# DYNAMICAL BEHAVIOUR OF THE JOSEPH-EGRI EQUATION

by

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*We investigate traveling wave solutions to the Joseph-Egri equation via extended auxiliary equation technique. We have determined stationary points of the dynamical systems by using bifurcation method. We also acquire cusp, periodic and homoclinic orbits. The investigated solutions are entirely different from the reported in the literature. However, some of the reported solutions are plotted to understand the physical application of the considered model using renowned mathematical software.*

Key words*: Joseph-Egri equation, extended auxiliary equation technique, soliton solutions,*

## Introduction

In fluid dynamics, a large number of natural theories are formed using soliton theory. Some non-linear equations were used to explain propagations of waves. For instance, the KdV equation modeled a one-way propagation of waves [1]. Boussinesq equation was employed to analyse wave propagation [2] and to study waves in anisotropy [3]. Some authors have examined problem formulations, applications, and stability analysis of the four and sixth-order non-linear Boussinesq equations propagation. The KdV equation is given:

$$u_t + u_x + uu_x + u_{xxx} = 0 \tag{1}$$

where $u_t$ and $uu_x$ are the perform a role in the evolution of time and $u_{xxx}$ is the used to identify the wave propagation [4-10]. Wave solutions to eq. (1) may be created by the mapping:

$$u = \overline{u}\mathrm{e}^{ik[x-c(k)t]} \tag{2}$$

If $k^2 > 1$, then $c(k) = 1 - k^2 < 0$, which contradicts with unidirectional propagation [11]. To address this problem, in [12] it was offered a substitution of eq. (1) called the BBM equation, which is given:

$$u_t + u_x + uu_x + u_{xxt} = 0 \tag{3}$$

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where $c(k) = (1 - k^2)^{-1}$. Now, if $u(x, 0)$ is ascertained, it is not possible to measure $u_t(x, 0)$. Now, if $u$ represents the movement with the speed $u_t$, then initially they are independent from one another. To prevent these, Joseph and Egri [11] proposed an alternative for eq. (1):

$$u_t + u_x + uu_x + u_{xtt} = 0 \tag{4}$$

subject to phase-field:

$$c(k) = \frac{-1+\sqrt{1+4k^2}}{2k^2} \tag{5}$$

that converges to $1 + 4k^2$ for small $k$ and to $1/k$ for large enough $|k|$. The bifurcation dynamical system method described the different classifications of wave solution of Joseph-Egri equation (JEE) [13]. Lie point symmetry of JEE is also analyzed based on the Lie symmetry in [14]. Moreover, concerning the exact and numerical solutions of the JEE, the methods are few in number. The Jacobi elliptic approach [15], the exact solution using tan-cot functions approach [16], exp[$-\Phi(\xi)$]-expansion [17], the $G'/G$-expansion [18], and the extended homogeneous balance principle [19]. However, for other interesting scientific phenomena and the exact wavesolutions for the JEE, we refer readers to [20-27] and the references therein. We employ the method of extended auxiliary to present explicit solitary wave solutions to the JEE. The method is simple and straightforward for implementation.

## Traveling wave properties

*Definition 1.* A bounded manifold for the flow of the wave propagation is defined:

$$\mathcal{M} = \left\{ (u, y) : -\infty < u < \infty, |y| \le \ell \right\}$$

where $\ell$ is a maximum value for the propagation in a cannel.

Letting $u(x, t) = h(\eta)$, $= kx + wt$. Substituting into eq. (4) gives a traveling wave equation:

$$(\kappa+\omega)h' + \kappa hh' + \kappa\omega^2 h''' = 0 \tag{6}$$

where $h' = h_\eta$. Integrating the previous equation once yields:

$$(\kappa+\omega)h + \frac{\kappa}{2}h^2 + \kappa\omega^2 h'' = c \tag{7}$$

where parameter $c$ is the integral constant. Consider the substitution $h' = f(h)$. Hence, $h''_\eta = f_\eta h'_\eta = f_\eta f$ and we can write eq. (7) as:

$$(\kappa+\omega)h + \frac{\kappa}{2}h^2 + \kappa\omega^2 h'' = (\kappa+\omega)h + \frac{\kappa}{2}h^2 + \frac{\kappa\omega^2}{2}\left(f_\eta^2\right)' = c \tag{8}$$

which can be expressed

$$\frac{\mathrm{dh}}{\mathrm{d}\eta} = \mathrm{y}$$
$$\frac{\mathrm{d}y}{\mathrm{d}\eta} = \frac{2}{\kappa\omega^2}\left[c - (\kappa+\omega)h - \frac{\kappa}{2}h^2\right] \tag{9}$$

