# NON-DIFFERENTIABLE FRACTIONAL ODD-SOLITON SOLUTIONS OF LOCAL FRACTIONAL GENERALIZED BROER-KAUP SYSTEM BY EXTENDING DARBOUX TRANSFORMATION 

by

Bo $X^{a, b}$, Pengchao SHI $^{a}$, and Sheng ZHANG ${ }^{a^{*}}$<br>${ }^{\text {a }}$ School of Mathematical Sciences, Bohai University, Jinzhou, China<br>${ }^{\text {b }}$ School of Educational Sciences, Bohai University, Jinzhou, China<br>Original scientific paper<br>https://doi.org/10.2298/TSCI23S1077X


#### Abstract

In this paper, a local fractional generalized Broer-Kaup (gBK) system is first derived from the linear matrix problem equipped with local space and time fractional partial derivatives, i.e, fractional Lax pair. Based on the derived fractional Lax pair, the second kind of fractional Darboux transformation (DT) mapping the old potentials of the local fractional gBK system into new ones is then established. Finally, non-differentiable frcational odd-soliton solutions of the local fractional gBK system are obtained by using two basic solutions of the derived fractional Lax pair and the established fractional DT. This paper shows that the DT can be extended to construct non-differentiable fractional soliton solutions of some local fractional non-linear evolution equations in mathematical physics.


Key words: non-differentiable fractional odd-soliton solution, fractional Lax pair, second kind of fractional DT, local fractional gBK system

## Introduction

Fractional calculus has many applications in various fields. It has been shown that fractional calculus plays an irreplaceable role in studying the heat transfer characteristics of textiles [1-3] and the mechnical and electrial properties of real materials [4]. Recently, the definitions of fractional derivatives and their physical meanings, as well as the construction and solution of fractional-order models have become important research topics [5-15]. In 2016, starting from Cattaneo constitutive relation with a Jeffrey's kernel, Hristov [16] developed the derivation of a transient heat diffusion equation with relaxation term expressed by Caputo-Fabrizio time-fractional derivative. More importantly, Hristov's approach not only allows seeing the physical background of Caputo-Fabrizio time-fractional derivative but also demonstrates how to modify other constitutive equations by using non-singular fading memories.

The main purpose of this paper is to extend DT to fractional-order non-linear evolution models. In order to illustrate the feasibility and effectiveness of extending DT, we derive the ( $1+1$ )-D local fractional gBK system:

$$
\begin{gather*}
D_{t}^{\alpha} v=D_{x}^{2 \alpha} v-2 v D_{x}^{\alpha} v-4 D_{x}^{\alpha} w \\
D_{t}^{\alpha} w=-D_{x}^{2 \alpha} w-2 D_{x}^{\alpha}(w v)-2 D_{x}^{\alpha} v \tag{1}
\end{gather*}
$$

for the first time and then construct its non-differentiable fractional odd-soliton solutions by extending DT. Here, $v$ and $w$ are two non-differentiable functions in the sense of integer-order

[^0]derivative, $D_{x}^{\alpha}$ and $D_{t}^{\alpha}$ are the local fractional partial derivative operators [17] with respect to $x$ and $t$, respectively, and the range of fractional order $\alpha$ is assumed to be $0<\alpha \leq 1$.

To the best of our knowledge, the local fractional gBK system (1) has not been reported. If we let $\alpha=1$ and $v$ and $w$ are two differentiable functions, then eq. (1) becomes the known gBK equations [18] which were first derived by Zhang et al. [15]. As for the known gBK equations [18], there are some results: Zhang and Liu [19] gave the third kind of DT and obtained even-soliton solutions, Zhang and Zheng [20] obtained the bilinear forms and constructed N -soliton solutions, and Zhang and Xu [21] verified Painleve integrability and derived a Bäcklund transformation and $N$-soliton solutions.

## Fractional Lax pair

Theorem 1. The ( $1+1$ )-D local fractional gBK system (1) can be derived from the fractional Lax pair, i.e, the fractional linear spectral equation:

$$
D_{x}^{\alpha} \phi=M \phi, M=\left(\begin{array}{cc}
-\lambda+\frac{1}{2} v & 1  \tag{2}\\
-2 w-2 & \lambda-\frac{1}{2} v
\end{array}\right)
$$

and its fractional time evolution equation:

$$
D_{t}^{\alpha} \phi=N \phi, N=\left(\begin{array}{cc}
2 \lambda^{2}+\frac{1}{2}\left(D_{x}^{\alpha} v-v^{2}\right) & -2 \lambda-v  \tag{3}\\
4 \lambda(1+w)+2 v+2 D_{x}^{\alpha} w+2 w v & -2 \lambda^{2}+\frac{1}{2}\left(v^{2}-D_{x}^{\alpha} v\right)
\end{array}\right)
$$

where the eigenfunction $\phi=\left[\phi_{1}(x, t, \lambda), \phi_{1}(x, t, \lambda)\right]^{T}$ and the potentials $v=v(x, t)$ and $w(x, t)$ are undifferentiable functions defined on a fractal set, the spectral parameter $\lambda$ is a constant.

