NS equations may be found in cases where these equations can be linearized and in cases where the

* Corresponding author, e-mail: tabdeljawad@psu.edu.sa

PDE are reduced to the ODE which have the possibilities of solutions. Polyanin and Zaistsev [9, 10], have dealt with the PDE in such manner. The flow of the fluid near the solid boundaries are called boundary-layer (BL) flow. The BL equations can be obtained from the NS equations by the order of magnitude analysis [11]. Thus the elliptical nature of the NS equations become parabolic which simplifies the solution of the problem. The exact solutions of BL equations are obtained by Polyanin. In [12] Polyanin and Zhurov have used generalized and functional separation of variables to deal with the BL equations. Labropulu [13] used the inverse method to find few more exact solutions for a second grade fluid. For the incompressible flow, as given in eqs. (1), (2), and (4), the momentum equations for the BL can be obtained by introducing the stream functions and putting them in the basic NS equations. Such a process reduces the NS equations (which are generally expressed in two unknown functions) to only one equation in a single unknown function. For more details see [14-18]. The objective of the present article is to investigate the steady/unsteady laminar 2-D BL flows of a Newtonian fluid over a flat plate in a Darcy porous medium. Using the generalized method of the separation of variables, we determine the general analytical solutions of the non-linear BL equations by considering the x- and y-components of these equations separately. Particular problems, described by appropriate initial-boundary conditions are studied based on the obtained general solutions. Further results can be traced as [19-22].

Boundary-layer equations in porous medium

The BL equations in porous medium (steady case) we assume the laminar, 2-D incompressible flow of a Newtonian fluid over a flat plate embedded in an isotropic, homogenous, fully saturated Darcy porous medium. The plate represents the (x, y) plane of the co-ordinate system $oxyz$ and y-axis is perpendicular on the plate. The corresponding fluid velocities in the x- and y-directions are $u(t, x, y)$, $v(t, x, y)$, respectively. In the case of 2-D flow, the velocity component in z-direction is ignored. Using the Darcy's law and introducing the boundary-layer approximations, the continuity equation and the momentum equations take the form:

$$\begin{gathered}\frac{\partial u(t,x,y)}{\partial x}+\frac{\partial v(t,x,y)}{\partial y}=0\\ \frac{\partial u(t,x,y)}{\partial t}+u(t,x,y)\frac{\partial u(t,x,y)}{\partial x}+v(t,x,y)\frac{\partial u(t,x,y)}{\partial y}=\nu\frac{\partial^2 u(t,x,y)}{\partial y^2}-\frac{\mu}{\kappa}u(t,x,y)-\frac{1}{\rho}\frac{\partial p(x,,y,t)}{\partial x}\\ \frac{\partial v(t,x,y)}{\partial t}+u(t,x,y)\frac{\partial v(t,x,y)}{\partial x}+v(t,x,y)\frac{\partial v(t,x,y)}{\partial y}=\nu\frac{\partial^2 v(t,x,y)}{\partial y^2}-\frac{\mu}{\kappa}v(t,x,y)-\frac{1}{\rho}\frac{\partial p(x,,y,t)}{\partial y}\end{gathered}\tag{1}$$

where ρ is the constant fluid density, μ – the dynamic viscosity, κ – the permeability of porous media, ν – the kinematic viscosity, and p – the pressure. The system of PDE will be considered along with the appropriate initial and boundary conditions.

