SIMULATION OF GENERALIZED TIME FRACTIONAL GARDNER EQUATION UTILIZING IN PLASMA PHYSICS FOR NON-LINEAR PROPAGATION OF ION-ACOUSTIC WAVES

by

Hajira IRSHAD [a], ***Mehnaz SHAKEEL*** [a], ***Imtiaz AHMAD*** [b], ***Hijaz AHMAD*** [c,d], ***Chutarat TEARNBUCHA*** [e*], ***and Weerawat SUDSUTAD*** [f]

[a] Shaheed Benazir Bhutto Women University, Peshawar, Pakistan
[b] Department of Mathematics, University of Swabi, Khyber Pakhtunkhwa, Pakistan
[c] Operational Research Center in Healthcare, Near East University, Nicosia/Mersin, Turkey
[d] Section of Mathematics, International Telematic University Uninettuno, Roma, Italy
[e] Department of General Education, Faculty of Science and Health Technology, Navamindradhiraj University, Bangkok, Thailand
[f] Department of Statistics, Faculty of Science, Ramkhamhaeng University, Bangkok, Thailand

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In this work, radial basis function collocation method (RBFCM) is implemented for generalized time fractional Gardner equation (GTFGE). The RBFCM is meshless and easy-to-implement in complex geometries and higher dimensions, therefore, it is highly demanding. In this work, the Caputo derivative of fractional order $\xi \in (0, 1]$ is used to approximate the first order time derivative whereas, Crank-Nicolson scheme is hired to approximate space derivatives. The numerical solutions are presented and discussed, which demonstrate that the method is effective and accurate.

Key words: *radial basis functions, time fractional Gardner equation, time fractional derivative*

Introduction

In the last century, many remarkable contributions have been made to the applications and theory of the fractional differential equation (FDE). The FDE are commonly used to model problems in research areas as diverse as chaos synchronization, wave propagation phenomenon, mixed convection flows, control theory, anomalous diffusive, unification of diffusion, dynamical systems, heat transfer, image processing, mixed convection flows, unification of diffusion, entropy theory and mechanical systems [1-9]. The most significant advantage of applying FDE in these and some other applications is their non-local structure, which means that the next data not only depends on the current information but also on the previous information. Therefore, fractional differential operators provide an admirable tool to describe the memory and hereditary properties of various mathematical, physical and engineering models. Fractional partial differential equations (FPDE) contains the unknown multivariable function and its fractional partial derivatives. The FPDE are used to model problems with functions of several variables, to find solution of many physical models. Mathematical methods to find a closed form solution for time FPDE (TFPDE) are considerable, amusing and in the wide sense, however, there does not exists any method that gives a closed form solution for non-linear FPDE because

* Corresponding author, e-mail: chutaratt@nmu.ac.th

of fractional derivatives in these equations [10]. Non-linear FPDE are commonly used for the description of different phenomena and dynamical processes in acoustics, engineering, material science, physics, viscoelasticity, electrochemistry and electromegnetics [11, 12].

The Gardner equation (GE) is an associate uniting of Korteweg-de Vries (KdV) and modified KdV equations, and that springs for example the outline of internal solitary waves in shallow water. The GE is widely utilized in numerous branches of physics, such as quantum field theory, plasma physics and fluid physics. It additionally, describes a spread of wave phenomena in plasma and solid state. In plasma physics time fractional GE (TFGE) is used to investigate the non-linear propagation of ion-acoustic waves in an unmagnetized plasma that consist of negative ions, non-thermal electrons, positive ions and negative-ion-beam featuring the Tsallis distribution [13]. In this investigation, we consider the generalized TFGE (GTFGE):

$$\frac{\partial^{\xi} w}{\partial t^{\xi}} + A_1 w w_x + A_2 w^2 w_x + A_3 w_{xxx} = \psi(x,t), \;\; t \in [0, T_0], \;\; 0 < \xi \le 1, \;\; x \in R \tag{1}$$

where $\psi(x, t)$ is the source term, ξ – the time fractional derivative order, and $\partial^{\xi} w/\partial t^{\xi}$ – the fractional derivative in the Caputo sense.