Proof. In view of the condition of complete integrability $D_{t}^{\alpha} D_{x}^{\alpha} \phi=D_{x}^{\alpha} D_{t}^{\alpha} \phi$, we can establish the fractional zero-curvature equation:

$$
\begin{equation*}
D_{t}^{\alpha} M-D_{x}^{\alpha} N+M N-N M=0 \tag{4}
\end{equation*}
$$

Directly comparing the four components of eq. (4) yields eq. (1).
Aforementioned fractional Lax pair eqs. (2) and (3) play important role in the construction of fractional DT for the local fractional gBK system (1). That is to construct a gauge transformation:

$$
\begin{equation*}
\bar{\phi}=T \phi \tag{5}
\end{equation*}
$$

of the fractional linear matrix problems (2) and (3). Here $T$ is a $2 \times 2$ matrix to be determined, $\bar{\phi}$ satisfies another linear matrix problems with the same forms as eqs. (2) and (3):

$$
\begin{align*}
& D_{t}^{\alpha} \bar{\phi}=\bar{M} \bar{\phi}, \bar{M}  \tag{6}\\
&=\left(D_{x}^{\alpha} T+T M\right) T^{-1}  \tag{7}\\
& D_{t}^{\alpha} \bar{\phi}=\bar{N} \bar{\phi}, \bar{N}
\end{align*}=\left(D_{t}^{\alpha} T+T N\right) T^{-1}
$$

where

$$
\bar{M}=\left(\begin{array}{cc}
-\lambda+\frac{1}{2} \bar{v} & 1  \tag{8}\\
-2 \bar{w}-2 & \lambda-\frac{1}{2} \bar{v}
\end{array}\right)
$$

$$
\bar{N}=\left(\begin{array}{cc}
2 \lambda^{2}+\frac{1}{2}\left(D_{x}^{\alpha} \bar{v}-\bar{v}^{2}\right) & -2 \lambda-\bar{v}  \tag{9}\\
4 \lambda(1+\bar{w})+2 \bar{v}+2 D_{x}^{\alpha} \bar{w}+2 \bar{w} \bar{v} & -2 \lambda^{2}+\frac{1}{2}\left(\bar{v}^{2}-D_{x}^{\alpha} \bar{v}\right)
\end{array}\right)
$$

## Fractional DT

Firstly, we assume that:

$$
T=T(\lambda)=a\left[\begin{array}{ll}
A(\lambda) & B(\lambda)  \tag{10}\\
C(\lambda) & D(\lambda)
\end{array}\right]
$$

where

$$
\begin{equation*}
A(\lambda)=\sum_{k=0}^{N-1} a_{k} \lambda^{k}, B(\lambda)=\sum_{k=0}^{N-1} b_{k} \lambda^{k}, C(\lambda)=\sum_{k=0}^{N-1} c_{k} \lambda^{k}, D(\lambda)=\lambda^{N}+\sum_{k=0}^{N-1} d_{k} \lambda^{k} \tag{11}
\end{equation*}
$$

while $a, a_{k}, b_{k}, c_{k}$, and $d_{k}(0 \leq k \leq N-1)$ are non-differentiable functions of $x$ and $t$.
When the spectral parameter $\lambda=\lambda_{j}$, the fractional linear matrix problems (2) and (3) have two basic solutions:

$$
\begin{align*}
& \phi\left(\lambda_{j}\right)=\left[\phi_{1}\left(x, t, \lambda_{j}\right), \phi_{2}\left(x, t, \lambda_{j}\right)\right]^{T}  \tag{12}\\
& \varphi\left(\lambda_{j}\right)=\left[\varphi_{1}\left(x, t, \lambda_{j}\right), \varphi_{2}\left(x, t, \lambda_{j}\right)\right]^{T} \tag{13}
\end{align*}
$$

In order to determine the matrix $T$ in eq. (10), we let constants $\gamma_{j}(1 \leq j \leq 2 N-1)$ satisfy:

$$
\begin{align*}
& A\left(\lambda_{j}\right) \phi_{1}\left(\lambda_{j}\right)+B\left(\lambda_{j}\right) \phi_{2}\left(\lambda_{j}\right)-\gamma_{j}\left[A\left(\lambda_{j}\right) \varphi_{1}\left(\lambda_{j}\right)+B\left(\lambda_{j}\right) \varphi_{2}\left(\lambda_{j}\right)\right]=0  \tag{14}\\
& C\left(\lambda_{j}\right) \varphi_{1}\left(\lambda_{j}\right)+D\left(\lambda_{j}\right) \varphi_{2}\left(\lambda_{j}\right)-\gamma_{j}\left[C\left(\lambda_{j}\right) \phi_{1}\left(\lambda_{j}\right)+D\left(\lambda_{j}\right) \phi_{2}\left(\lambda_{j}\right)\right]=0 \tag{15}
\end{align*}
$$

