NUMERICAL SOLUTION FOR STOCHASTIC HEAT EQUATION WITH NEUMANN BOUNDARY CONDITIONS

by

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In this article, we propose a new technique based on 2-D shifted Legendre polynomials through the operational matrix integration method to find the numerical solution of the stochastic heat equation with Neumann boundary conditions. For the proposed technique, the convergence criteria and the error estimation are also discussed in detail. This new technique is tested with two examples, and it is observed that this method is very easy to handle such problems as the initial and boundary conditions are taken care of automatically. Also, the time complexity of the proposed approach is discussed and it is proved to be $O[k(N+1)^4]$ where N denotes the degree of the approximate function and k is the number of simulations. This method is very convenient and efficient for solving other partial differential equations.

Key words: *stochastic PDE, heat equation, error analysis, operational matrices, shifted legendre polynomials*

Introduction

Stochastic differential equations and stochastic integral equations arise as a result of the addition of one or more random elements to deterministic differential and integral equations, respectively. Such random elements are often considered as *noise*. Such models help in studying the random phenomena arising in various physical, biological, or economic changes in fields like mechanics, medicine, population dynamics, finance, *etc*. [1]. Stochastic partial differential equations (SPDE) find wide applications in the fields of mathematical physics, engineering, financial mathematics, and financial physics, such as random interface growth, random evolution of surfaces, fluids subject to random forcing, asset pricing theory, and pricing of financial derivatives, *etc*. The applications of stochastic functional equations in various fields can be found in [2-6]. In most of the aforementioned situations, finding the analytic solution the problem modelled is not an easy task. An alternate method to solve such functional equations (both linear and non-linear) is to approximate the unknown function in terms of a linear combination of the basis functions of suitable polynomials [2].

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The stochastic heat equation that we consider in this article has been obtained by replacing the space-time white noise $\dot{W}(t, x)$ as given in [7] by time white noise $\mathrm{d}B(t)/\mathrm{d}t$. Hence, the new equation takes the form:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \left[b + \sigma \frac{\mathrm{d}B(t)}{\mathrm{d}t}\right] u(x,t) \tag{1}$$

where u is a function of (x, t) and $t \geq 0$, $x \in [0, 1]$ and the Neumann boundary conditions given:

$$u(x,0) = u_0(x); \frac{\partial u(t,0)}{\partial x} = \frac{\partial u(t,1)}{\partial x} = 0, t > 0 \tag{2}$$

The constants b and σ belong to real numbers, are the drift and diffusion co-efficient, respectively. Also, $u_0(x): [0, 1] \to R$ is the initial condition. The b, σ are Lipschitz continuous for some constants $C_{b,\sigma}$. The space $(\Omega, \mathcal{F}, P)$ is called the probability space. Here, $u(x, t)$ is the unknown function. The $B(t)$ is the 1-D Brownian motion process.

The generalisation of the previous equation is an SPDE with reflection, in particular, a stochastic heat equation given:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + f\left[u(x,t)\right] \dot{W}_{x,t} \tag{3}$$

with the aforementioned Neumann Boundary conditions. In this equation, authors have considered a space-time white noise on a probability space $(\Omega, \mathcal{F}, P)$. It has been proved that the solutions of the previous equation depend continuously on the function f. The stability properties of the solution are also discussed for the function f. Next, we discuss some of the literature pertaining to stochastic heat equations.

Fundamental solutions of parabolic equations with time-dependent coefficients have been studied in [8], along with the existence and uniqueness of the solution for the stochastic reaction-diffusion equation. Computational methods based on wavelets were studied by Aidoo and Wilson in [9] and they considered three different types of stochastic equations with random inputs. It has also been established in that article that the solution obtained through the wavelets method is stable. Chen *et al.* [10], the boundedness parameter has been studied for the stochastic heat equation. The stochastic heat equation with non-homogeneous Neumann boundary conditions has been investigated in [11] where the finite element method has been implemented to discretize the spatial co-ordinates and the linear implicit Euler method has been implemented to discretize the temporal co-ordinates. An implicit Euler scheme with non-uniform time discretization has been adopted where the error bound is achieved in terms of the number of evaluations of 1-D components of Brownian motion. The path-wise uniqueness of the solution the stochastic heat equation has also been discussed in [12]. Also, the stability properties of the stochastic heat equation have been carried-out in various perspectives.