Steady-state flow without pressure gradient in x-direction

In this case, the second equation of the system (1) becomes:

$$u(x,y)\frac{\partial u(x,y)}{\partial x}+v(x,y)\frac{\partial u(x,y)}{\partial y}=\nu\frac{\partial^2 u(x,y)}{\partial y^2}-\frac{\mu}{\kappa}u(x,y)\tag{2}$$

We introduce the stream function $w(x, y)$:

$$u(x,y)=\frac{\partial w(x,y)}{\partial y},\ \ v(x,y)=-\frac{\partial w(x,y)}{\partial x}\tag{3}$$

and we will determine a set of functions which are solutions of the non-linear eq. (2). The obtained parametric solutions will be particularized:

$$u(x,0)=\frac{\mu}{\kappa}x+u_0,\ \left[\frac{\partial u(x,y)}{\partial y}+\frac{\partial v(x,y)}{\partial x}\right]\Big|_{y=0}=\frac{\mu x}{\kappa}\sqrt{\frac{2\rho}{\kappa}}-u_0\sqrt{\frac{2\rho}{\kappa}}$$
$$v(x,0)=0,\ \lim_{y\to\infty}v(x,y)<\infty,\ \frac{\partial v(x,y)}{\partial x}\Big|_{y=0}=-\frac{\mu}{\kappa} \tag{4}$$

Using (3), eq. (1) is identically satisfied, and the third equation in eq. (1) will give the pressure gradient in the y-direction, after velocities u and v and are determined. Equation (2) reduces at the non-linear equation:

$$\frac{\partial w(x,y)}{\partial y}\frac{\partial^2 w(x,y)}{\partial x\partial y}-\frac{\partial w(x,y)}{\partial x}\frac{\partial^2 w(x,y)}{\partial y^2}=\nu\frac{\partial^3 w(x,y)}{\partial y^3}-\frac{\mu}{\kappa}\frac{\partial w(x,y)}{\partial y} \tag{5}$$

To solve (4), we assume the form of the stream functions as [21]:

$$w(x,y)=xF(y)+G(y) \tag{6}$$

where F and G are arbitrary functions of y. Thus eq. (5) becomes:

$$x\left[\left(F'_y\right)^2-FF''_{yy}-\nu F'''_{yyy}+\frac{\mu}{\kappa}\right]F'_y+\left[G'_yF'_y-FG''_{yy}-\nu G'''_{yyy}+\frac{\mu}{\kappa}G'_y\right]=0 \tag{7}$$

with $\nu=\mu/\rho$, eq. (7) gives:

$$F'''_{yyy}+\frac{1}{\nu}FF''_{yy}-\frac{1}{\nu}\left(F'_y\right)^2-\frac{\rho}{\kappa}F'_y=0 \tag{8}$$

$$G'_y\,F'_y-FG''_{yy}-\nu G'''_{yyy}+\frac{\mu}{\kappa}G'_y=0 \tag{9}$$

Equation (8) is an autonomous equation and its order can be decreased by one. If a particular solution of this equation is known, then by reduction of order, the corresponding eq. (9) can be reduced to the second order. The particular solution of eq. (8) is given, see [22]:

$$F=c_1\mathrm{e}^{c_2y}-\frac{c_2^2-\dfrac{\rho}{\kappa}}{\dfrac{1}{\nu}c_2} \tag{10}$$

where c_1, c_2 are constants. We use eq. (10) to solve eq. (9), therefore:

$$G'''_{yyy}+\left(\frac{c_1\mathrm{e}^{c_2y}}{\nu}-\frac{c_2^2-\dfrac{\rho}{k}}{c_2}\right)G''_{yy}-\frac{1}{\nu}\left(c_1c_2\mathrm{e}^{c_2y}+\frac{\mu}{\kappa}\right)G'_y=0 \tag{11}$$

Using reduction of order and putting $G'_y(y)=U(y)$ in eq. (11), we obtain:

$$U''_{yy}+\left(\frac{c_1e^{c_2y}}{\nu}-\frac{c_2^2-\dfrac{\rho}{\kappa}}{c_2}\right)U'_y-\frac{1}{\nu}\left(c_1c_2\mathrm{e}^{c_2y}+\frac{\mu}{\kappa}\right)U=0 \tag{12}$$