We rewrite eqs. (14) and (15) as the linear algebraic equations:

$$
\begin{align*}
& A\left(\lambda_{j}\right)+\sigma_{j} B\left(\lambda_{j}\right)=0  \tag{16}\\
& C\left(\lambda_{j}\right)+\sigma_{j} D\left(\lambda_{j}\right)=0 \tag{17}
\end{align*}
$$

or

$$
\begin{gather*}
\sum_{k=0}^{N-1}\left(a_{k}+\sigma_{j} b_{k}\right) \lambda_{j}^{k}=0  \tag{18}\\
\sum_{k=0}^{N-1}\left(c_{k}+\sigma_{j} d_{k}\right) \lambda_{j}^{k}=-\sigma_{j} \lambda_{j}^{N} \tag{19}
\end{gather*}
$$

with

$$
\begin{equation*}
\sigma_{j}=\frac{\phi_{2}\left(\lambda_{j}\right)-\gamma_{j} \varphi_{2}\left(\lambda_{j}\right)}{\phi_{1}\left(\lambda_{j}\right)-\gamma_{j} \varphi_{1}\left(\lambda_{j}\right)},(1 \leq j \leq 2 N-1) \tag{20}
\end{equation*}
$$

where $\lambda_{\mathrm{j}}\left(\lambda_{\mathrm{j}} \neq \lambda_{k}\right.$ for $\left.k \neq j\right)$ and $\gamma_{j}$ are constants properly chosen so that the determinants of the coefficients of eqs. (18) and 19) are not zeros. Thus, $a_{k}, b_{k}, c_{k}$, and $d_{k}(0 \leq k \leq N-1)$ can be uniquely determined by eqs. (18) and (19), while $a$ will be further determined later.

We can see from eqs. (10) and (11) that:

$$
\begin{equation*}
\operatorname{det} T\left(\lambda_{j}\right)=a^{2}\left[A\left(\lambda_{j}\right) D\left(\lambda_{j}\right)-B\left(\lambda_{j}\right) C\left(\lambda_{j}\right)\right] \tag{21}
\end{equation*}
$$

which is a polynomial of degree $2 N-1$ about $\lambda$. In view of eqs. (16) and (17), we then have $\operatorname{det} T\left(\lambda_{\mathrm{j}}\right)=0$ and from this know that $\lambda_{\mathrm{j}}(1 \leq j \leq 2 N-1)$ are $2 N-1$ roots of $\operatorname{det} T\left(\lambda_{\mathrm{j}}\right)$, namely:

$$
\begin{equation*}
\operatorname{det} T\left(\lambda_{j}\right)=\prod_{j=1}^{2 N}\left(\lambda-\lambda_{j}\right) \tag{22}
\end{equation*}
$$

Proposition 1. If let the matrix $\bar{M}$ in eq. (8) has the same form as $M$ in eq. (2) with the relationships between the old potentials $v$ and $w$ and the new ones $\bar{v}$ and $\bar{w}$ :

$$
\begin{align*}
& \bar{v}=D_{x}^{\alpha} \ln a_{N-1}+v  \tag{23}\\
& \bar{w}=-D_{x}^{\alpha} d_{N-1}+w
\end{align*}
$$

then $a$ has to solve the local fractional differential equation:

$$
\begin{equation*}
D_{x}^{\alpha} \ln a=\frac{v}{2}-\frac{\bar{v}}{2} \tag{24}
\end{equation*}
$$

Here the assumptions had been used:

$$
\begin{equation*}
b_{N-1}=\frac{1}{2}, c_{N-1}=-w-1 \tag{25}
\end{equation*}
$$

Proof. Since $T^{-1}=T^{*} / \operatorname{det} T$ and

$$
\left(D_{x}^{\alpha} T+T U\right) T^{*}=\left[\begin{array}{ll}
f_{11}(\lambda) & f_{12}(\lambda)  \tag{26}\\
f_{21}(\lambda) & f_{22}(\lambda)
\end{array}\right]
$$

It can be checked that $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are polynomials of $2 N$ degree with respect to $\lambda$ while $f_{12}(\lambda)$ and $f_{21}(\lambda)$ are polynomials of $2 N-1$ degree with respect to $\lambda$. Taking $\lambda=\lambda_{\mathrm{j}}(1 \leq j \leq 2 N-1)$ and using eqs. (2), (12), (13), and (20) yields a local fractional Riccati equation:

$$
\begin{equation*}
D_{x}^{\alpha} \sigma_{j}=-2 w-2+2\left(\lambda_{j}-\frac{v}{2}\right) \sigma_{j}-\sigma_{j}^{2} \tag{27}
\end{equation*}
$$

A direct computation shows that all $\lambda=\lambda_{j}(1 \leq j \leq 2 N-1)$ are the roots of the polynomials $f_{k l}(\lambda)(k, l=1,2)$. Based on this fact, we suppose that eq. (26) can be written:

$$
\begin{equation*}
\left(D_{x}^{\alpha} T+T M\right) T^{*}=(\operatorname{det} T) P(\lambda) \tag{28}
\end{equation*}
$$

where the matrix:

$$
P(\lambda)=\left(\begin{array}{cc}
P_{11}^{(1)} \lambda+P_{11}^{(0)} & P_{12}^{(0)}  \tag{29}\\
P_{21}^{(0)} & P_{22}^{(1)} \lambda+P_{22}^{(0)}
\end{array}\right)
$$

and $P_{k l}^{(m)}(k, l=1,2 ; m=0,1)$ are independent of $\lambda$. For convenience, we further rewrite eq. (28):

$$
\begin{equation*}
D_{x}^{\alpha} T+T M=P(\lambda) T \tag{30}
\end{equation*}
$$

Comparing the coefficients of $\lambda^{N+1}, \lambda^{N}$, and $\lambda^{N-1}$ in eq. (30) and using eqs. (23)-(25), we have:

$$
\begin{equation*}
P_{11}^{(0)}=-1, P_{11}^{(0)}=\frac{\bar{v}}{2}, P_{12}^{(0)}=1, P_{12}^{(0)}=-2 \bar{w}-2, P_{22}^{(1)}=1, P_{22}^{(0)}=-\frac{\bar{v}}{2} \tag{31}
\end{equation*}
$$

which imply that $\bar{M}$ is equal to $P(\lambda)$ determined by eqs. (29) and (31).