Radial basis functions (RBF) methodology is one of the best techniques to find the numerical solution of fractional order models. The most important property of an RBF methodology is it's meshfree nature because there is no need to create any mesh. Therefore, it can be applied easily to high dimensional problems since the computation of distance in any dimensions is straightforward. Kansa [14] was the one who introduced RBF collocation method (RBFCM) in order to solve PDE. In the same manner, Zerroukat *et al*. [15, 16] applied the MultiQuadrics (MQ) to find the solution of the heat transfer problem and to find the solution of linear advection-diffusion equations by applying the thin pate splines (TPS) and also discussed the stability. Not only integer order PDE [15] but also Kansa's approach has been used to find the solution of fractional order PDE [17, 18]. In [19] the head-on collision of time fractional shock waves is discussed and also local RBFCM is applied to find the numerical solution of two sided time fractional KdV Burgers equations.

Formulation of the numerical scheme

This section is devoted to the formulation of the suggested numerical scheme.

Time fractional derivative

The time fractional derivative $\partial^{\xi} w(x, t)/\partial t^{\xi}$ in eq. (1), is the Caputo fractional derivative [20], which can be written:

$$\frac{\partial^{\xi} w(x,t)}{\partial t^{\xi}} = \begin{cases} \dfrac{1}{\Gamma(1-\xi)} \displaystyle\int_0^t \frac{\partial w(x,\delta)}{\partial \delta} \frac{1}{(t-\delta)^{\xi}} \mathrm{d}\delta, & 0 < \xi < 1 \\[2ex] \dfrac{\partial w(x,t)}{\partial t}, & \xi = 1 \end{cases} \tag{2}$$

where ξ is the fractional order derivative, $t_m = m\Delta t$, $m = 0, 1, 2,..., N$ and Δt – the time step. The finite difference scheme is hired to discretize the classical derivative term:

$$\frac{\partial^{\xi} w(x,t_{m+1})}{\partial t^{\xi}} = \frac{1}{\Gamma(1-\xi)} \int_0^{t_{m+1}} \frac{\partial w(x,\delta)}{\partial \delta} (t_{m+1}-\delta)^{-\xi} \mathrm{d}\delta = \frac{1}{\Gamma(1-\xi)} \sum_{l=0}^{m} \int_{l\Delta t}^{(l+1)\Delta t} \frac{\partial w(x,\delta_l)}{\partial \delta} (t_{m+1}-\delta)^{-\xi} \mathrm{d}\delta \tag{3}$$

the first-order time derivative appearing in eq. (3) is approximated:

$$\frac{\partial w(x,\delta_l)}{\partial \delta}=\frac{w(x,\delta_{l+1})-w(x,\delta_l)}{\delta}+O(\Delta t) \tag{4}$$

then

$$\frac{\partial^{\xi} w(x,t_{m+1})}{\partial t^{\xi}}\approx\frac{1}{\Gamma(1-\xi)}\sum_{l=0}^{m}\int\limits_{l\Delta t}^{(l+1)\Delta t}\left[\frac{w(x,\delta_{l+1})-w(x,\delta_l)}{\delta}+O(\Delta t)\right](t_{m+1}-\delta)^{-\xi}\,\mathrm{d}\delta$$
$$\frac{\partial^{\xi} w(x,t_{m+1})}{\partial t^{\xi}}=a_0\sum_{l=0}^{m}b_l\left(w^{m-l+1}-w^{m-l}\right)+\left[O(\Delta t)^{2-\xi}\right] \tag{5}$$

where

$$a_0=\frac{\Delta t^{-\xi}}{\Gamma(2-\xi)},\; b_l=(l+1)^{1-\xi}-l^{1-\xi},\; l=0,1,2,\cdots,m$$

and $w^0 = w(x, t = 0) = w_0(x)$ is initial condition (IC).

