THE HAAR WAVELETS BASED NUMERICAL SOLUTION OF RECCATI EQUATION WITH INTEGRAL BOUNDARY CONDITION

by

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A Haar wavelet collocation method (HWCM) is presented for the solution of Riccati equation subject to the two-point and integral boundary condition. The quasilinearization technique is applied to linearized the Riccati equation and then the linearized equation with boundary condition is solved by converting into system of algebraic equation with the help of Haar wavelets. We have considered three different form of Reccati equation, two having integral boundary condition and one with two-point boundary condition. The numerical results obtained by HWCM are stable, efficient and convergent.

Key words: Haar wavelet, Riccati equation, collocation method, integral boundary, linearization

Introduction

The Riccati equation has importance in a variety of fields of engineering and applied research like phenomena related to transmission-line, optimal control theory, random process theory, diffusion and convection problems [1, 2]. Therefore, engineers and scientists are trying to solve accurately the Riccati differential equations. Because a large number of Riccati equations cannot be handle using conventional analytical techniques, alternatively they must be solved using numerical techniques or approximation methodologies. Different iterative and perturbation numerical procedures like Adomian's decomposition method [3], variational iteration method [4], homotopy perturbation method [5], decomposition method [6], and piecewise variational iteration [7] method are presented recently for the solution of Riccati equation. We consider the first-order Riccati equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x}+f(x)y^2+g(x)y+h(x)=0 \tag{1}$$

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subject to the integral boundary condition:

$$y(a)+y(b)+\int_a^b y(x)dx=\alpha \tag{2}$$

where $f(x)$, $g(x)$, $h(x)$ are functions of independent variable and α is given known constant.

Riccati differential equation is a subclass of non-linear differential equations that play an important role in a variety of sectors of applied research. A 1-D static Schrodinger equation, for example, is strongly connected to the Riccati differential equation. A polynomial in two elementary functions meeting a projective Riccati equation can be represented as a solitary wave solution of a non-linear partial differential equation [8].

The Haar wavelet scheme is a collocation-based method that has gained popularity in recent years and is widely used to solve a variety of problems in numerical analysis, signal processing, and other applicable domains. The Haar wavelet is presented in weak and strong formulations like Daubechies wavelet method [9], wavelet collocation techniques [10, 11], wavelet mesh-less schemes [12], and wavelet-Galerkin method [13]. Many Haar function-based methodologies have been used by many researchers to tackle various challenges in science and engineering [14, 15]. The Haar wavelet approach has lately been utilized to solve linear and non-linear direct and inverse problems [16-21]. In current technology, the Haar function is often employed to identify software piracy [22].

Haar wavelets as a strong computing tool for the solution of differential equation has also been used for the Riccati differential equation. In [23], the Haar wavelets are applied to eq. (1) with the simple boundary condition $y(0) = \alpha$. The Riccati differential equation is solved by Haar wavelets operational matrix method in [2] with simple boundary condition $y(0) = \alpha$. The Haar wavelets and high order Haar wavelets methods are also used for eq. (1) with the simple boundary condition $y(0) = \alpha$, where the results of the higher order methods are far better the the previous one. To solve eq. (1) by Haar wavelets method with the integral boundary condition defined in eq. (2) is challenging and classical one, therefore, it is considered in this study.

In this paper, a technique based on Haar wavelet is introduced such that the highest order derivative in the Reccati equation is approximated by Haar series. Due to the discontinuity of the Haar wavelets, one time integration of the Haar wavelet series is preferred to get the expression for y. Through quasilinearization and then the Haar wavelets series, we can express the Reccati equation into Haar matrix of order $2M \times 2M$, which can be easily solved.

Haar wavelets

For $x \in [a, b]$ the Haar function can be described:

$$h_i(x)=\begin{cases}1 & \text{if} \quad x\in[\nu_1(i),\nu_2(i)) \\ -1 & \text{if} \quad x\in[\nu_2(i),\nu_3(i)) \\ 0 & \quad\text{otherwise}\end{cases} \tag{3}$$

where

$$\nu_1(i)=a+\frac{(b-a)k}{m},\ \nu_2(i)=a+\frac{(b-a)(k+0.5)}{m},\ \text{and}\ \nu_3(i)=a+\frac{(b-a)(k+1)}{m} \tag{4}$$

where k is the translation parameter and m – the level of the wavelet such that $i = m + k + 1$, $k = 0, 1,...,m - 1$, $m = 2^j$, $j = 0, 1, ...$Considering the n^{th} time integration of the Haar function, we get:

$$p_{i,n}(x)=\int_a^x\int_a^x\ldots\int_a^x h_i(s)\mathrm{d}s^n=\begin{cases}0, & x<\nu_1(i)\\ \dfrac{(x-\nu_1(i))^n}{n!}, & x\in[\nu_1(i),\nu_2(i))\\ \dfrac{(x-\nu_1(i))^n-2(x-\nu_2(i))^n}{n!}, & x\in[\nu_2(i),\nu_3(i))\\ \dfrac{(x-\nu_1(i))^n-(-2(x-\nu_2(i)))^n+(x-\nu_3(i))^n}{n!} & \text{for}\quad x\geq\nu_3(i)\end{cases}\tag{5}$$