Integrating, we get:

$$h' = \pm\sqrt{\frac{1}{3\omega^2}\left[\frac{6\rho}{\kappa} + \frac{6c}{\kappa}h - 3\left(1+\frac{\omega}{\kappa}\right)h^2 - h^3\right]} \tag{10}$$

where $\rho$ is a conservative energy level of eq. (6).

Bifurcation analyses is employed to illustrate the effects of constants $c$, $\kappa$, and $\omega$ on the wave propagation and the presence of a complex duping occurrence. The points $(\bar{h}, \bar{y}) \in \mathcal{M}$ for those determinant of the Jacobian matrix is non-zero, are refered to generaric points, whereas the points for which the determinant of the Jacobian matrix is zero are known as deenarate. Both of these points may be virtual if $(\bar{h}, \bar{y}) \notin \mathcal{M}$.

Solitary wave solutions and homoclinic orbits induced by bifurcation theory hold an important role in the qualitative analysis of the dynamical systems. In the existence of homoclinic orbits, the flow entering the channel is moved to the trapping mode, but there exists not homoclinic orbits, the flow entering the channel exists it without touching to the center of the channel. Thus one has the following propositions.

*Proposition 1:* The set of points in that make the change in the sign of the motion of wave propagation are called the equilibrium points of system (9). Thus, the traveling wave solutions, which is a cubic polynomial in the r.h.s. of (10), be denoted by $\phi(h)$. Thus, we have the following possibilities of equilibrium points of system (9):

– For

$$\frac{\omega}{\kappa}+1=\sqrt{2\left|\frac{c}{k}\right|}$$

system (9) has a degenerate point is a center

$$E_3=\left(\sqrt{2\left|\frac{c}{k}\right|},0\right)=\left(h_3,0\right)$$

which is a cusp point. In this case $\phi(h)$ has one real root $\alpha$.

– For

$$\sqrt{2\left|\frac{c}{\kappa}\right|}<\frac{\omega}{\kappa}+1$$

system (9) has two equilibras

$$E_{1,2}=\left(\frac{(\kappa+\omega)\pm\sqrt{(\kappa+\omega)^2+2c\kappa}}{\kappa},0\right)=\left(h_{1,2},0\right)$$

where $E_1$ is the center while and $E_2$ – the saddle point, which is unstable. Thus, $\phi(h)$ has three real roots satisfying the conditions for $\gamma<\beta<\alpha$, $\gamma=\beta<0<\alpha$, and $\gamma<\beta=\alpha$.

– For

$$\sqrt{2\left|\frac{c}{\kappa}\right|}>\frac{\omega}{\kappa}+1$$

system (9) has no equilibrium point.

Note first that, by the system (9), by taking the derivative of $y$ to be zero, we get aquadratic equation with the discriminant given:

$$h_{1,2}=\frac{(\kappa+\omega)\pm\sqrt{(\kappa+\omega)^2+2c\kappa}}{\kappa}$$

which gives the equilibra $E_1$ and $E_2$ of system (9) for

$$\sqrt{2\left|\frac{c}{\kappa}\right|}<\frac{\omega}{\kappa}+1$$

From system (9), we have the coefficient of a linearized matrix at $h_{1,2}$ given:

$$J(h_1)=\begin{pmatrix} 0 & 1 \\ -\dfrac{2(\kappa+\omega)+\sqrt{(\kappa+\omega)^2+2c\kappa}}{\kappa\omega^2} & 0 \end{pmatrix}$$

and

$$J(h_2)=\begin{pmatrix} 0 & 1 \\ -\dfrac{2(\kappa+\omega)-\sqrt{(\kappa+\omega)^2+2c\kappa}}{\kappa\omega^2} & 0 \end{pmatrix}$$

with eigenvalues

$$\lambda_{\pm}=\frac{2(\kappa+\omega)\pm\sqrt{(\kappa+\omega)^2+2c\kappa}}{\kappa\omega^2}$$