Finally, eq. (5) can be written in precise form:

$$\frac{\partial^{\xi} w(x,t_{m+1})}{\partial t^{\xi}}=\begin{cases}a_0\left(w^{m+1}-w^{n}\right)+a_0\sum\limits_{l=1}^{m}b_l\left(w^{m-l+1}-w^{m-l}\right), & m\ge 1\\ a_0(w^1-w^0), & m=0\end{cases} \tag{6}$$

Space fractional derivative

In the next step, Kansa's method is applied and collocate$w(x, t^{m+1})$ by RBF. The solution is interpolated at M different collocation points $x_j|j = 1, 2,..., M$, where $x_j|j \in \Omega$ are interior points while x_1 and x_M are boundary points, Ω represents a bounded domain and $\partial\Omega$ is its boundary. The numerical solution of $w(x, t^{m+1})$ can be expressed in terms of RBF:

$$w\left(x,t^{m+1}\right)=\sum_{j=1}^{M}\lambda_j^{m+1}\varphi\left(\|x_i-x_j\|\right) \tag{7}$$

where $i = 1, 2,..., M$ and λ_j^{m+1} are the unknown coefficients at the $(m + 1)^{\text{th}}$ time level, $\phi(r_{ij})$ is the RBF, and $\|\cdot\|$ – the Euclidean norm and $r_{ij} = \|x_i - x_j\|$.

Equation (7) can be written in matrix form:

$$w^{m+1}=S\lambda^{m+1} \tag{8}$$

where

$$w^{m+1}=[w_1^{m+1},w_2^{m+1},\cdots,w_M^{m+1}]^T,\; \lambda^{m+1}=[\lambda_1^{m+1},\lambda_2^{m+1},\cdots,\lambda_M^{m+1}]^T$$

and the collocation matrix S^0 is given:

$$S^{(0)}=\begin{bmatrix}\varphi_{11} & \cdots & \varphi_{1j} & \cdots & \varphi_{1M}\\ \vdots & \ddots & \vdots & \vdots & \vdots\\ \varphi_{i1} & \cdots & \varphi_{ii} & \cdots & \varphi_{iM}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \varphi_{M1} & \cdots & \varphi_{Mj} & \cdots & \varphi_{MM}\end{bmatrix} \tag{9}$$

→

$$S^{(1)} = \left[\varphi'\left(r_{ij}\right)\right],\ i,j = 1,\cdots,M$$
$$S^{(2)} = \left[\varphi''\left(r_{ij}\right)\right],\ i,j = 1,\cdots,M$$
$$S^{(3)} = \left[\varphi'''\left(r_{ij}\right)\right],\ i,j = 1,\cdots,M$$

Generalized time fractional Gardner equation

Consider eq. (1) with the boundary conditions (BC):

$$w(x,t) = g(x,t), \quad t > 0, \quad x \in \partial\Omega$$

and IC

$$w(x,0) = w_0(x) \tag{10}$$

Time derivative is discretized by Eq. (6) and space derivatives are discretized by Crank-Nicolson scheme (CNS). Equation (1) can be written:

$$\begin{aligned} &a_0 w^{m+1} - a_0 w^m + a_0 \sum_{l=1}^{m} b_l \left(w^{m-l+1} - w^{m-l}\right) + \frac{1}{2} A_1 (w w_x)^{m+1} + \frac{1}{2} A_1 (w w_x)^m + \\ &+ \frac{1}{2} A_2 \left(w^2 w_x\right)^{m+1} + \frac{1}{2} A_2 \left(w^2 w_x\right)^m + \frac{1}{2} A_3 (w_{xxx})^{m+1} + \frac{1}{2} A_3 (w_{xxx})^m = \psi^{m+1} \end{aligned} \tag{11}$$

In eq. (11) non-linear terms $(ww_x)^{m+1}$ and $(w^2w_x)^{m+1}$ are linearized, respectively:

$$(w w_x)^{m+1} = w^{m+1} w_x^m + w^m w_x^{m+1} - w^m w_x^m$$

and

$$\left(w^2 w_x\right)^{m+1} = 2 w^{m+1} w^m w_x^m + \left(w^2\right)^m w_x^{m+1} - 2\left(w^2\right)^m w_x^m$$