For $i = 1$, we define:

$$h_1(x)==\begin{cases}1, & x\in[a,b]\\ 0 & \text{elsewhere}\end{cases}\ \text{mother wavelet,}\quad\text{and}\quad p_{1,n}(x)=\frac{(x-a)^n}{n!}$$

Haar wavelet collocation method

The non-linear term in eq. (1) can be approximated by using the quasilinearization technique [24]:

$$(y^2)^{n+1}\approx 2y^n y^{n+1}-(y^2)^n\tag{6}$$

From eqs. (1) and (6) we obtain:

$$\frac{\mathrm{d}y}{\mathrm{d}x}+f(x)(2y^n y^{n+1})+g(x)y=f(x)(y^2)^n-h(x)\tag{7}$$

Now to approximate the linearized eq. (7) with the given boundary condition (2) the wavelets expansion can be written:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\sum_{i=1}^{2M}a_i h_i(x)\tag{8}$$

Integrating eq. (8) with respect to x, we get:

$$y(x)=\sum_{i=1}^{2M}a_i p_{i,1}(x)+c_0\tag{9}$$

where c_0 is the unknown integration constant. By putting eqs. (8) and (9) in eq. (7), we get:

$$\sum_{i=1}^{2M}a_i(h_i(x)+f(x)(2y^n(p_{i,1}(x)))+g(x)p_{i,1}(x))+c_0(f(x)(2y^n)+g(x))=f(x)(y)^n-h(x)\tag{10}$$

Introducing the collocation points:

$$x_j=a+\frac{(b-a)(j-0.5)}{2M}\qquad j=1,2,\ldots,2M$$

we get the following $2M$ equation in $2M+1$ unknowns:

$$\sum_{i=1}^{2M}a_i(h_i(x_j)+f(x_j)(2y^n(p_{i,1}(x_j)))+g(x_j)p_{i,1}(x_j))+\\+c_0(f(x_j)(2y^n)+g(x_j))=f(x_j)(y)^n-h(x_j)\tag{11}$$

The remaining one equation can be obtained from the given boundary condition (2):

$$y(0)+y(1)+\int_0^1 y(x)dx=\alpha$$
$$\sum_{i=1}^{2M} a_i(p_{i,1}(1)+p_{i,2}(1))+3c_0=\alpha \tag{12}$$

Now we have a complete system of $2M+1$ equations with $2M+1$ unknowns including $2M$ Haar coefficient, a_i, and single constant of integration, c_0. After finding a_i and c_0 and then using them in eq. (9) the required solution based on Haar wavelet to the Reccati equation can be obtained.

Results and discussion

We apply the HWCM to Reccati equation and used the experimental convergence order and the maximum error to observe the performance and accuracy of the current approach, which are defined:

$$C_R(M)=\log(L_\infty(M/2)/L_\infty(M))/\log(2)$$
$$L_\infty=\max_{a\le x\le b}(|y(x)-y_M(x)|)$$

To present the time efficiency, we have used the second unit of time in different tables.

Test Problem 1. Consider the Riccati equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=1-y^2 \tag{13}$$

subject to the integral boundary condition:

$$\int_0^1 y(x)dx=0.4337 \tag{14}$$

The exact solution of the aforementioned problem is

$$y(x)=\frac{e^{2x}-1}{e^{2x}+1}$$

The approximate solution of Riccati equation are interpolated using eq. (9) for different points with different values of J, which are given in tab. 1. The point wise approximate solution along with the absolute error are shown in fig. 1, where the numerical results obtained after 10 number of iterations are in good agreement with the exact solutions.

