DYNAMICS OF UNSTEADY FLUID-FLOW CAUSED BY A SINUSOIDALLY VARYING PRESSURE GRADIENT THROUGH A CAPILLARY TUBE WITH CAPUTO-FABRIZIO DERIVATIVE

by

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This paper presents a study of the unsteady flow of second grade fluid through a capillary tube, caused by sinusoidally varying pressure gradient, with fractional derivative model. The fractional derivative is taken in Caputo-Fabrizio sense. The analytical solution for the velocity profile has been obtained for non-homogenous boundary conditions by employing the Laplace transform and the finite Hankel transform. The influence of order of Caputo-Fabrizio time-fractional derivative and time parameter on fluid motion is discussed graphically.

Key words: *Caputo-Fabrizio time-fractional derivative, second grade fluid, oscillating flow, Laplace transform, finite Hankel transform*

Introduction

The study of non-Newtonian fluid-flow through a capillary tube is of great interest due to remarkable applications in many physical phenomena, *i.e.* blood flow in capillaries, flow in blood oxygenator, oil transport in pipe-lines, flow of refringent, air conditioning system, heat pumps system *etc*. The well-known Navier Stokes equation successfully describes the flow of Newtonian fluid, however, it is unable to describe some unusual characteristics (*e.g.* shear thinning/thickening) exhibited by many fluids, such as, ketchup, shampoo, paints, slurries, *etc*. Several rheological non-Newtonian models are presented to describe the fluid-flow problem related to such materials. Amongst the different interesting proposed models, the differential type fluids [1] have caught the interest of many researchers.

The second grade fluid model is proved to be helpful to successfully describe various non-Newtonian effects of fluids [2]. It was first proposed by Coleman and Noll [3]. A non-Newtonian fluid is called a second grade fluid if the velocity field contains up to two derivatives in stress strain tensor relationship [4]. The ability of this fluid model to describe the normal stress effects and its relatively simpler structure is beneficial in the investigation of various problems in fluid dynamics. Sajid *et al*. [5] obtained the analytical solution of a second grade fluid model

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with unsteady flow by employing homotopy analysis method. Faraz and Khan [6] discussed a second grade model for MHD rotating flow through a porous shrinking surface. Siddiqui *et al*. [7] explored a second grade fluid-flow model for porous medium to observe the influence of time-dependent stenosis. Marinca and Marinca [8] presented a novel approach to construct exact solutions of a modified fluid model subject to thermal radiation taking into account the absorption/generation of heat. Shojaei *et al*. [9] studied a second grade fluid-flow subject to thermal radiation using an analytic approach and observed how different physical parameters affect the velocity profile. The flow was considered along a stretching cylinder. Alamri *et al*. [10] studied the Cattaneo-Christov heat flux model and investigated the effect of mass transfer for a second grade fluid-flow problem subject to magnetic field effects.

Recently, fractional calculus has gained popularity to describe mathematical models corresponding to various physical phenomena. Fractional order operators are more generalized and flexible due to their ability to describe memory and hereditary properties. Many interesting applications and investigations related to fractional order differential equations are being reported in physics, biology, engineering and other fields of science. The fractional order governing equations are used to study the physical phenomena related to melts and polymer solutions. The fractional calculus has achieved a special status to describe viscoelastic properties [11, 12]. The use of fractional order governing equations is reported in many interesting investigations [13-19].

The aim of this study is to investigate the effects of the fractional order derivative on the analytical solution of the mathematical model for the unsteady flow of second grade fluid in a capillary tube. The analytical solution is determined with the help of finite Hankel and Laplace transforms.