Equation (11) becomes:

$$\begin{aligned} &a_0 w^{m+1} - a_0 w^m + a_0 \sum_{l=1}^{m} b_l \left(w^{m-l+1} - w^{m-l}\right) + \\ &+ \frac{1}{2} A_1 \left[w^{m+1} w_x^m + w^m w_x^{m+1} - w^m w_x^m\right] + \frac{1}{2} A_1 (w w_x)^m + \\ &+ \frac{1}{2} A_2 \left[2 w^{m+1} w^m w_x^m + \left(w^2\right)^m w_x^{m+1} - 2(w^2)^m w_x^m\right] + \frac{1}{2} A_2 \left(w^2 w_x\right)^m + \\ &+ \frac{1}{2} A_3 (w_{xxx})^{m+1} + \frac{1}{2} A_3 (w_{xxx})^m = \psi^{m+1} \end{aligned} \tag{12}$$

$$\begin{aligned} &a_0 w^{m+1} + \frac{1}{2} A_1 w^{m+1} w_x^m + \frac{1}{2} A_1 w^m w_x^{m+1} + A_2 w^{m+1} w^m w_x^m + \frac{1}{2} A_2 (w^m)^2 w_x^{m+1} + \frac{1}{2} A_3 (w_{xxx})^{m+1} = \\ &= a_0 w^m + \frac{1}{2} A_2 (w^m)^2 w_x^m - \frac{1}{2} A_3 (w_{xxx})^m - a_0 \sum_{l=1}^{m} b_l \left(w^{m-l+1} - w^{m-l}\right) + \psi^{m+1} \end{aligned}$$

Now from eq. (8) it is known that $w^{m+1} = \lambda^{m+1}S^{(0)}$. Thus, eq. (12) can be written:

$$a_0\lambda^{m+1}S^0 + \frac{1}{2}A_1\lambda^{m+1}S^0\lambda^m S^1 + \frac{1}{2}A_1\lambda^{m+1}S^1\lambda^m S^0 + A_2\lambda^{m+1}S^0\lambda^m S^0\lambda^m S^1 +$$

$$+\frac{1}{2}A_2\lambda^{m+1}S^1\lambda^m S^0\lambda^m S^0 + \frac{1}{2}A_3\lambda^{m+1}S^3 = a_0\lambda^m S^0 + \frac{1}{2}A_2\lambda^m S^0\lambda^m S^0\lambda^m S^1 -$$

$$-\frac{1}{2}A_3\lambda^m S^3 - a_0\sum_{l=1}^{m} b_l\left(w^{m-l+1} - w^{m-l}\right) + \psi^{m+1} \tag{13}$$

$$\lambda^{m+1}\left[a_0 S^0 + \frac{1}{2}A_1 S^0\lambda^m S^1 + \frac{1}{2}A_1 S^1\lambda^m S^0 + A_2 S^0\lambda^m S^0\lambda^m S^1 + \frac{1}{2}A_2 S^1\lambda^m S^0\lambda^m S^0 + \frac{1}{2}A_3 S^3\right] =$$

$$= \lambda^m\left[a_0 S^0 + \frac{1}{2}A_2 S^0\lambda^m S^0\lambda^m S^1 - \frac{1}{2}A_3 S^3\right] - a_0\sum_{l=1}^{m} b_l\left(w^{m-l+1} - w^{m-l}\right) + \psi^{m+1}$$

Rewrite eq. (13) in the matrix form:

$$\lambda^{m+1}P = \lambda^m Q + G^{m+1},\ \ \lambda^{m+1} = \lambda^m QP^{-1} + G^{m+1}P^{-1} \tag{14}$$

where

$$P = a_0 S^0 + \frac{1}{2}A_1 S^0\lambda^m S^1 + \frac{1}{2}A_1 S^1\lambda^m S^0 + A_2 S^0\lambda^m S^0\lambda^m S^1 + \frac{1}{2}A_2 S^1\lambda^m S^0\lambda^m S^0 + \frac{1}{2}A_3 S^3$$

$$Q = a_0 S^0 + \frac{1}{2}A_2 S^0\lambda^m S^0\lambda^m S^1 - \frac{1}{2}A_3 S^3$$

$$G^{m+1} = G_1^{m+1} + G_2^{m+1}$$

$$G_1^{m+1} = \left[-a_0\sum\nolimits_{l=1}^{m} b_l\left(w^{m-l+1} - w^{m-l}\right) + \psi^{m+1}\right]^T$$

where

$$G_2^{m+1} = \left[g_1^{m+1}, 0, \cdots, g_2^{m+1}\right]^T$$

where G_1^{m+1} is a column vector of order $N\times 1$.