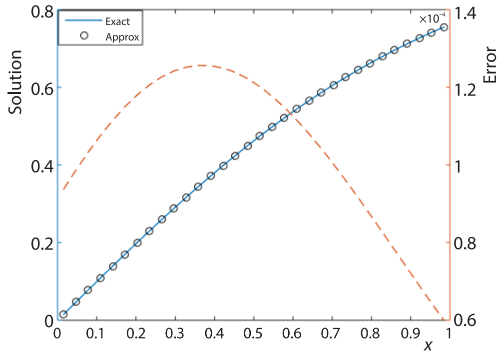

Figure 1. Comparison of exact solution and error for Test *Problem 1*

Table 1. The L_∞ error of HWCM for Test *Problem 1*

x	L_∞ HWCM		
	$M = 8$	$M = 16$	$M = 32$
0.1	$1.0229 \cdot 10^{-4}$	$9.8588 \cdot 10^{-5}$	$1.0521 \cdot 10^{-4}$
0.2	$1.0483 \cdot 10^{-4}$	$1.1633 \cdot 10^{-4}$	$1.0167 \cdot 10^{-4}$
0.3	$1.0668 \cdot 10^{-4}$	$1.2139 \cdot 10^{-4}$	$9.8156 \cdot 10^{-5}$
0.4	$2.1834 \cdot 10^{-4}$	$9.6383 \cdot 10^{-5}$	$9.8646 \cdot 10^{-5}$
0.5	$1.2042 \cdot 10^{-4}$	$3.2526 \cdot 10^{-5}$	$7.0735 \cdot 10^{-5}$
0.6	$2.0839 \cdot 10^{-4}$	$7.9084 \cdot 10^{-5}$	$8.3862 \cdot 10^{-5}$
0.7	$6.4956 \cdot 10^{-5}$	$9.6859 \cdot 10^{-5}$	$6.7263 \cdot 10^{-5}$
0.8	$3.9687 \cdot 10^{-5}$	$8.3192 \cdot 10^{-5}$	$5.8005 \cdot 10^{-5}$
0.9	$1.1879 \cdot 10^{-4}$	$4.1710 \cdot 10^{-5}$	$5.5953 \cdot 10^{-5}$
0.0	$2.1502 \cdot 10^{-4}$	$2.0288 \cdot 10^{-5}$	$2.8361 \cdot 10^{-5}$

Test Problem 2. Consider the non-linear equation of the form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 2y - y^2 \tag{15}$$

subject to the integral boundary condition:

$$y(0) + \int_0^1 y(x)\mathrm{d}x = 0.7891 \tag{16}$$

The exact solution of eq. (15) is given:

$$y(x) = \frac{\left(-1+\sqrt{2}\right)\left(-1+e^{2\sqrt{2}x}\right)}{1+\left(3-2\sqrt{2}\right)e^{2\sqrt{2}x}} \tag{17}$$

The maximum absolute error, convergence rate and CPU time for *Test Problem 2* are given in tab. 2. It is clear from the table that the error decreases when the value of J increases that means that by increasing the collocation point the accuracy increases. The comparison of numerical solution obtained by 10 iteration is performed with the exact solutions in fig. 2, where the absolute errors are also shown having magnitude of 10^{-4}.

Table 2. Results of HWCM for Test *Problem 2*

M	HWCM		
	L_∞	C_R	CPU time
2	$1.4448 \cdot 10^{-2}$	–	0.0027
4	$3.7005 \cdot 10^{-3}$	1.9651	0.0073
8	$9.0340 \cdot 10^{-4}$	2.0343	0.0098
16	$2.1018 \cdot 10^{-4}$	2.1037	0.0414
32	$3.6116 \cdot 10^{-5}$	2.5409	0.0929

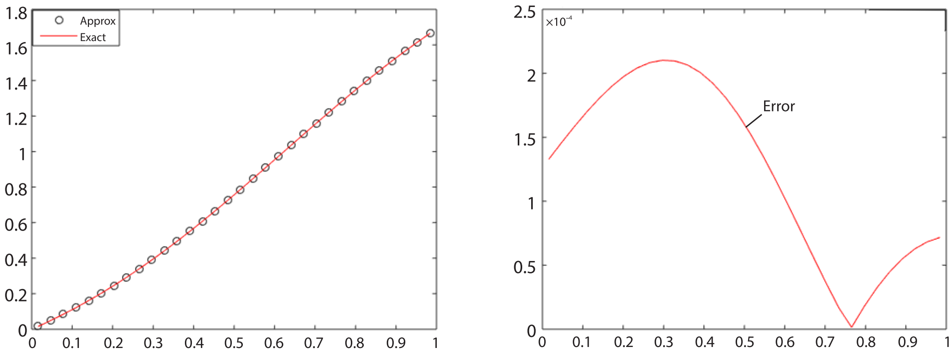

Figure 2. Comparison of exact solution with HWCM at *J* = 4 for Test *Problem 2*

Test Problem 3. Considering the variable coefficients non-linear differential equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=2x-\frac{1}{x}y+\frac{1}{x^3}y^2 \tag{18}$$

with two point boundary condition $y(1) = y(1/2)$. The exact solution of the previous problem:

$$y(x)=\frac{2x^3+30x^2}{2x+15} \tag{19}$$

For Test *Problem 3* the error is calculated in the internal points of the interval as shown in tab. 3. The numerical and exact solution are computed for two different values of J ($J = 4$ and $J = 5$). From the table it is clear that by increasing the values of J the error decreases. The comparison of exact and numerical solution are present at 32 points, which have very small amount of difference of 10^{-5} as seen in fig. 3. The The maximum absolute error, convergence rate and CPU time are given in tab. 4, which shows that the HWCM is time efficient and convergent.