Mathematical formulation

The pulsating flow of an incompressible second grade fluid in a capillary tube of internal radius r_0 is considered, which is caused by sinusoidally varying pressure gradient:

$$\nabla p = \vec{e}_z \left[B_0 + B_1 \exp(\imath\omega t)\right] \tag{1}$$

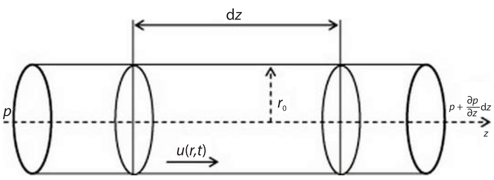

Figure 1. The physical configuration

where $\vec{e}_z$ is the unit vector along z-direction, B_0 and B_1 are the amplitudes of pressure gradient, $\imath$ – the imaginary constant, t – the time, and ∇p – the pressure gradient with frequency ω. The real part of the pressure gradient defined in eq. (1) provides the cosine oscillations, whereas the imaginary part provides the sine oscillations. The physical aspects of fluid-flow are shown by fig. 1. The Cauchy stress tensor T_1 with pressure p, and dynamic viscosity μ can be expressed:

$$T_1 = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \tag{2}$$

for material moduli α_1, α_2, unit tensor I and kinematic tensors A_1, A_2:

$$A_1 = (\nabla \vec{V}) + (\nabla \vec{V})^T \tag{3}$$

$$A_2 = \frac{\mathrm{d}}{\mathrm{d}t} A_1 + A_1(\nabla \vec{V}) + (\nabla \vec{V})^T A_1 \tag{4}$$

where $\nabla \vec{V}$ is the gradient of velocity field $\vec{V} = [0, 0, v(r, t)]$ and T – the matrix transpose. The velocity field for the problem under assumption:

$$\vec{V}(r,t) = v(r,t)\vec{e}_z \tag{5}$$

where r is the distance from the axial direction. Using eq. (5) into eq. (2), gives the non-zero component of the shear stress $S_{r,z}$:

$$S_{r,z} = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right)\frac{\partial v(r,t)}{\partial r} \tag{6}$$

For the gradient of pressure is in the axial direction, the linear momentum equation:

$$\rho \frac{\partial v}{\partial t} = \left[B_0 + B_1 \exp(\iota\omega t)\right] + \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)S_{r,z} \tag{7}$$

where the body forces are neglected. Eliminating $S_{r,z}$ from eqs. (6) and (7), gives:

$$\rho \frac{\partial v}{\partial t} = \left[B_0 + B_1 \exp(\iota\omega t)\right] + \mu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) \tag{8}$$

The initial and boundary conditions:

$$v(r,0) = 0, \quad \forall \quad 0 \le r \le r_0 \tag{9}$$

$$\frac{\partial v(0,t)}{\partial r} = 0, \quad \forall \quad t \ge 0 \tag{10}$$

$$v(r_o,t) = 0, \quad \forall \quad t \ge 0 \tag{11}$$

The dimensionless quantities are considered:

$$v^* = \frac{v}{v_m}, \; r^* = \frac{r}{r_0}, \; t^* = \frac{\mu t}{\rho r_0^2}, \; \beta^* = \frac{\alpha_1}{\rho r_0^2}, \; \omega^* = \frac{\omega r_0^2}{\nu}$$

By dropping the notation *, the dimensionless initial-value problem can be written:

$$\frac{\partial v}{\partial t} = \gamma\left[B_0 + B_1 \exp(\iota\omega t)\right] + \left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) + \beta \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) \tag{12}$$

where

$$\gamma = \frac{r_0^2}{\mu v_m}$$

with

$$v_m = \frac{r_0^2}{\mu}\left(\frac{\partial p}{\partial z}\right)$$

The constant γ is related to the pressure fluctuation amplitude, whereas v_m denotes the cross-sectional mean velocity of flow averaged over time. The suitable conditions can be expressed:

$$v(r,0) = 0, \quad \forall \quad 0 \le r \le 1 \tag{13}$$

$$\frac{\partial v(0,t)}{\partial r} = 0, \quad \forall \quad t \ge 0 \tag{14}$$

$$v(1,t) = 0, \quad \forall \quad t \ge 0 \tag{15}$$

Calculation of the velocity field

The constitutive equation corresponding to an incompressible second grade fluid with fractional derivative can be written:

$$D_t^{\alpha} v = \gamma\left[B_0 + B_1 \exp(\imath\omega t)\right] + \left(1+\beta D_t^{\alpha}\right)\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r}\right) \tag{16}$$

where the fractional derivative operator of order α is defined in the Caputo-Fabrizio sense [20]:

$$D_t^{\alpha} u = \begin{cases} \dfrac{M(\alpha)}{1-\alpha}\displaystyle\int_0^t \exp\left(\frac{-\alpha(t-\tau)}{(1-\alpha)}\right) f'(\tau), & 0<\alpha<1 \\ \dfrac{\partial f(r,t)}{\partial t}, & \alpha = 1 \end{cases} \tag{17}$$