Not only motivated by the aforementioned works but also due to the limited availability of literature on the study of approximate solutions to the stochastic heat equation, we have employed the shifted version of Legendre polynomials called 2-D shifted Legendre polynomials (2-DSLP) to obtain an approximate solution of eq. (1) in the interval (0, 1) × (0, 1). The usage of shifted Legendre polynomials has provided fruitful results for various other works undertaken in [13]. The operational matrices of integration are coupled with the salient properties of these polynomials to convert the given problem into a system of simultaneous algebraic equations. Solving these equations provides the required numerical solution.

Mathematical background

The basic definitions of stochastic calculus [14], which are required for our further study, have been highlighted in this section. The concept of Brownian motion in 1-D, which is considered to be an important tool in the development of stochastic calculus, is available in [15, 16].

Definition 1. Let $p \geq 2$ and $L^p(\Omega, H)$ be the collection of all strongly measurable random variables and if:

$$\| V \|_{L^p} = \left[E | V |^p \right]^{1/p} = \left(\int_{\Omega} | V |^p \, \mathrm{d}P \right)^{1/p}$$

for each $V \in L^p(\Omega, H)$ then $L^p(\Omega, H)$ is a Banach space.

Definition 2. Let $A, B \in [0, T] \to \mathbb{R}$ and if:

$$A(t) \leq \lambda + \int_0^t B(s)A(s)\mathrm{d}s \quad \text{for} \quad t \in [0,T] \quad \text{then} \quad A(t) \leq \lambda \left[\int_0^t B(s)\mathrm{d}s \right]$$

for all $t \in [0, T]$ with $\lambda \geq 0$.

Definition 3. The sequence X_n converges to X in L^2 if:

$$E\left(| X_n |^2 \right) < \infty \quad \text{and} \quad E\left(\| X_n - X \| \right)^2 \to 0 \quad \text{when} \quad n \to \infty$$

Lemma 4. The Ito isometry of $f \in v(s, T)$ f is given:

$$E\left(\int_s^T \left[f(t,w)\mathrm{d}B(t)(w) \right]^2 \right) = E\left(\int_s^T \left[f^2(t,w)\mathrm{d}t \right] \right)$$

Shifted Legendre polynomials

Preliminaries and properties

The Legendre polynomials $P_n(z)$ are the solutions of Legendre's differential equation. The orthogonal property of Legendre polynomials is defined as:

$$\int_{-1}^{1} P_n(z)P_m(z)\mathrm{d}z = \frac{2}{2n+1}\delta_{nm}$$

where δ_{nm} is the Kronecker delta. The shifted Legendre polynomials are derived from $P_n(z)$ by replacing z by 2t-1, denoted by $L_n(t)$ thereby refined interval is [0,1]. The detailed information about the orthogonality of shifted Legendre polynomials, its analytic form and the vector form of L(t) can be found in [13].

The matrix form of $L(t)$ which is of degree N can be represented:

$$\begin{bmatrix} L_0(t) \\ L_1(t) \\ L_2(t) \\ \vdots \\ L_N(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ (-1)^1 \dfrac{(1)!}{(1)!(0!)^2} & (-1)^2 \dfrac{(2)!}{(0)!(1!)^2} & & 0 \\ (-1)^2 \dfrac{(2)!}{(2)!(0!)^2} & (-1)^3 \dfrac{(3)!}{(1)!(1!)^2} & & 0 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^N \dfrac{(N)!}{(N)!(0!)^2} & (-1)^{N+1} \dfrac{(N+1)!}{(N-1)!(1!)^2} & & (-1)^{2N} \dfrac{(2N)!}{(0)!(N!)^2} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^N \end{bmatrix} \tag{4}$$

Thus:

$$L(t) = DY(t) \tag{5}$$

The 2-DSLP are defined on $\Delta = [0, K_1] \times [0, K_2]$:

$$L_{ij}(x,y) = L_i\left(\frac{2}{K_1}t-1\right)L_j\left(\frac{2}{K_2}s-1\right), \quad i,j = 0,1,2,...$$