Proposition 2. Suppose that $a$ satisfies the following time-dependence:

$$
\begin{gather*}
D_{t}^{\alpha}(\ln a)=-D_{x}^{\alpha} d_{N-1}+2 a_{N-2}+4 b_{N-3}-2 d_{N-2}+v a_{N-1}- \\
-2 d_{N-1}\left(a_{N-1}+2 b_{N-2}-d_{N-1}\right)-\frac{1}{2}\left(v^{2}-v_{x}\right) \tag{32}
\end{gather*}
$$

then the matrix $\bar{N}$ determined by eq. (7) arrives at eq. (9) with the same form as $N$ in eq. (3), mapping the two old potentials $v$ and $w$ into the two new potentials $\bar{v}$ and $\bar{w}$ through the same fractional DT in eqs. (5) and (23).

Proof. Using $T^{-1}=T^{*} / \operatorname{det} T$ and letting

$$
\left(D_{t}^{\alpha} T+T N\right) T^{*}=\left[\begin{array}{ll}
g_{11}(\lambda) & g_{12}(\lambda)  \tag{33}\\
g_{21}(\lambda) & g_{22}(\lambda)
\end{array}\right]
$$

we can see that $g_{11}(\lambda)$ and $g_{22}(\lambda)$ are polynomials of $2 N+1$ degree with respect to $\lambda$ while $g_{12}(\lambda)$ and $g_{21}(\lambda)$ are polynomials of $2 N$ degree with respect to $\lambda$.

Taking $\lambda=\lambda_{\mathrm{j}}(1 \leq j \leq 2 N)$ and employing eqs. (3), (12), (13), and (20), we can obtain another Riccati equation:

$$
\begin{equation*}
D_{t}^{\alpha} \sigma_{j}=4 \lambda_{j}(1+w)+2 v+2 D_{x}^{\alpha} w+2 w v+\left[-4 \lambda_{j}^{2}+\left(v^{2}-D_{x}^{\alpha} v\right)\right] \sigma_{j}+\left(2 \lambda_{j}+v\right) \sigma_{j}^{2} \tag{34}
\end{equation*}
$$

Obviously, all $\lambda_{\mathrm{j}}(1 \leq j \leq 2 N)$ are the roots of $g_{k l}(\lambda)(k, l=1,2)$. Thus, we can write eq. (33) as:

$$
\begin{equation*}
\left(D_{t}^{\alpha} T+T N\right) T^{*}=(\operatorname{det} T) Q(\lambda) \tag{35}
\end{equation*}
$$

with

$$
Q(\lambda)=\left(\begin{array}{cc}
Q_{11}^{(2)} \lambda^{2}+Q_{11}^{(1)} \lambda+Q_{11}^{(0)} & Q_{12}^{(1)} \lambda+Q_{12}^{(0)}  \tag{36}\\
Q_{21}^{(1)} \lambda+Q_{21}^{(0)} & Q_{22}^{(2)} \lambda^{2}+Q_{22}^{(1)} \lambda+Q_{22}^{(0)}
\end{array}\right)
$$

where $Q_{k l}^{m}(\lambda)(k, l=1,2, m=0,1)$ are independent of $\lambda$. Then eq. (35) can be re-written:

$$
\begin{equation*}
D_{t}^{\alpha} T+T N=Q(\lambda) T \tag{37}
\end{equation*}
$$

Comparing the coefficients of $\lambda^{N+2}, \lambda^{N+1}, \lambda^{N}$, and $\lambda^{N-1}$ in eq. (37) yields:

$$
\begin{gather*}
Q_{22}^{(2)}=-2, Q_{11}^{(2)}=2, Q_{12}^{(1)}=-2, Q_{11}^{(1)}=0, Q_{22}^{(1)}=0, Q_{12}^{(1)}=-2  \tag{38}\\
Q_{12}^{(0)}=-2 a_{N-1}-4 b_{N-2}+2 d_{N-1}  \tag{39}\\
Q_{11}^{(0)}=D_{t}^{\alpha}(\ln \alpha)+\frac{1}{2}\left(v^{2}-v_{x}\right)-8 b_{N-3}-2 v a_{N-1}-4 a_{N-2}+4 d_{N-2}-2 Q_{12}^{(0)} d_{N-1}  \tag{40}\\
Q_{21}^{(1)} a_{N-1}=4 c_{N-2}+4(w+1) d_{N-1}+2 v+2 w_{x}+2 w v  \tag{41}\\
Q_{21}^{(0)}=2 d_{N-1} D_{t}^{\alpha}(\ln a)+2 D_{t}^{\alpha} d_{N-1}+\left(v^{2}-D_{x}^{\alpha} v\right) d_{N-1}- \\
-2 v c_{N-1}-2 Q_{21}^{(1)} b_{N-2}-4 c_{N-2}-Q_{22}^{(0)} d_{N-1}  \tag{42}\\
Q_{22}^{(0)}=-2 C_{N-1}+D_{t}^{\alpha}(\ln a)-Q_{21}^{(1)} b_{N-1}+\frac{1}{2}\left(v^{2}-v_{x}\right) \tag{43}
\end{gather*}
$$

where $a_{-1}=b_{-1}=c_{-1}=d_{-1}=0$ have been used.