In order to obtain a solution (12), we assume $U(y)=\mathrm{e}^{c_2}V(y)$, thus eq. (12), becomes:

$$\nu V''_{yy}+\left[\nu c_2+\frac{\mu}{\kappa c_2}+c_1\mathrm{e}^{c_2y}\right]V'_y=0 \tag{13}$$

with the solution:

$$V(y)=b_1\int_0^y \exp\left\{-\left[\frac{c_1}{\nu c_2}\mathrm{e}^{c_2\eta}+\left(\frac{\rho}{\kappa c_2}+c_2^2\right)y\right]\mathrm{d}\eta\right\}+b_2 \tag{14}$$

Integrating two times, we obtain the function $G(y)$ in the form:

$$G(y)=\frac{b_1}{c_2}\mathrm{e}^{c_2 y}+b_2\int_0^y \mathrm{e}^{c_2\eta}\int_0^\eta \exp\left[-\left\{\frac{c_1}{\nu c_2}\mathrm{e}^{c_2\xi}+\left(\frac{\rho}{\kappa c_2}+c_2^2\right)\xi\right\}\mathrm{d}\xi\mathrm{d}\eta\right]+b_3 \tag{15}$$

Now, using eq. (6), the stream function is written:

$$\begin{aligned}w(x,y)=x\left(c_1\mathrm{e}^{c_2 y}-\nu c_2+\frac{\mu}{\kappa c_2}\right)+\frac{b_1}{c_2}\mathrm{e}^{c_2 y}+\\+b_2\int_0^y \mathrm{e}^{c_2\eta}\int_0^\eta \exp\left[-\left\{\frac{c_1}{\nu c_2}\mathrm{e}^{c_2\xi}+\left(\frac{\rho}{\kappa c_2}+c_2^2\right)\xi\right\}\mathrm{d}\xi\mathrm{d}\eta\right]+b_3\end{aligned} \tag{16}$$

The boundary conditions (4) are written in the form:

$$F(0)=0,\ F'(0)=\frac{\mu}{\kappa},\ G'(0)=u_0,\ G''(0)=-u_0\sqrt{\frac{2\rho}{\kappa}} \tag{17}$$

Using eqs. (16) and (17) we find:

$$c_1=-\frac{\nu}{2}\sqrt{\frac{2\rho}{\kappa}},\ c_2=-\sqrt{\frac{2\rho}{\kappa}},\ F(y)=\frac{\nu}{2}\sqrt{\frac{2\rho}{\kappa}}\left\{1-\exp\left(-y\sqrt{\frac{2\rho}{\kappa}}\right)\right\} \tag{18}$$

respectively:

$$G(y)=-u_0\sqrt{\frac{\kappa}{2\rho}}\exp\left(-y\sqrt{\frac{2\rho}{\kappa}}\right)+b_3 \tag{19}$$

$$w(x,y)=x\frac{\nu}{2}\sqrt{\frac{2\rho}{\kappa}}\left\{1-\exp\left(-y\sqrt{\frac{2\rho}{\kappa}}\right)\right\}-u_0\sqrt{\frac{\kappa}{2\rho}}\exp\left(-y\sqrt{\frac{2\rho}{\kappa}}\right)+b_3 \tag{20}$$

Steady-state flow without pressure gradient in y-direction

The y-component of the BL equations takes the form:

$$u(x,y)\frac{\partial v(x,y)}{\partial x}+v(x,y)\frac{\partial v(x,y)}{\partial y}=\nu\frac{\partial^2 v(x,y)}{\partial y^2}-\frac{\mu}{\kappa}v(x,y) \tag{21}$$

Using the stream function $w(x, y)$ given by eq. (3), we get:

$$\frac{\partial w(x,y)}{\partial y}\frac{\partial^2 w(x,y)}{\partial x^2}-\frac{\partial w(x,y)}{\partial x}\frac{\partial^2 w(x,y)}{\partial y\partial x}=\nu\frac{\partial^3 w(x,y)}{\partial y^2\partial x}-\frac{\mu}{\kappa}\frac{\partial w(x,y)}{\partial x} \tag{22}$$