From eq. (8):

$$w^{m+1} = \lambda^{m+1}S,\ \text{also}\ \ w^m = \lambda^m S \Rightarrow \lambda^m = w^m S^{-1}$$

Thus, from eqs. (8) and (14):

$$w^{m+1} = \left[\lambda^m QP^{-1} + G^{m+1}P^{-1}\right]S$$

$$w^{m+1} = W^m S^{-1}QP^{-1}S + G^{m+1}P^{-1}S$$

This scheme can be used to find the numerical solution at any time level m. The W^0 is taken from the IC (10). The collocation matrix S is non-singular for distinct collocation points [21].

Numerical experiments

In this section, the derived RBF collocation scheme (12) is applied to solve the governing eq. (1).

Test Problem 1. Consider $A_1 = 1$, $A_2 = 0$, $A_3 = 0$, and $\psi(x, t) = x + xt^2$, in eq. (1), which is the non-linear time fractional advection equation, subject to the IC:

$$w(x, 0) = 0$$

In tab. 1, the results obtained by RBF collocation scheme are compared with the exact solution $w(x, t) = xt$ as well as with results of adomian decomposition method (ADM) and variational iteration method (VIM) at $\xi = 1$for different t and x. From tab. 1, it can be noted that the solution obtained by RBFCM is more accurate as compared to that of VIM and less accurate to that of ADM. However, with the passage of time the results obtained by ADM become less accurate than the results obtained by RBFCM. In fig. 1, numerical values for different values of ξ are plotted, which demonstrate that, by increasing the value of time fractional order ξ, the amplitude and steepness decreases. In fig. 2, the numerical values at different values of t are plotted, which indicates that, the amplitude and steepness increases by increasing the timet t. Figures 3 and 4, the 3-D plot of numerical solutions for different fractional order ξ are plotted.

Table 1. Comparison of numerical values when $\xi = 1.0$, $\Delta t = 0.0001$, $c = 1500$, and $M = 21$, for Test *Problem 1*

t	x	w_{APP}	w_{ADM}	w_{VIM}	w_{EXACT}
	0.25	0.050024	0.050000	0.050309	0.050000
0.2	0.50	0.100049	0.100000	0.100619	0.100000
	0.75	0.150073	0.150001	0.150928	0.150000
	1.0	0.200098	0.200001	0.201237	0.200000
	0.25	0.100023	0.100023	0.101894	0.100000
0.4	0.50	0.200046	0.200046	0.203787	0.200000
	0.75	0.300069	0.300069	0.305681	0.300000
	1.0	0.400093	0.400092	0.407575	0.400000
	0.25	0.150005	0.150411	0.153094	0.150000
0.6	0.50	0.300074	0.300823	0.306188	0.300000
	0.75	0.450008	0.451234	0.459282	0.450000
	1.0	0.600189	0.601646	0.612376	0.600000

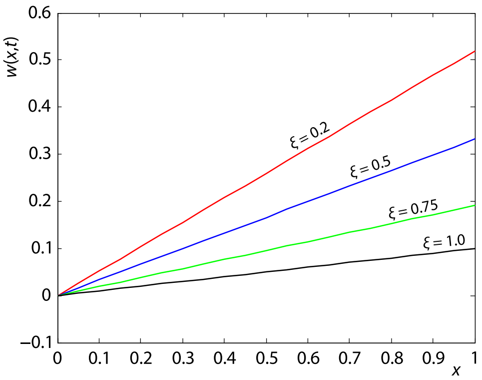

Figure 1. Numerical solution for different values of ξ at $t = 0.2$, $\Delta t = 0.0001$, $c = 1500$, and $M = 21$, for *Test Problem 1*