On the other hand, comparing the coefficient of $\lambda^{N-1}$ in eq. (30) we gain the relation expressions:

$$
\begin{gather*}
D_{x}^{\alpha} a_{N-1}=(\bar{v}-v) a_{N-1}  \tag{44}\\
\bar{v}=2 a_{N-1}+4 b_{N-2}-2 d_{N-1}  \tag{45}\\
D_{x}^{\alpha} c_{N-1}=2 c_{N-2}-v c_{N-1}+2(w+1) d_{N-1}-2(\bar{w}+1) a_{N-1}  \tag{46}\\
D_{x}^{\alpha} b_{N-2}=-a_{N-2}-2 b_{N-3}+\bar{v} b_{N-2}+d_{N-2} \tag{47}
\end{gather*}
$$

With the help of eqs. (23), (25), (32), and (39)-(47), we can derive:

$$
\begin{gather*}
Q_{12}^{(0)}=-\bar{v}, Q_{11}^{(0)}=\frac{1}{2}\left(D_{x}^{\alpha} \bar{v}-\bar{v}^{2}\right)  \tag{48}\\
Q_{21}^{(1)}=4(\bar{w}+1), Q_{21}^{(0)}=2 \bar{v}+2 D_{x}^{\alpha} \bar{w}+2 \bar{w} \bar{v}, Q_{22}^{(0)}=\frac{1}{2}\left(\bar{v}^{2}-D_{x}^{\alpha} \bar{v}\right) \tag{49}
\end{gather*}
$$

In fact, it is obvious that eqs. (39) and (45) lead to the first expression of eq. (48). For the second expression, we substitute eq. (32) and the first expression of eq. (48) into eq. (40) and use eqs. (44) and (47), then one has:

$$
\begin{align*}
& Q_{11}^{(0)}=-D_{x}^{\alpha} d_{N-1}-2 a_{N-2}-4 b_{N-3}+2 d_{N-2}-\bar{v} a_{N-1}+D_{x}^{\alpha} a_{N-1}+\bar{v} d_{N-1}= \\
&=-D_{x}^{\alpha} d_{N-1}+2 D_{x}^{\alpha} b_{N-2}+D_{x}^{\alpha} a_{N-1}-2 \bar{v} b_{N-2}-\bar{v} a_{N-1}+\bar{v} d_{N-1}=  \tag{50}\\
&=\frac{D_{x}^{\alpha} \bar{v}}{2}-\frac{\bar{v}}{2}\left(2 a_{N-1}+4 b_{N-2}-2 d_{N-1}\right)
\end{align*}
$$

which can be written as the second expression of eq. (48) by using eq. (45). Taking $c_{N-1}=-w-1$ from eq. (25) and substituting it into eq. (46), we have:

$$
\begin{equation*}
w_{x}=-2 c_{N-2}-v(w+1)-2(w+1) d_{N-1}+2(\bar{w}+1) a_{N-1} \tag{51}
\end{equation*}
$$

Inserting eq. (51) into eq. (41) reaches the first expression of eq. (49). By a similar way to eq. (50), we can derive the third expression of eq. (49) from eqs. (32), (43), and (47) and the first two expressions of eqs. (48) and (49). The substitution of eqs. (23), (32), (45), (47), and the first and third expressions of eq. (49) into eq. (42) gives:

$$
\begin{align*}
& Q_{21}^{(0)}=2 d_{N-1}\left(\frac{\bar{v}^{2}}{2}-\frac{D_{x}^{\alpha} \bar{v}}{2}-2 D_{x}^{\alpha} d_{N-1}\right)+D_{t}^{\alpha} d_{N-1}-  \tag{52}\\
& -2 v c_{N-1}-8(\bar{w}+1)-4 c_{N-2}+\left(\bar{v}^{2}-D_{x}^{\alpha} \bar{v}\right) d_{N-1}
\end{align*}
$$

At the same time, using eqs. (1), (23), and (46), we have:

$$
\begin{gather*}
D_{t}^{\alpha} D_{x}^{\alpha} d_{N-1}=D_{t}^{\alpha} w-D_{t}^{\alpha} \bar{w}=D_{x}^{\alpha}\left[-D_{x}^{\alpha} w-2(w v)-2 v\right]- \\
 \tag{53}\\
-D_{x}^{\alpha}\left[-D_{x}^{\alpha} \bar{w}-2(\bar{w} \bar{v})-2 \bar{v}\right]  \tag{54}\\
-4 c_{N-2}=-2 D_{x}^{\alpha} c_{N-1}-2 v c_{N-1}+4(w+1) d_{N-1}-4(\bar{w}+1) a_{N-1}  \tag{55}\\
4(w+1)=4(\bar{w}+1)+4 D_{x}^{\alpha} d_{N-1}
\end{gather*}
$$

It is easy to see from eq. (53) that:

$$
\begin{equation*}
D_{t}^{\alpha} d_{N-1}=-D_{x}^{\alpha} w-2(w v)-2 v+D_{x}^{\alpha} \bar{w}+2(\bar{w} \bar{v})+2 \bar{v} \tag{56}
\end{equation*}
$$

Then the substitution of eqs. (54)-(56) into eq. (52) arrives at the third expression of eq. (49).