To find a solution, we assume the form of the stream function as [15]:

$$w(x,y)=F(x)+\nu\lambda y$$

The λ is any and $F(x)$ is general function. Substituting in eq. (21) we get:

$$\nu\lambda F''_{xx}=-\frac{\mu}{\kappa}F'_x$$

Putting $F_x' = U$, reduces the preceding equation:

$$\nu\lambda U' = -\frac{\mu}{\kappa}U$$

Integration yields:

$$U = b_1 \mathrm{e}^{-\left(\frac{\rho}{\kappa\lambda}\right)x}$$

as $\mu = \nu\rho$. Now since $U = F_x'$, therefore

$$F(x) = -\beta_1 \mathrm{e}^{-\frac{x}{\alpha_1}} + b_2$$

where $\alpha_1 = \kappa\lambda/\rho$, $\beta_1 = b_1\alpha_1$, b_2 are the constants of integration. Thus, the solution has the form:

$$w(x,y) = -\beta_1 \mathrm{e}^{-\frac{x}{\alpha_1}} + \nu\lambda y + b_2 \tag{23}$$

Let us to use the general solution (17) for the flow with the boundary conditions:

$$u(0,y) = \nu\lambda,\ \ v(0,y) = v_0 \tag{24}$$

In this case we obtain:

$$\begin{aligned} u(x,y) &= \nu\lambda,\ \ v(x,y) = v_0 \mathrm{e}^{\left(-\frac{\rho x}{\kappa\lambda}\right)} \\ w(x,y) &= \nu\lambda y - v_0 \frac{\kappa\lambda}{\rho} \mathrm{e}^{\left(-\frac{\rho x}{\kappa\lambda}\right)} \end{aligned} \tag{25}$$

Now, using the x-component of BL equation, we determine the pressure gradient in x-direction:

$$\frac{\partial p}{\partial x} = \frac{\mu^2\lambda}{\kappa} \tag{26}$$

respectively, the pressure:

$$p(x,y) = \frac{\mu^2\lambda}{\kappa}x + p_0,\ \ p_0 = \text{const} \tag{27}$$

Unsteady 2-D boundary-layer flow

In this section, we determine general analytical solutions to eq. (1). The obtained solutions will be particularized for given initial and boundary conditions.

Unsteady boundary-layer flow without pressure gradient in the x-direction

The desired equation is obtained from the NS eq. (1), by neglecting the forcing term and considering the equation in the porous medium. Thus we have:

$$\frac{\partial u(t,x,y)}{\partial t} + u(t,x,y)\frac{\partial u(t,x,y)}{\partial x} + v(t,x,y)\frac{\partial u(t,x,y)}{\partial y} = \nu\frac{\partial^2 u(t,x,y)}{\partial y^2} - \frac{\mu}{\kappa}u(t,x,y) \tag{28}$$

Introducing the stream function eq. (28), we get:

$$\frac{\partial^2 w(t,x,y)}{\partial t\partial y} + \frac{\partial w(t,x,y)}{\partial y}\frac{\partial^2 w(t,x,y)}{\partial x\partial y} - \frac{\partial w(t,x,y)}{\partial x}\frac{\partial^2 w(t,x,y)}{\partial y^2} = \nu\frac{\partial^3 w(t,x,y)}{\partial y^3} - \frac{\mu}{\kappa}\frac{\partial w(t,x,y)}{\partial y} \tag{29}$$