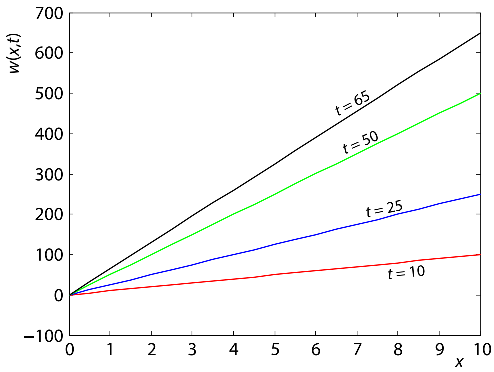

Figure 2. Numerical solution for different values of t at $\xi = 0.75$, $\Delta t = 0.1$, $c = 1500$, and $M = 21$, for Test *Problem 1*

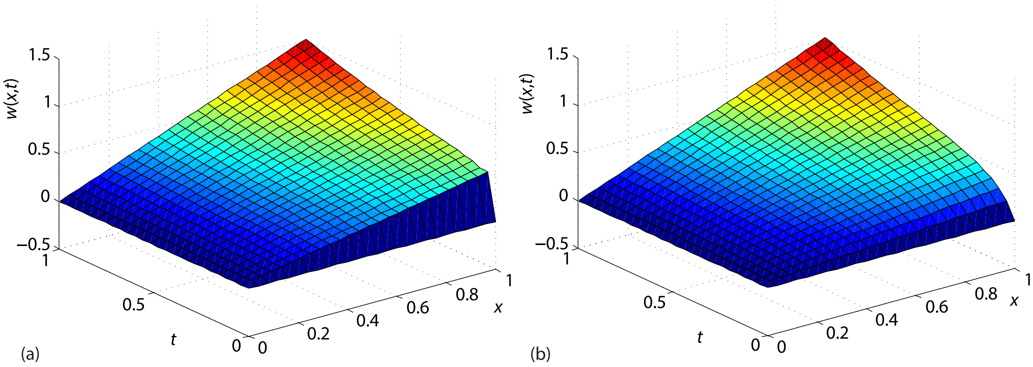

Figure 3. Numerical solution against the position *x* and time *t* at Δt = 0.04, *c* = 1500, and *M* = 26, for ξ = 0.2 (a) and ξ = 0.5 (b), for Test *Problem 1*

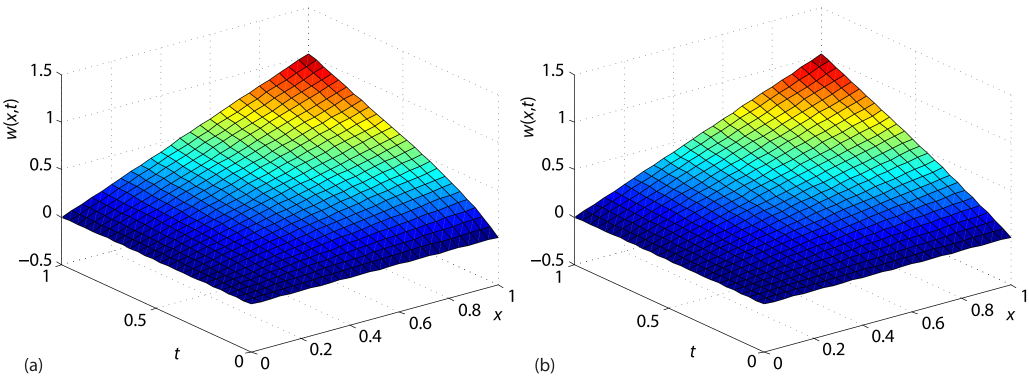

Figure 4. Numerical solution against the position *x* and time *t* at Δt = 0.04, *c* = 1500, and *M* = 26, for ξ = 0.75 (a) and ξ = 0.9 (b), for Test *Problem 1*

Conclusion

In this work, RBFCM is applied to find the numerical solution of GTFGE. The time derivative is considered in Caputo sense and the scheme is derived for $0 < \xi < 1$. Different test problems are included to check the efficiency and accuracy of the scheme, and that the current method is straightforward and simple. Results obtained by RBFCM are compared with ADM and VIM, which demonstrates the accuracy of the scheme. The numerical solutions are plotted for different values of ξ to show the influence of the fractional order ξ on the solution.

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