Table 3. Comparison of numerical solution, exact solution and error of HWCM at *J* = 4 and *J* = 5 for Test *Problem 3*

x	$J = 4$			$J = 4$		
	HWCM	Exact	Error	HWCN	Exact	Error
0.1	0.0196986	0.0198684	$1.6976\text{E} \cdot 10^{-4}$	0.0198643	0.0198684	$4.0712 \cdot 10^{-6}$
0.2	0.0789472	0.0789610	$1.3799 \cdot 10^{-5}$	0.0789200	0.0789610	$4.0941 \cdot 10^{-5}$
0.3	0.1765265	0.1765384	$1.1888 \cdot 10^{-5}$	0.1764985	0.1765384	$3.9950 \cdot 10^{-5}$
0.4	0.3117423	0.3118987	$1.5635 \cdot 10^{-4}$	0.3118958	0.3118987	$2.9065 \cdot 10^{-6}$
0.5	0.4839347	0.4843750	$4.4023 \cdot 10^{-4}$	0.4842649	0.4843750	$1.1006 \cdot 10^{-4}$
0.6	0.6931775	0.6933333	$1.5583 \cdot 10^{-4}$	0.6933294	0.6933333	$3.8710 \cdot 10^{-6}$
0.7	0.9381507	0.9381707	$1.9947 \cdot 10^{-5}$	0.9381313	0.9381707	$3.9393 \cdot 10^{-5}$
0.8	1.2182872	1.2183132	$2.5974 \cdot 10^{-5}$	1.2182727	1.2183132	$4.0499 \cdot 10^{-5}$
0.9	1.5330463	1.5332142	$1.6789 \cdot 10^{-4}$	1.5332056	1.5332142	$8.5900 \cdot 10^{-6}$
1.0	1.8819127	1.8823529	$4.4023 \cdot 10^{-4}$	1.8822428	1.8823529	$1.1006 \cdot 10^{-4}$

Table 4. Results of HWCM for Test *Problem 3*

M	HWCM		
	L_∞	C_R	CPU time
2	$9.9696 \cdot 10^{-4}$	–	0.0104
4	$3.4456 \cdot 10^{-4}$	1.5328	0.0049
8	$9.9388 \cdot 10^{-5}$	1.7935	0.0093
16	$2.6594 \cdot 10^{-5}$	1.9019	0.0191
32	$6.8729 \cdot 10^{-6}$	1.9521	0.0518

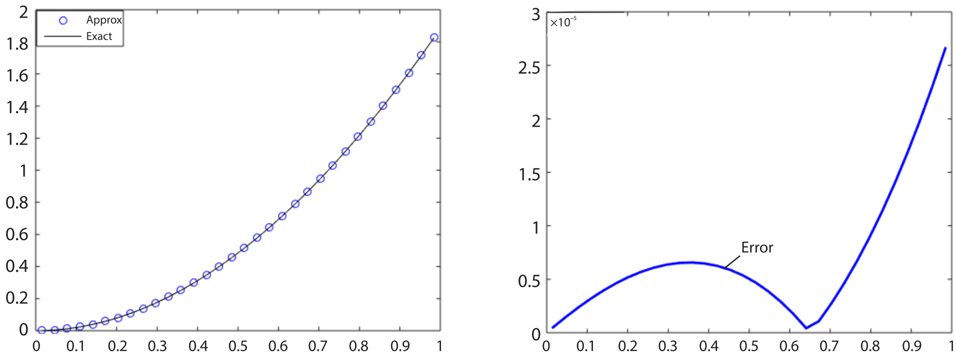

Figure 3. Comparison of exact solution with HWCM at J = 4 for Test *Problem 3*

Conclusion

In this work, HWCM is applied to solve Reccati equation with constant and variable coefficients along with different types of boundary conditions. To linearize the Reccati equation the quasilinearization technique has been introduced and then Haar wavelets are used to approximate the differential equation. The results of the proposed method are numerically stable. The HWCM is efficient and gives acceptable and accurate solutions. The HWCM is simple and accurate numerical technique which is applied to non-linear ODE and can be extended to solve PDE with various boundary conditions without any difficulty, as compared to the other numerical methods.

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