Using the method of generalized separable variable [10], we assume:

$$w(t,x,y) = xF(y,t) + G(y,t) \tag{30}$$

where F and G are general functions. Thus eq. (30) gives:

$$\frac{\partial^2 F}{\partial t \partial y} + \left(\frac{\partial F}{\partial y}\right)^2 - F\frac{\partial^2 F}{\partial y^2} - \nu\frac{\partial^3 F}{\partial y^3} + \frac{\mu}{\kappa}\frac{\partial F}{\partial y} = 0 \tag{31}$$

$$\frac{\partial^2 G}{\partial t \partial y} + \frac{\partial F}{\partial y}\frac{\partial G}{\partial y} - F\frac{\partial^2 G}{\partial y^2} - \nu\frac{\partial^3 G}{\partial y^3} + \frac{\mu}{\kappa}\frac{\partial G}{\partial y} = 0 \tag{32}$$

To find solution of eq. (31), let us consider:

$$F = \phi(t)\mathrm{e}^{\lambda y} + \theta(t) \tag{33}$$

where ϕ and θ are general functions to be determined and λ is any constant [12]. Substituting (33) in eq. (31), we get:

$$\phi_t' - \lambda\phi\theta - \nu\lambda^2\phi + \frac{\mu}{k}\phi = 0 \tag{34}$$

Now, we solve eq. (34) by separation of functions. We have:

$$\frac{\phi_t'(t)}{\phi(t)} = \lambda\theta(t) + \nu\lambda^2 - \frac{\mu}{\kappa} = \beta,\ \ \beta = \text{constant} \tag{35}$$

Introducing the notation:

$$\beta_2 = \nu\lambda^2 - \frac{\mu}{\kappa}$$

we obtain:

$$\theta(t) = \frac{\beta - \beta_2}{\lambda} = \text{constant},\ \ \phi(t) = \beta_1\mathrm{e}^{\beta t} \tag{36}$$

respectively:

$$F(y,t) = \beta_1\mathrm{e}^{\beta t + \lambda y} + \frac{\beta - \beta_2}{\lambda} \tag{37}$$

Using eq. (37) and making:

$$\frac{\partial G(y,t)}{\partial y} = \mathrm{e}^{\lambda y}H(t)$$

Equation (32) becomes:

$$\frac{H_t'(t)}{H(t)} = \beta = \text{constant} \tag{38}$$

with the general solution:

$$H(t) = d_1\mathrm{e}^{\beta t} \tag{39}$$

respectively:

$$G(y,t) = \frac{d_1}{\lambda}\mathrm{e}^{\beta t + \lambda y} + d_2(t) \tag{40}$$

Finally, the stream function is given:

$$w(t,x,y)=x\left(\beta_1 e^{\beta t+\lambda y}+\frac{\beta-\beta_2}{\lambda}\right)+\frac{d_1}{\lambda}\mathrm{e}^{\beta t+\lambda y}+d_2(t) \tag{41}$$

and the pressure $p(t, x, y)$ is obtained from the eq. (1). Now, we determine the solution for the unsteady BL flow with the initial and boundary conditions:

$$\begin{gathered}u(x,y,0)=\nu\lambda^2 x\mathrm{e}^{\lambda y},\ \ v(x,y,0)=\nu\lambda(1-\mathrm{e}^{\lambda y})\\ u(x,0,t)=\frac{\mu}{\kappa}x\mathrm{e}^{-\frac{\mu}{\kappa}},\ \ \lim_{y\to\infty}u(t,x,y)=0\end{gathered} \tag{42}$$

Using eqs. (3), (41), and (42) we obtain:

$$u(t,x,y)=\frac{\mu x}{\kappa}\exp\left(-\frac{\mu}{\kappa}t-\sqrt{\frac{\rho}{\kappa}}y\right),\ \ v(t,x,y)=\nu\sqrt{\frac{\rho}{\kappa}}\left[\exp\left(-\frac{\mu}{\kappa}t-\sqrt{\frac{\rho}{\kappa}}y\right)-1\right] \tag{43}$$