By the qualitative theories of dynamical systems, it is true that $E_1$ is a center while $E_2$ is a saddle point. Similarily for:

$$\left(\frac{\omega}{\kappa}+1\right)^2+\frac{2c}{\kappa}=0$$

we have a linearized matrix

$$J(h_3)=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with eigenvalue $\lambda = 0$, which gives rise a degenerate cusp point at $E_3$. Lastly, if

$$\left(\frac{\omega}{\kappa}+1\right)^2+\frac{2c}{\kappa}<0$$

we have no equilibrium point for system (9).

Next, phase portriates of the system (9) is given.

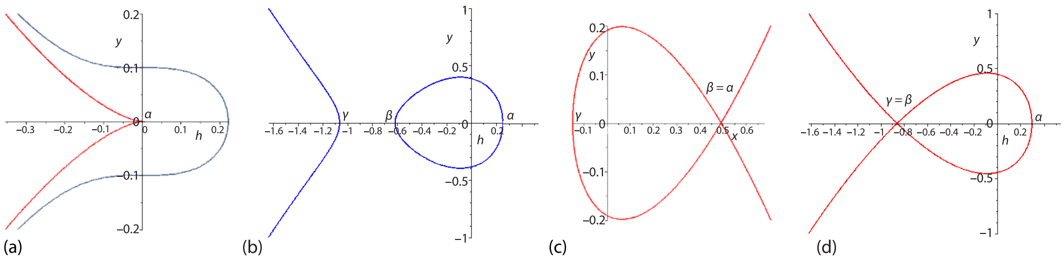


**Figure 1. The phase portriates of the system (9)**

We shall observe from the bifurcation of phase portriates that there exists one real root, three distict real roots with different ways of expressions. Since we are looking bounded traveling wave solutions of eq. (8), we can exclude some cases by qualitative theories of dynamical systems. Let Case 1 be the equilibrium point corresponding to fig. 1(a). In this case, we have a degenerate singular soliton, thus any solution approaches to $\alpha = 0$ as $t$ doverges to infinity and moves to ±∞ negative infinity. Let Case 2 corresponds to fig. 1(b), that leads to the so-called a periodic wave solution (snoidal waves). In this case $\phi(h)$ has three real solutions given:

$$\phi(h)=\frac{6\rho}{\kappa}+\frac{6c}{\kappa}h-3\left(1+\frac{\omega}{\kappa}\right)h^2-h^3=(h-\gamma)(h-\beta)(\alpha-h)$$

then

$$\sqrt{\phi(h)}=\sqrt{(h-\gamma)(h-\beta)(\alpha-h)}$$

for $\gamma<\beta<\alpha$. Thus, eq. (8) can be written

$$\frac{\eta}{\sqrt{3\omega^2}}=\int\frac{\mathrm{d}h}{\sqrt{(h-\gamma)(h-\beta)(\alpha-h)}}$$

Hence, we get:

$$h(\eta)=\gamma+\frac{\beta-\gamma}{1-k_1^2 sn^2\left(\sqrt{\frac{A}{3\omega^2}}g\eta,k_1\right)}\quad\text{where } k_1^2=\frac{\alpha-\beta}{\alpha-\gamma}\ \text{ and }\ g=\frac{2}{\sqrt{\alpha-\gamma}}$$

Corresponding to the third and fourth cases of figs. 1(c) and 1(d), we have three distinct roots. Now considering Case 3 we have from eq. (10):

$$\phi(h)=6\rho+6ch-3(\kappa+\omega)h^2-\kappa h^3=(h-\gamma)^2(\alpha-h)$$

then

$$\sqrt{\phi(h)}=(h-\gamma)\sqrt{(\alpha-h)}$$

for $\gamma=\beta<0<\alpha$. Thus, eq. (10) can be written:

$$\frac{\eta}{\sqrt{3\omega^2}}=\int\frac{\mathrm{d}h}{(h-\gamma)\sqrt{\alpha-h}}$$

Let $\upsilon=h-\gamma$ and $A=\alpha-h$ $(0<\upsilon<b)$, getting:

$$\frac{\eta}{\sqrt{3\omega^2}}=\int\frac{\mathrm{d}\upsilon}{\upsilon\sqrt{A-\upsilon}}$$