Application of the Laplace transform of sequential fractional derivatives [21] on eq. (16), yields the transformed problem:

$$\frac{s\overline{v}(r,s) - \overline{v}(r,0)}{(1-\alpha)s+\alpha} = \gamma\left(\frac{B_0}{s} + \frac{B_1}{s-\imath\omega}\right) + \left(\frac{\partial^2 \overline{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v}}{\partial r}\right) + \frac{\beta s}{(1-\alpha)s+\alpha}\left(\frac{\partial^2 \overline{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v}}{\partial r}\right) \tag{18}$$

where s is the transform parameter and

$$\overline{v}(r,s) = \int_0^{\infty} v(r,t) e^{-st} \mathrm{d}t$$

is the image function of $v(r, t)$. Using the initial condition, eq. (18) reduces to:

$$\frac{s\overline{v}(r,s)}{(1-\alpha)s+\alpha} = \gamma\left(\frac{B_0}{s} + \frac{B_1}{s-\imath\omega}\right) + \left(1 + \frac{\beta s}{(1-\alpha)s+\alpha}\right)\left(\frac{\partial^2 \overline{v}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v}}{\partial r}\right) \tag{19}$$

Using the zeroth order finite Hankel transform [22] and taking into account the boundary conditions, the finite Hankel transform of $v(r, t)$ is obtained from eq. (19):

$$\overline{v}_H(r_n,s) = \left[\left(\frac{\gamma(1-\alpha)}{\alpha r_n^2 + s(1+r_n^2(1-\alpha+\beta))}\right)\left(B_0 + B_1\frac{s}{s-\imath\omega}\right) + \right.$$
$$\left. + \left(\frac{\gamma\alpha}{\alpha r_n^2 + s(1+r_n^2(1-\alpha+\beta))}\right)\left(\frac{B_0}{s} + \frac{B_1}{s-\imath\omega}\right)\right]\frac{J_1(r_n)}{r_n} \tag{20}$$

where

$$\overline{v}_H(r_n,s) = \int_0^1 r\overline{v}(r,s) J_0(r r_n)\mathrm{d}r$$

where r_n is the positive roots of $J_0(x)$ for all natural numbers n, and J_0 – the zeroth order Bessel function of first kind. The inverse Laplace transform is applied to eq. (20), which yields:

$$v_H(r_n,t) = \frac{J_1(r_n)}{k(i\omega+d_n)r_n}\left[\left(\gamma(1-\alpha) - \frac{\gamma\alpha}{d_n}\right)\left[B_0(i\omega+d_n) + B_1 d_n\right]\mathrm{e}^{-d_n t} + \right.$$
$$\left. + \frac{\gamma\alpha B_0(i\omega+d_n)}{d_n} + \left[\gamma(1-\alpha)i\omega + \gamma\alpha\right]B_1\mathrm{e}^{i\omega t}\right] \tag{21}$$

where

$$k = 1 + (1 - \alpha + \beta) r_n^2 \quad \text{and} \quad d_n = \frac{\alpha r_n^2}{k}$$

Applying the inverse Hankel transform:

$$v(r,t) = 2\sum_{n=1}^{\infty} \frac{J_0(r r_n)}{J_1^2(r_n)} v_H(r_n, t), \quad 0 \le r \le 1 \tag{22}$$

the suitable expression for velocity field is obtained:

$$v(r,t) = 2\sum_{n=1}^{\infty} \frac{J_0(r r_n)}{r_n J_1(r_n)} \Bigg[\left(\gamma(1-\alpha) - \frac{\gamma\alpha}{d_n} \right) \left[B_0 \left(i\omega + d_n \right) + B_1 d_n \right] \mathrm{e}^{-d_n t} + \\ + \frac{\gamma\alpha B_0 \left(i\omega + d_n \right)}{d_n} + \left[\gamma(1-\alpha) i\omega + \gamma\alpha \right] B_1 \mathrm{e}^{i\omega t} \Bigg] \tag{23}$$

Graphical illustration

The 3-D plots of the solution are graphically expressed in fig. 2. The variation of velocity of the fluid is shown in fig. 3 for changes in the time parameter, corresponding to sine and cosine pressure gradient. The graphical representation clearly indicates how the fluid velocity, v, changes with the increase of time parameter in the boundary-layer region, for both sine and cosine pressure gradient. The effects of fractional order α on the velocity of fluid are shown in fig. 4.