Unsteady boundary-layer flow without pressure gradient in the y-direction

The y-component of the momentum equation for the unsteady BL equations is given by:

$$\frac{\partial v(t,x,y)}{\partial t}+u(t,x,y)\frac{\partial v(t,x,y)}{\partial x}+v(t,x,y)\frac{\partial v(t,x,y)}{\partial y}=\nu\frac{\partial^2 v(t,x,y)}{\partial y^2}-\frac{\mu}{\kappa}v(t,x,y) \tag{44}$$

applying stream functions to eq. (44), we obtain:

$$\frac{\partial^2 w(t,x,y)}{\partial t\partial x}+\frac{\partial w(t,x,y)}{\partial y}\frac{\partial^2 w(t,x,y)}{\partial x^2}-\frac{\partial w(t,x,y)}{\partial x}\frac{\partial^2 w(t,x,y)}{\partial y\partial x}=\nu\frac{\partial^3 w(t,x,y)}{\partial y^2\partial x}-\frac{\mu}{\kappa}\frac{\partial w(t,x,y)}{\partial x} \tag{45}$$

To solve eq. (45), we assume:

$$w(t,x,y)=yF(x,t)+G(x,t) \tag{46}$$

where F and G are general function as used in [10]. Substituting (46) in eq. (45), we obtain:

$$\frac{\partial^2 F}{\partial x\partial t}+F\frac{\partial^2 F}{\partial x^2}-\left(\frac{\partial F}{\partial x}\right)^2+\frac{\mu}{\kappa}\frac{\partial F}{\partial x}=0 \tag{47}$$

$$\frac{\partial^2 G}{\partial x\partial t}+F\frac{\partial^2 G}{\partial x^2}-\frac{\partial G}{\partial x}\frac{\partial F}{\partial x}+\frac{\mu}{\kappa}\frac{\partial G}{\partial x}=0 \tag{48}$$

First, we solve eq. (47), and for this we let use the technique been applied in [12]:

$$F=\phi(t)\mathrm{e}^{\lambda x}+\theta(t) \tag{49}$$

Using F and considering eq. (48), we get:

$$\phi_t'+\lambda\phi\theta+\frac{\mu}{\kappa}\phi=0 \tag{50}$$

Solving eq. (50) for θ, we obtain:

$$\theta=-\frac{1}{\lambda}\frac{\phi_t'}{\phi}-\frac{\mu}{\lambda\kappa} \tag{51}$$

Thus:

$$F=\phi\mathrm{e}^{\lambda x}-\frac{1}{\lambda}\frac{\phi_t'}{\phi}-\frac{\mu}{\lambda\kappa} \tag{52}$$

We can also solve eq. (50) for ϕ, therefore, we rewrite it:

$$\phi_t' = -\phi\left(\lambda\theta + \frac{\mu}{\kappa}\right) \tag{53}$$

Separating the variable and integrating w.r.t. t, we have:

$$\phi = \gamma_1 \mathrm{e}^{-\left(\frac{\mu}{\kappa}t + \lambda\int\theta \mathrm{d}t\right)} \tag{54}$$

Therefore, F takes the form:

$$F = \gamma_1 \mathrm{e}^{-\left(\frac{\mu}{\kappa}t + \lambda\int\theta \mathrm{d}t\right) + \lambda x} + \theta \tag{55}$$

Now, we use the value of F to solve eq. (48), therefore, first let $\partial G/\partial x = U$ in eq. (48) we get:

$$\frac{\partial U}{\partial t} + \left(\phi \mathrm{e}^{\lambda} x - \frac{1}{\lambda}\frac{\phi_t'}{\phi} - \frac{\mu}{\lambda\kappa}\right)\frac{\partial U}{\partial x} - \lambda\phi \mathrm{e}^{\lambda x} U + \frac{\mu}{\kappa} U = 0 \tag{56}$$

To solve this we assume $U = \mathrm{e}^{\lambda x} H(t)$, H is a general function of t, in eq, (38) and simplifying, we get:

$$H_t' - \frac{\phi_t'}{\phi} H = 0 \tag{57}$$

The integral to eq. (57):

$$H = \gamma_2 \phi \tag{58}$$

Thus:

$$U = \frac{\partial G}{\partial x} = \gamma_2 \phi \mathrm{e}^{\lambda x} \tag{59}$$