We consider the space $L^2(\Delta)$ equipped with the following inner product and norm:

$$\langle u(x,y), v(x,y)\rangle = \int_0^{K_1}\int_0^{K_2} u(x,y)v(x,y)\mathrm{d}x\mathrm{d}y$$

$$\|u(x,y)\| = \langle u(x,y), u(x,y)\rangle^{1/2} = \left[\int_0^{K_1}\int_0^{K_2} |u(x,y)|^2 \,\mathrm{d}x\mathrm{d}y\right]^{1/2}$$

where $u(x, y)$, $v(x, y)$ are arbitrary functions. The set of 2-DSLP forms a complete $L^2(\Delta)$ orthogonal systems such that the orthogonality condition:

$$\int_0^{K1}\int_0^{K2} L_{ij}(x,y)L_{mn}(x,y)\mathrm{d}x\mathrm{d}y = \begin{cases} \dfrac{K_1K_2}{(2i+1)(2j+1)}, & \text{for } i=m, j=n \\ 0 \ \text{otherwise} \end{cases}$$

Any arbitrary function $u(t) \in L^2[0, 1]$ can be approximated in terms of $L_n(t)$:

$$u(t) = \sum_{n=0}^{\infty} u_n L_n(t) \tag{6}$$

from which the coefficients u_j are given:

$$u_j = (2j+1)\int_0^1 u(x)L_j(x)\mathrm{d}x, \quad j = 0,1,... \tag{7}$$

Approximating $u(t)$ by the first $(N + 1)$ terms:

$$u(t) \simeq \sum_{n=0}^{N} u_n L_n(t) = U^T L(t) = L^T(t)U \tag{8}$$

where U is the shifted Legendre coefficient vector given by $U = [u_0, u_1,..., u_N]^T$.

Similarly, an arbitrary function $u(x, y) \in L^2(\Delta)$ can be expressed by 2-DSLP:

$$u(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{ij} L_i(x)L_j(y) = L^T(x)UL(y) \tag{9}$$

Approximating the previous equation by the first $N + 1$ terms:

$$u(x,y) \simeq \sum_{i=0}^{N}\sum_{j=0}^{N} u_{ij} L_i(x)L_j(y) = L^T(x)UL(y) \tag{10}$$

Operational matrices

This section discusses the concept of constructing the operational matrices for the shifted Legendre polynomials using integration. In addition, the stochastic operational matrix of integration of order $(N+1) \times (N+1)$ is derived.

Operational matrices of integration

The integrals of $L_n(s)$ are evaluated with the aid of recurrence property of $L_n(t)$:

$$\int_0^t L_n(s)\mathrm{d}s = \frac{1}{2(2n+1)}\left[L_{n+1}(t) - L_{n-1}(t)\right] \tag{11}$$

Therefore:

$$\int_0^t L(s)ds = PL(t) - \frac{1}{2(2n+1)}L_{n+1}(t) \tag{12}$$

where P is the matrix, which denotes the integration matrix of polynomials. The integration of the vector $L(t)$ can be approximated from eq. (12):

$$\int_0^t L(s)\mathrm{d}s \simeq PL(t) \tag{13}$$

Hence any function $u(t)$ can be approximated:

$$\int_0^t u(s)\mathrm{d}s \simeq \int_0^t U^T L(s)\mathrm{d}s = U^T PL(t) \tag{14}$$

Stochastic operational matrix of shifted Legendre polynomials

For the the vector $L(t)$, we define its Ito integral with stochastic matrix which is derived with the help of integration and is given:

$$\int_0^1 L(s)\mathrm{d}W(s) = P_s L(t) \tag{15}$$

where P_s is the stochastic operational matrix of integration whose order is $(N+1) \times (N+1)$ and they are calculated:

$$\begin{aligned}\int_0^t L(s)\mathrm{d}W(s) &= \int_0^t DX(s)\mathrm{d}W(s) \\ &= D\left[\int_0^t \mathrm{d}W(s) \ \int_0^t s\mathrm{d}W(s) \ldots \int_0^t s^N \mathrm{d}W(s)\right]^T \\ &= D\left[W(t)Y(t) - \left[0 \ \int_0^t \mathrm{d}W(s) \ldots N\int_0^t s^{N-1}\mathrm{d}W(s)\right]^T\right]\end{aligned} \tag{16}$$

$$= D\left(\lambda_i\right) \tag{17}$$

where each

$$\lambda_i = t^i W(t) - \int_0^t s^{i-1} W(s)\mathrm{d}s,\ \ i = 0,1,...,N$$

Evaluating the integral for each i, we get:

$$\lambda_i = t^i W(t) - \frac{t^i}{4}\left[2\left(\frac{t}{2}\right)^{i-1} W\left(\frac{t}{2}\right) + t^{i-1} W(t)\right] = \left[\left(1 - \frac{i}{4} W(t) - \frac{i}{2} W\left(\frac{t}{2}\right)\right)\right] t^i$$