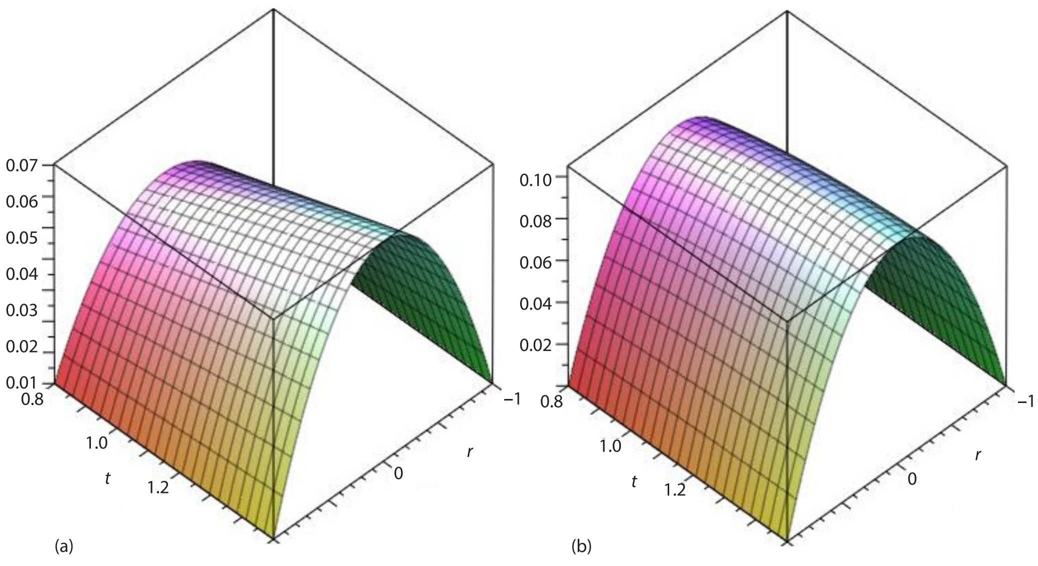

Figure 2. The 3-D plots of velocity profile when γ = 0.4, B_0 = 0.7, B_1 = 0.8, ω = π/7, α = 0.55, β = 0.55; (a) sine pressure gradient and (b) cosine pressure gradient

Results and discussion

Fractional order mathematical models are more generalized and flexible due to their ability to describe the memory and hereditary properties. The idea of fractionalized differen-