Therefore, with the help of eqs. (7), (36), (48), and (49) we can make the conclusion that $\bar{N}=Q(\lambda)$. Thus, the proof is completed.

With previous Propositions 1 and 2 as preparation, we know $(\phi, v, w) \rightarrow(\bar{\phi}, \bar{v}, \bar{w})$ gives the second kind of fractional DT of the fractional gBK system (1). That is the following Theorem 2.

Theorem 2. Suppose that $a_{N-1}$ and $d_{N-1}$ can be determined by the linear algebraic eqs. (18) and (19) with $b_{N-1}=1 / 2$ and $c_{N-1}=-w-1$, then the old fractional solutions ( $v, w$ ) of the fractional gBK sysem (1) are mapped into the new fractional solutions $(\bar{v}, \bar{w})$ under the fractional DT consisting of eqs. (5) and (23).

## Non-differentiable fractional odd-soliton solutions

We employ, in this section, the second kind of fractional $N$-fold DT (5) and (23) to obtain the fractional odd-solution solutions of the fractional gBK system (1). For this purpose, we select the seed solutions $(v, w)$ as constants. Then the fractional Lax pair (4) and (5) give two basic solutions:

$$
\begin{equation*}
\phi\left(\lambda_{j}\right)=\binom{\cosh _{\alpha} \xi_{j}}{c_{j} \sinh _{\alpha} \xi_{j}+k_{j} \cosh _{\alpha} \xi_{j}}, \varphi\left(\lambda_{j}\right)=\binom{\sinh _{\alpha} \xi_{j}}{c_{j} \cosh _{\alpha} \xi_{j}+k_{j} \sinh _{\alpha} \xi_{j}} \tag{57}
\end{equation*}
$$

where

$$
\begin{equation*}
\xi_{j}=\frac{c_{j}\left(x^{\alpha}+b_{j} t^{\alpha}\right)}{\Gamma(1+\alpha)}, c_{j}=\sqrt{k_{j}^{2}-2 w-2}, b_{j}=-2 \lambda_{j}-v, k_{j}=\lambda_{j}-\frac{1}{2} v,(1 \leq j \leq 2 N-1) \tag{58}
\end{equation*}
$$

From eq. (20) we have:

$$
\begin{equation*}
\sigma_{j}=\frac{c_{j}\left(\tanh _{\alpha} \xi_{j}-\gamma_{j}\right)}{1-\gamma_{j} \tanh _{\alpha} \xi_{j}}+k_{j} \quad(1 \leq j \leq 2 N-1) \tag{59}
\end{equation*}
$$

In eqs. (57)-(59), $\sinh _{a} \xi_{j}, \cosh _{a} \xi_{j}$, and, $\tanh _{\alpha} \xi_{j}$, are the generalized hyperbolic functions defined in fractal set [18]. Substituting eqs. (55) and (56) into eqs. (18) and (19) and using Cramer's rule, we have the following Theorem 3.

Theorem 3. Suppose that the second kind of fractional $N$-fold DT $(\phi, v, w) \rightarrow(\bar{\phi}, \bar{v}, \bar{w})$ determined by eqs. (5) and (23), then the fractional gBK system (1) have the fractional ( $2 N-1$ )-soliton solutions:

$$
\begin{align*}
& \bar{v}[N]=D_{x}^{\alpha} \ln a_{N-1}+v \\
& \bar{w}[N]=-D_{x}^{\alpha} d_{N-1}+w \tag{60}
\end{align*}
$$

with

$$
\begin{gather*}
a_{N-1}=-\frac{\tilde{\Delta}}{2 \Delta}, d_{N-1}=\frac{\tilde{\Delta}_{d_{N-1}}}{\tilde{\Delta}}  \tag{61}\\
\Delta=\left|\begin{array}{ccccccc}
1 & \sigma_{1} & \lambda_{1} & \sigma_{1} \lambda_{1} & \cdots & \sigma_{1} \lambda_{1}^{N-2} & \lambda_{1}^{N-1} \\
1 & \sigma_{2} & \lambda_{2} & \sigma_{2} \lambda_{2} & \cdots & \sigma_{2} \lambda_{2}^{N-2} & \lambda_{2}^{N-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \sigma_{2 N-1} & \lambda_{2 N-1} & \sigma_{2 N-1} \lambda_{2 N-1} & \cdots & \sigma_{2 N-1} \lambda_{2 N-1}^{N-2} & \lambda_{2 N-1}^{N-1}
\end{array}\right| \tag{62}
\end{gather*}
$$