We assume that $W(0.5)$ and $W(0.25)$ are the approximate value of $W(t)$ and $W(t/2)$, respectively for any value of t in [0,1]. Hence:

$$D(\lambda_i) =$$

$$= D\begin{pmatrix} W(0.5) & 0 & & 0 \\ 0 & \frac{3}{4}W(0.5) - \frac{1}{2}W(0.25) & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & \left(1 - \frac{N}{4}\right) W(0.5) - \frac{N}{2^N} W(0.25) \end{pmatrix} \begin{pmatrix} 1 \\ t \\ \vdots \\ t^N \end{pmatrix} =$$

$$= D\Gamma_s Y(t) = D\Gamma_s D^{-1} L(t) = P_s L(t),\ \text{where}\ P_s = D\Gamma s D^{-1}$$

Therefore, the Ito integral of an arbitrary function $u(t)$ can be represented:

$$\int_0^t u(s)\mathrm{d}W(s) = \int_0^t U^T L(s)\mathrm{d}W(s) = U^T P_s L(t) \tag{18}$$

Proposed methodology

Consider the stochastic heat equation introduced in section *Introduction*:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \left[b + \sigma \frac{\mathrm{d}B(t)}{\mathrm{d}t}\right] u(x,t) \tag{19}$$

with assumptions and boundary conditions as in eq. (2). Integrating eq. (19) with respect to t, together with the boundary conditions, we get:

$$u(x,t) = u_0(x) + \int_0^t \frac{\partial^2 u(x,\tau)}{\partial x^2}\mathrm{d}\tau + b\int_0^t u(x,\tau)\mathrm{d}\tau + \sigma\int_0^t u(x,\tau)\mathrm{d}B\tau \tag{20}$$

where the third integral is an Ito integral. Approximating $\partial^2 u(x, t)/\partial x^2$ using 2-DSLP:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \simeq L^T(x) U L(t) \tag{21}$$

where $U = [u_{ij}]$ is a square matrix of order $(N + 1)$which is to be determined. Integrating eq. (21) twice with respect to x successively:

$$\frac{\partial u(x,t)}{\partial x} \simeq L^T(x) P^T U L(t) \tag{22}$$

and

$$u(x,t) \simeq u(0,t) + L^T(x)\left(P^T\right)^2 UL(t) \tag{23}$$

Using the boundary conditions, it is understood that $u(0,t) = k$ (a constant):

$$u(x,t) \simeq L^T(x)KL(t) + L^T(x)(P^T)^2 UL(t)$$

therefore:

$$u(x,t) \simeq L^T(x)ML(t),\ M = K + \left(P^T\right)^2 U \tag{24}$$

Using the aforementioned approximation, eq. (20) becomes:

$$u(x,t) = L^T(x)U_0E^T L(t) + L^T(x)UPL(t) + bL^T(x)MPL(t) + \sigma L^T(x)MP_s L(t) \tag{25}$$

which in turn, gives:

$$u(x,t) = L^T(x)[U_0E^T + UP + bMP + \sigma MP_s]L(t) = L^T(x)\tilde{U}L(t) \tag{26}$$

where

$$\tilde{U} = \left[U_0E^T + UP + bMP + \sigma MP_s\right]$$

and E, the shifted Legendre coefficient vector of the unit step function.

We generate $(N + 1)^2$ linear algebraic equations based on the connection coefficients, $u_{i,j}$, $i, j = 0$ to N, using the following equation and to find the solution of the equation given in (1) numerically:

$$\int_0^1\int_0^1 \left[L^T(x)\left(M - \bar{U}\right)L(t)\right]L_i(x)L_j(t)w(x,t)\mathrm{d}x\mathrm{d}t = 0;\ i,j = 0 \text{ to } N \tag{27}$$

Convergence analysis

The error function is defined and the convergence of the error function zero is established in the following theorem.