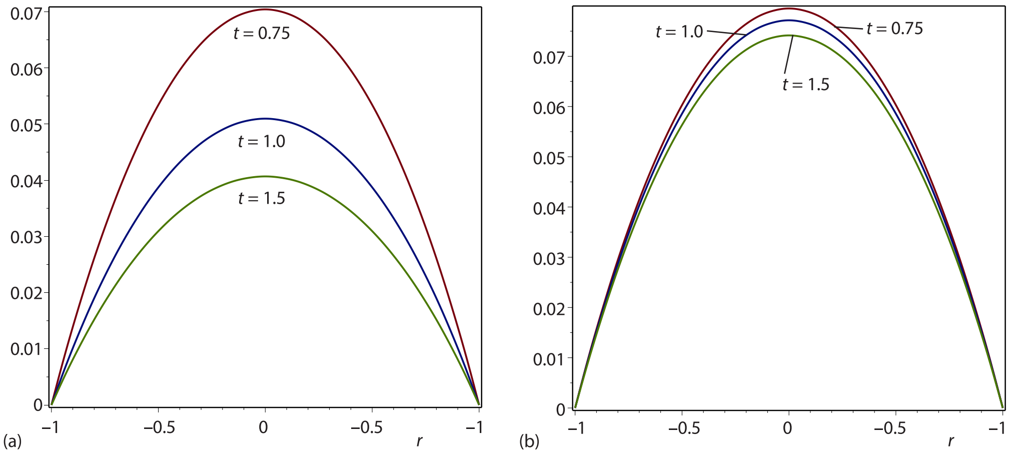

Figure 3. The effect of time parameter *t* on plot of velocity profile when $\gamma = 0.4$, $B_0 = 0.7$, $B_1 = 0.8$, $\omega = \pi/7$, $\alpha = 0.55$, $\beta = 0.55$; (a) sine pressure gradient and (b) cosine pressure gradient

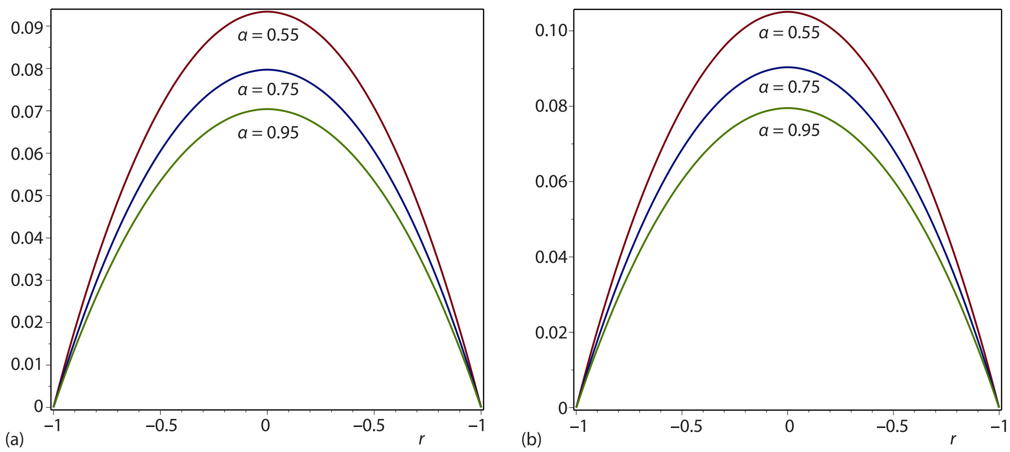

Figure 4. The effect of variations in α on plot of velocity profile when $\gamma = 0.4$, $B_0 = 0.7$, $B_1 = 0.8$, $\omega = \pi/7$, $\beta = 0.55$; (a) sine pressure gradient and (b) cosine pressure gradient

tial models is not only mathematically intriguing and interesting, it has also been reported to provide a deeper insight into various real life phenomena. Motivated by the recently increasing interest to explore the fractional order models, this work is aimed to study a fractional order unsteady flow of second grade fluid through a capillary tube caused by sinusoidally varying pressure gradient.

The fractional derivative is taken in the Caputo-Fabrizio sense. This definition of the fractional derivative is found beneficial in many studies and is widely accepted. The proposed flow model is solved using the finite Hankel and Laplace transforms. The velocity profile is examined for varying time as well as the changing fractional order. It is observed that the velocity profile changes continuously with the change in the fractional parameter. Graphical illustrations of the obtained results have been presented to explain the obtained results.

The results presented in this work are novel and not reported elsewhere to the best of our knowledge. In future, the proposed model maybe explored using the recently developed definitions of fractional derivatives including beta-derivative, M-truncated derivative and Atangana-Beleanu derivative. The obtained solutions and observations will be helpful to understand the dynamical framework of the related flow problems.

Conclusion

The flow of an incompressible fractional second grade fluid through a capillary tube, caused by a sinusoidally varying pressure gradient, has been investigated to obtain the analytical solution for the velocity field $v(r, t)$ by employing Laplace and finite Hankel transform techniques. This solution presents the fluid motion for large and small time. It also depicts the variations in the fluid velocity for changes in the fractional parameter α. The graphical illustrations are used to illustrate the changes in the the velocity field for an increase in the time parameter t, corresponding to sine as well as cosine pressure gradient. The figures also depict the variations in the fluid velocity with an increase in fractional parameter α, for both sine and cosine pressure gradient.

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