$$
\begin{gather*}
\tilde{\Delta}=\left|\begin{array}{ccccccc}
1 & \sigma_{1} & \lambda_{1} & \sigma_{1} \lambda_{1} & \cdots & \sigma_{1} \lambda_{1}^{N-2} & \sigma_{1} \lambda_{1}^{N-1} \\
1 & \sigma_{2} & \lambda_{2} & \sigma_{2} \lambda_{2} & \cdots & \sigma_{2} \lambda_{2}^{N-2} & \sigma_{2} \lambda_{2}^{N-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \sigma_{2 N-1} & \lambda_{2 N-1} & \sigma_{2 N-1} \lambda_{2 N-1} & \cdots & \sigma_{2 N-1} \lambda_{2 N-1}^{N-2} & \sigma_{2 N-1} \lambda_{2 N-1}^{N-1}
\end{array}\right|  \tag{63}\\
\tilde{\Delta}_{d_{N-1}}=\left|\begin{array}{ccccccc}
1 & \sigma_{1} & \lambda_{1} & \sigma_{1} \lambda_{1} & \cdots & \sigma_{1} \lambda_{1}^{N-2} & (w+1) \lambda_{1}^{N-1}-\sigma_{1} \lambda_{1}^{N} \\
1 & \sigma_{2} & \lambda_{2} & \sigma_{2} \lambda_{2} & \cdots & \sigma_{2} \lambda_{2}^{N-2} & (w+1) \lambda_{2}^{N-1}-\sigma_{2} \lambda_{2}^{N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \sigma_{2 N-1} & \lambda_{2 N-1} & \sigma_{2 N-1} \lambda_{2 N-1} & \cdots & \sigma_{2 N-1} \lambda_{2 N-1}^{N-2} & (w+1) \lambda_{2 N-1}^{N-1}-\sigma_{2 N-1} \lambda_{2 N-1}^{N}
\end{array}\right| \tag{64}
\end{gather*}
$$

where $\sigma_{j}(j=1,2, \ldots, 2 N-1)$ are expressed by eq. (59).
For the fractional one- and three-soliton solutions, we let $N=1$ and then from eqs. (25), (61)-(64) we get:

$$
\begin{equation*}
a_{0}=\frac{\sigma_{1}}{2}, b_{0}=\frac{1}{2}, c_{0}=-w-1, d_{0}=\frac{w+1}{\sigma_{1}}-\lambda_{1} \tag{65}
\end{equation*}
$$

Thus one obtains the fractional one-soliton solutions of the local fractional gBK system (1):

$$
\begin{gather*}
\bar{v}[1]=\frac{D_{x}^{\alpha} \sigma_{1}}{\sigma_{1}}+v \\
\bar{w}[1]=-D_{x}^{\alpha}\left(\frac{w+1}{\sigma_{1}}\right)+w \tag{66}
\end{gather*}
$$

where

$$
\begin{equation*}
\sigma_{1}=\frac{c_{1}\left(\tanh _{\alpha} \xi_{1}-\gamma_{1}\right)}{1-\gamma_{1} \tanh _{\alpha} \xi_{1}}+k_{1}, \xi_{1}=\frac{c_{1}\left(x^{\alpha}+b_{1} t^{\alpha}\right)}{\Gamma(1+\alpha)}, c_{1}=\sqrt{k_{1}^{2}-2 w-2}, b_{1}=-2 \lambda_{1}-v, k_{1}=\lambda_{1}-\frac{v}{2} \tag{67}
\end{equation*}
$$

Setting $N=2$, from eq. (25) we have $b_{1}=1 / 2$ and $c_{1}=-w-1$. Then using eqs. (58)(64) we obtain the following fractional three-soliton solutions of the fractional gBK system (1):

$$
\begin{gather*}
\bar{v}[2]=D_{x}^{\alpha} \ln a_{1}+v  \tag{68}\\
\bar{w}[2]=-D_{x}^{\alpha} d_{1}+w
\end{gather*}
$$

where

$$
\begin{gather*}
a_{1}=-\frac{\tilde{\Delta}}{2 \Delta}, d_{1}=\frac{\tilde{\Delta}_{d_{1}}}{\tilde{\Delta}}, \Delta=\left|\begin{array}{lll}
1 & \sigma_{1} & \lambda_{1} \\
1 & \sigma_{2} & \lambda_{2} \\
1 & \sigma_{3} & \lambda_{3}
\end{array}\right|, \tilde{\Delta}=\left|\begin{array}{lll}
1 & \sigma_{1} & \lambda_{1} \sigma_{1} \\
1 & \sigma_{2} & \lambda_{2} \sigma_{2} \\
1 & \sigma_{3} & \lambda_{3} \sigma_{3}
\end{array}\right| \\
\Delta_{d_{1}}=\left|\begin{array}{lll}
1 & \sigma_{1} & (w+1) \lambda_{1}+\sigma_{1} \lambda_{1}^{2} \\
1 & \sigma_{2} & (w+1) \lambda_{2}+\sigma_{2} \lambda_{2}^{2} \\
1 & \sigma_{3} & (w+1) \lambda_{3}+\sigma_{3} \lambda_{3}^{2}
\end{array}\right| \tag{69}
\end{gather*}
$$

where $\sigma_{j}, \xi_{j}, c_{j}, b_{j}$, and $k_{j}$ for $j=1,2,3$ are expressed by:

$$
\begin{align*}
\sigma_{j} & =\frac{c_{j}\left(\tanh _{\alpha} \xi_{j}-\gamma_{j}\right)}{1-\gamma_{j} \tanh _{\alpha} \xi_{j}}+k_{j}, \xi_{j}=\frac{c_{j}\left(x^{\alpha}+b_{j} t^{\alpha}\right)}{\Gamma(1+\alpha)}  \tag{70}\\
c_{j} & =\sqrt{k_{j}^{2}-2 w-2}, b_{j}=-2 \lambda_{j}-v, k_{j}=\lambda_{j}-\frac{v}{2}
\end{align*}
$$

## Discussion

Replacing $A(\lambda)$ in eq. (11) with $A(\lambda)+\lambda^{N}$, then by the similar operations we can derive the third kind of fractional DT and even-soliton solutions for systme (1). On this basis, further replacing $D(\lambda)$ in eq. (11) with $D(\lambda)-\lambda^{N}$ and, similarly we can construct the first kind of fractional DT and another pair of non-differentiable fractional odd-soliton solutions for system (1). Constructing the first and third DT and their corresponding non-differentiable fractional odd-soliton and even-soliton solutions are worth studying.