Theorem 5. Let $u(x, t)$ and $u_N(x, t)$ be the exact and approximate solutions of (19) with boundary conditions (2). Let e_N denote the error function $e_N(x, t) = u(x, t) - u_N(x, t)$. Also, assume that:

$$\| u(x, t)\| < \infty$$

$$\left\|\frac{\partial^2 e_N(x,t)}{\partial x^2}\right\| < M_1 e_N$$

then for large values of N, $\|u(x, t) - u_N(x, t)\|$ tends to 0 where $\|u\|^2 = E[|u|^2]$.

Proof. We begin with eq. (20)

$$e_N(x,t) = \int_0^t \frac{\partial^2 e_N(x,\tau)}{\partial x^2}\mathrm{d}\tau + b\int_0^t e_N(x,\tau)\mathrm{d}\tau + \sigma\int_0^t e_N(x,\tau)\mathrm{d}B\tau + R_N(x)$$

where $R_N(x)$ is the residual error of $u_0(x)$. Let this be written as $e_N(x, t) = I_1 + I_2 + I_3 + R_N(x)$.

Considering:

$$I_1 = \int_0^t \frac{\partial^2 e_N(x,\tau)}{\partial x^2}\mathrm{d}\tau,\ |I_1|^2 = |\alpha\int_0^t \frac{\partial^2 e_N(x,\tau)}{\partial x^2}\mathrm{d}\tau|^2 \le \alpha^2\left[\int_0^t\left|\frac{\partial^2 e_N(x,\tau)}{\partial x^2}\right|\mathrm{d}\tau\right]^2 \le \alpha^2 M_1^2 e_N^2$$

Taking expectation results:

$$\|I_1\| \le M_1 e_N \tag{28}$$

Similarly, taking expectation results by considering I_2 and I_3, we have:

$$\|I_2\| \le b e_N \tag{29}$$

$$\|I_3\| \le \sigma e_N \tag{30}$$

Equations (28)-(30) and the property $(l + m + n + o)^2 \le 4(l^2 + m^2 + n^2 + o^2)$ results in:

$$\|e_N(x,t)\|^2 \le 4\left[M_1^2 e_N^2 + b^2 e_N^2 + \sigma^2 e_N^2 + R_N^2(x)\right] \le \frac{4\left\|R_N^2(x)\right\|}{1-4\left(M_1^2 + b^2 + \sigma^2\right)}$$

For large values of N, the residual error tends to 0 and hence $e_N(x, t)$ becomes 0.

Time complexity

The total number of arithmetic operations required for the proposed method is discussed in the following theorem.

Theorem 6. Suppose that N and k are the degree of the approximate function and the number of simulations, respectively, then the time complexity of this proposed method is $O[k(N + 1)^4]$.

Proof. The total number of arithmetic operations required to compute the each of the matrices D, P, Γ_s, P_s, U, M, $\tilde{U}$ is $(N + 1)^4$. In addition, $(N + 1)^4$ operations are required to determine the values of the unknown coefficients. Hence, the overall time complexity of this proposed method is $(N + 1)^4$ to perform K runs.

Numerical examples

To illustrate the applicability, effectiveness, and reliability of the proposed method, some illustrative examples are considered in this section. The computational work has been carried out using MATLAB. The N and k represent the highest degree of the approximate function and the number of simulations, respectively. A simulation study was carried out by assigning different values of N and k.

Example 1. We consider the stochastic heat equation of the type

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \left[b + \sigma \frac{\mathrm{d}B(t)}{\mathrm{d}t}\right] u(x,t)$$

with regard to the constraints

$$u(x,0) = 3 - 2\cos(\pi x), \quad u_x(0,t) = u_x(1,t) = 0$$

The $u(x,t) = 3 - 2\cos(\pi x)\exp(-\pi^2 t + bt)$ is the exact (deterministic) solution of the problem for $\sigma = 0$. The graph of the approximate solution for $\sigma = 0$ obtained by the proposed methodology is shown in fig. 1(a). We observe that the results obtained through the proposed algorithm match the exact solution while increasing the values of the parameters involved in the function approximation. The figures reveal that the approximate solution is very close to the exact solution. The graph of the approximate solution for $\sigma = 1$ and $b = 1$ obtained by the proposed methodology is shown in fig. 1(b).