Integrating both sides w.r.t x, we get:

$$G(x,t) = \frac{\gamma_2}{\lambda}\phi \mathrm{e}^{\lambda x} + \gamma_3 \tag{60}$$

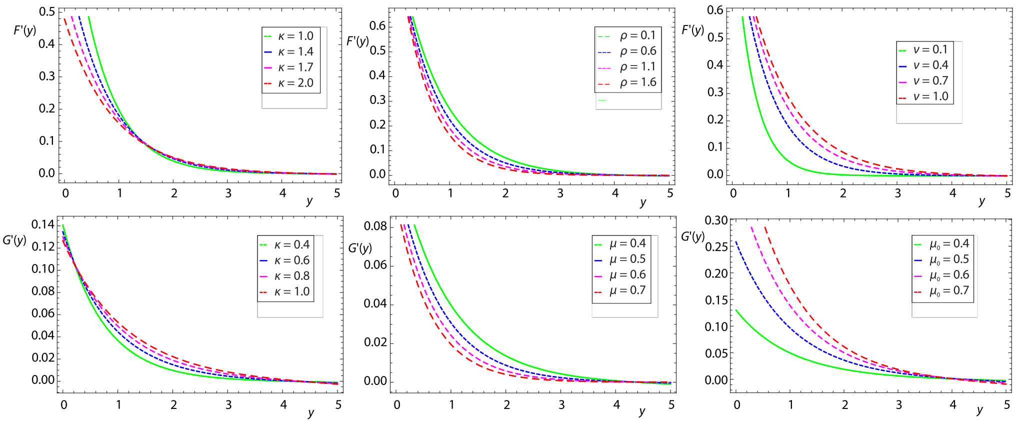
Therefore, the general solution becomes:

$$w(t,x,y) = \gamma_1\left(y + \frac{\gamma_2}{\lambda}\right)\mathrm{e}^{\frac{\mu}{\kappa}t - \lambda\left(x + \int\theta \mathrm{d}t\right)} + \theta y + \gamma_3 \tag{61}$$

where y_1, y_2, and y_3 are the constants obtained by integration of eqs. (53), (57), and (59), respectively. Appropriate initial-boundary conditions for the problem studied in this section can be considered in a similar manner as in the previous section.

Numerical solution of the transformed coupled systems (8) and (9)

Here in this section by the help of shooting method, we plot the solutions corresponding to the different values of the parameters in fig. 1.



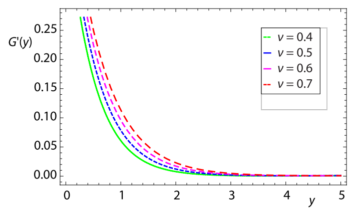


Figure 1. Plots of the solution at different values of the parameters for the proposed problem

Conclusion

We have found the exact solutions for the BL equations in a porous medium. The x- and y- components of these equations are treated independently. Moreover, the solutions are found for both the steady and unsteady cases. We have used the inverse method which is one of the famous method for finding the exact solutions of differential equations. The solutions are given in equations (20) for component and equation (25) for component of the steady case. Similarly, for the unsteady case the solutions are given in equation (43) and equation (61) for component and component, respectively.