## Acknowledgment

This work was supported by Liaoning BaiQianWan Talents Program of China (2020921037), the Natural Science Foundation of Education Department of Liaoning Province of China (LJ2020002) and the National Science Foundation of China (11547005).

## References

[1] Fan, J., Liu, Y., Heat Transfer in the Fractal Channel Network of Wool Fiber, Materials Science and Technology, 26 (2010), 11, pp. 1320-1322
[2] Fan, J., Shang, X. M., Fractal Heat Transfer in Wool Fiber Hierarchy, Heat Transfer Research, 44 (2013), 5, pp. 399-407
[3] Fan, J., et al., Influence of Hierarchic Structure on the Moisture Permeability of Biomimic Woven Fabricusing Fractal Derivative Method, Advances in Mathematical Physics, 2015 (2015), Apr., ID817437
[4] He, J. H., A Tutorial Review on Fractal Spacetime and Fractional Calculus, International Journal of Theoretical Physics, 53 (2014), 11, pp. 3698-3718
[5] He, J, H., Fractal Calculus and its Geometrical Explanation, Results in Physics, 10 (2018), Sept., pp. 272-276
[6] He, J. H., A New Fractal Derivation, Thermal Science, 15 (2011), Suppl. 1, pp. S145-S147
[7] Khalil, R., et al., A New Definition of Fractional Derivative, Journal of Computational and Applied Mathematics, 264 (2014), July, pp. 65-70
[8] Xu, B., et al., Analytical Methods for Non-linear Fractional Kolmogorov-Petrovskii-Piskunov Equation: Soliton Solution and Operator Solution, Thermal Science, 25 (2021), 3B, pp. 2159-2166
[9] Xu, B., et al., Line Soliton Interactions for Shallow Ocean-Waves and Novel Solutions with Peakon, Ring, Conical, Columnar and Lump Structures Based on Fractional KP Equation, Advances in Mathematical Physics, 2021 (2021), ID6664039
[10] Zhang, S., et al., Variable Separation for Time Fractional Advection-Dispersion Equation with Initial and Boundary Conditions, Thermal Science, 20 (2016), 3, pp. 789-792
[11] Xu, B., et al., Variational Iteration Method for Two Fractional Systems with Boundary Conditions, Thermal Science, 26 (2022), 3B, pp. 2649-2657
[12] Xu, B., et al., Fractional Isospectral and Non-isospectral AKNS Hierarchies and Their Analytic Methods for N-fractal Solutions with Mittag-Leffler Functions, Advances in Difference Equations, 2021 (2021), ID223
[13] Xu, B., et al., Fractional Rogue Waves with Translational Coordination, Steep Crest and Modified Asymmetry, Complexity, 2021, (2021), ID6669087
[14] Xu, B., Zhang, S., Riemann-Hilbert Approach for Constructing Analytical Solutions and Conservation Laws of a Local Time-Fractional Non-linear Schrödinger Equation, Symmetry, 13 (2021), 9, ID13091593
[15] Zhang, S., Zheng, X. W., Non-Differentiable Solutions for Non-linear Local Fractional Heat Conduction Equation, Thermal Science, 25 (2021), Special Issue 2, S309-S314
[16] Hristov, J., Transient Heat Diffusion with a Non-singular Fading Memory from the Cattaneo Constitutive Equation with Jeffrey's Kernel to the Caputo-Fabrizio Time-Fractional Derivative, Thermal Science, 20 (2016), 2, pp. 757-762
[17] Yang, X. J., et al., Local Fractional Integral Transforms and Their Applications, Academic Press, London, UK, 2015
[18] Zhang, Y. F., et al., An Integrable Hierachy and Darboux Transformations, Bilinear Backlund Transformations of a Reduced Equation, Applied Mathematics and Computation, 219 (2013), 11, pp. 5837-5848
[19] Zhang, S., Liu, D. D., The Third Kind of Darboux Transformation and Multisoliton Solutions for Generalized Broer-Kaup Equations, Turkish Journal of Physics, 39 (2015), 2, pp. 165-177
[20] Zhang, S., Zheng, X. W., N-Soliton Solutions and Non-linear Dynamics for Two Generalized Bro-er-Kaup Systems, Non-Linear Dynamics, 107 (2022), Jan., pp. 1179-1193
[21] Zhang, S., Xu, B., Painleve Test and Exact Solutions for (1+1)-Dimensional Generalized Broer-Kaup Equations, Mathematics, 10 (2022), 3, ID 486


[^0]:    *Corresponding author, e-mail: szhangchina@126.com