Example 2: Next, we consider the stochastic heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2} + \left[1 + \sigma \frac{\mathrm{d}B(t)}{\mathrm{d}t}\right] u(x,t)$$

with respect to the constraints

$$u(x,0) = x^2(15 - x), \; u_x(0,t) = u_x(1,t) = 0, \quad 0 < x < 1, \quad t > 0$$

The graph of the approximate solution for $\sigma = 0$ obtained by the proposed methodology, is shown in fig. 2(a). The graph of the approximate solution corresponding to $\sigma = 1$ and $b = 1$ is shown in fig. 2(b).

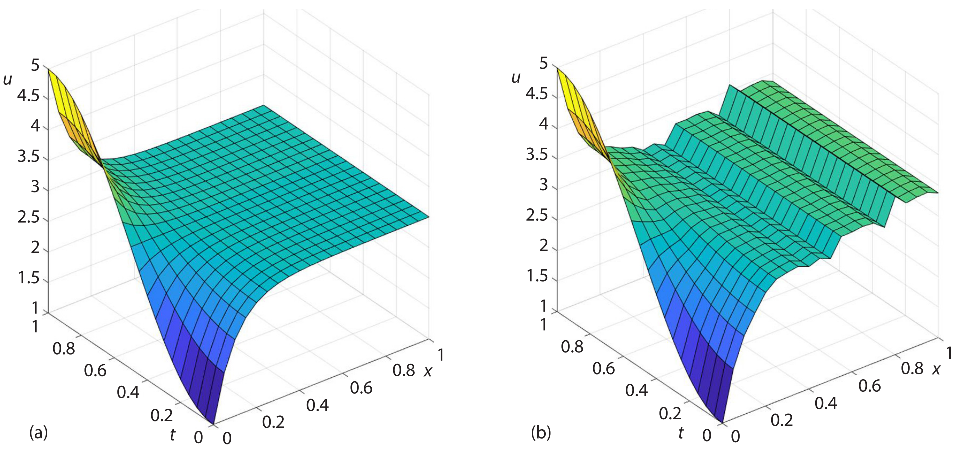

Figure 1. The Graph of the approximate solution for *Example 1*

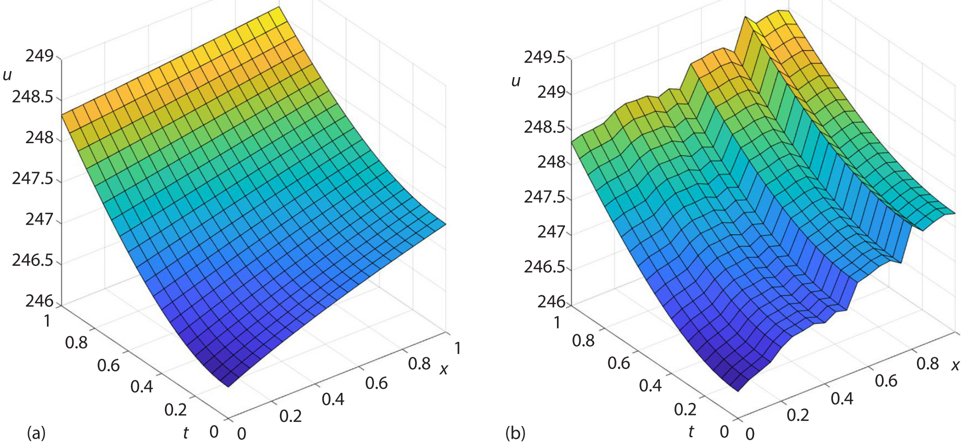

Figure 2. The Graph of the approximate solution for *Example 2*

Conclusion

This paper discusses a fast approximation method for solving a stochastic heat equation with Neumann boundary conditions. To solve the given equation, stochastic operational matrices for stochastic integration and fractional stochastic integration have been constructed. The shifted Legendre polynomial matrix is a triangular matrix. As a result, the dual matrix is found to be diagonal. This is a noteworthy characteristic when working with the shifted Legendre polynomial. The proposed methodology has undergone theoretical research, and the method's applicability has been statistically validated by using numerical examples. The proposed approach's time complexity is also discussed, and it is proven to be $O[k(N+1)^4]$, where N is the degree of the approximate function and k is the number of simulations.This method is easy to implement and to handle other types of partial differential equations governing various parameters encountered in different disciplines of science and engineering.

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