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References

[1] Andreev, V. K., *et al.*, *Applications of Group-Theoretical Methods in Hydrodynamic*, Kluwer, Dordrecht, The Netherlands, 1998
[2] Burde, G. I., The Construction of Special Explicit Solutions of the Boundary-Layer Equations, Unsteady Flows, *Q. J. Mech. Appl. Math.*, *48* (1995), 4, pp. 611-633
[3] Burde, G. I., New Similarity Reductions of the Steady-State Boundary-Layer Equations, *Journal Phys. A*: *Math. Gen.*, *29* (1996), 8, pp. 1665-1683
[4] Ludlow, D. K., *et al.*, New Similarity Solutions of the Unsteady Incompressible Boundary-Layer Equations, *Q. J. Mech. Appl. Math.*, *53* (2000), 2, pp. 175-206
[5] Polyanin, A. D., Exact Solutions and Transformations of the Equations of a Stationary Laminar Boundary-layer, *Theo. Foun. Chem. Engg.*, *35* (2001), July, pp. 319-328
[6] Ludlow, D. K., *et al.*, Non-Classical Symmetry Reductions of the 3-D Incompressible Navier' Stokes Equations, *Journal Phys. A: Math. Gen.*, *31* (1998), 39, pp. 7965-7980
[7] Ludlow, D. K., *et al.*, Non-Classical Symmetry Reductions of the 2-D Incompressible Navieras Stokes Equations, *Stud. Appl. Math.*, *103* (1999), 3, pp. 183-240

[8] Saccomandi, G., A Remarkable Class of Non-Classical Symmetries of the Steady 2-D Boundary-Layer Equations, *Journal Phys. A: Math. Gen*., *37* (2004), 27, pp. 7005-7017
[9] Polyanin, A. D, Zaitsev, V. F., Transformations and Exact Solutions Containing Arbitrary Functions for Boundary-Layer Equations, *Dok. Phy*., *46* (2001), 7, pp. 526-531
[10] Polyanin, A. D., Zaitsev, V. F., Equations of an Unsteady-State Laminar Boundary-Layer: General Transformation and Exact Solutions, *Theo. Foun. Chem. Engg*., *35* (2001), Nov., pp. 529-539
[11] Papanastasiou, T. C., *et al., Viscous Fluid-Flow*, CRC Press, Boca Raton, Fla., USA, 2000
[12] Polyanin, A. D., Zhurov, A. I., Methods of Generalized and Functional Separation of Variables in Hydrodynamics and Heat and Mass Transfer Equations, *Theo. Foun. Chem. Engg*., *36* (2002), May, pp. 201-213
[13] Labropulu, F., A Few More Exact Solutions of a Second Grade Fluid via Inverse Method, *Mech. Res*. *Comm*., *27* (2000), 6, pp. 713-720
[14] Loitsyanskiy, L.G., *Mechanics of Liquids and Gases*, Begell House, New York, USA, 1996
[15] Schlichting, H., *Boundary-Layer Theory*, McGraw-Hill, New York, USA, 1981
[16] Kamel, M. T., Hamdan, M. H., Riabounchisky Flow through Porous Media, *Int. J. Pure. Appl. Math*., *27* (2005), 1, pp. 113-125
[17] Merabet, N., *et al*., Analytical Approach to the Darcy-Lapwood-Brinkman Equation, *Appl. Math. and Comput*., *195* (2008), 2-1, pp. 579-585
[18] Vafai, K., Tien, C. L., Boundary and Inertia Effects on Flow and Heat Transfer in Porous Media, *Int. J. Heat Mass Trans.*, *24* (1981), 2, pp. 211-220
[19] Vafai, K., Kim, S. J., On the Limitations of Brinkman-Forchheimer-Extended Darcy Equation, *Int. J. Heat Fluid-Flow*, *16* (1995), 1, pp. 11-15
[20] Kaviany, M., Laminar Flow Through a Porous Channel Bounded by Isothermal Parallel Plates, *Int. J. Heat Mass Trans*., *28* (1985), 4, pp. 851-858
[21] Polyanin, A. D., *et al*., Exact Solutions of Navier' Stokes, Boundary-Layer, and Heat-Conduction Equations, *Theo. Foun. Chem. Engg*., *36* (2002), July, pp. 346-352
[22] Polyanin, A. D., Zaitsev, V. F., *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, New York, USA, 2002

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