Next, we substitute $\tau=(A-\upsilon)^{1/2}$, hence $\upsilon^2=A-\tau^2$ and $\mathrm{d}\tau=\mathrm{d}\eta/2\,(A-\upsilon)^{1/2}$ so that the aforementioned will be transformed:

$$\frac{\eta}{\sqrt{3\omega^2}}=2\int\frac{\mathrm{d}\tau}{\tau^2-A}$$

Using partial fraction method we get:

$$\frac{\eta}{\sqrt{3\omega^2}}=\frac{1}{\sqrt{A}}\ln\frac{\sqrt{A}-\tau}{\sqrt{A}+\tau}$$

for $0 < \tau < \sqrt{A}$. Solving for $\omega$ we have:

$$\tau = \frac{\sqrt{A}\left(1-\exp\left(\sqrt{\frac{A}{3\omega^2}}\eta\right)\right)}{\left(\exp\left(\sqrt{\frac{A}{3\omega^2}}\eta\right)-1\right)} = -\frac{\sqrt{A}\sinh\left(\sqrt{\frac{A}{3\omega^2}}\eta\right)}{\cosh\left(\sqrt{\frac{A}{3\omega^2}}\eta\right)}$$

Returning to original variables $\tau^2 = A - \upsilon = \alpha - \gamma - h + \gamma = \alpha - h$ and using the hyperbolic identity $\cosh^2\theta - \sinh^2\theta = 1$, we get:

$$h(\eta) = \alpha - \tau^2(\eta) = \alpha - (\alpha-\gamma) - \frac{\sqrt{A}\sinh^2\left(\sqrt{\frac{A}{3\omega^2}}\eta\right)}{\cosh^2\left(\sqrt{\frac{A}{3\omega^2}}\eta\right)} = \gamma + (\alpha-\gamma)\operatorname{sech}^2\left(\sqrt{\frac{A}{3\omega^2}}\eta\right)$$

Obviously, $h(\eta) \to \gamma$ as $\eta \to \pm\infty$, so that the traveling wave solution is a solitary wave solution. For the relation:

$$\phi(h) = \frac{6\rho}{\kappa} + \frac{6c}{\kappa}h - 3\left(1+\frac{\omega}{\kappa}\right)h^2 - h^3 = -h^3 - (2\gamma+\alpha)h^2 - \left(2\alpha\gamma+\gamma^2\right)h + \alpha\gamma^2$$

we get

$$\frac{\kappa+\omega}{\kappa} = \frac{(2\gamma+\alpha)}{3} = \frac{(\alpha-\gamma)}{3} + \gamma$$

Hence the solitary wave solution eq. (10):

$$u(x,t) = 3\left(\frac{\kappa+\omega}{\kappa}\right)\operatorname{sech}^2\left(\sqrt{\frac{A}{3\omega^2}}(kx+\omega t)\right)$$

Similarily, we can find the traveling wave solution of Case 4 by following the same procedure as given previously.

## The method of extended auxiliary equation

In this section, we will investigate the solutions of eq. (6) using extended auxiliary equation technique (EAET) described in [28]. Now setting the integration constant to zero eq. (8) yields:

$$(w+k)h + k\frac{h^2}{2} + kw^2h'' = 0 \tag{11}$$

Balancing $h^2$ with $h''$ gives $N = 2$ and suggest the solution:

$$h(\eta) = n_0 + n_1\Psi(\eta) + n_2\Psi^2(\eta) \tag{12}$$

$$\left(\frac{\mathrm{d}\Psi}{\mathrm{d}\eta}\right)^2 = m_1\Psi^2(\eta) + m_2\Psi^4(\eta) + m_3\Psi^6(\eta) \tag{13}$$

where $m_1$, $m_2$, $m_3$, $n_0$, $n_1$, $n_2$ are determined constants. Proceeding by substituting eq. (10) into eq. (12) by setting the coefficient of $\Psi_i(i = 0, 1, 2,..., 6)$ to 0, we get a system of linear equations whose solutions are:

$$n_0 = 0,\ n_1 = 0,\ n_2 = -12w^2 m_2,\ m_1 = -\frac{1}{4}\frac{w+k}{kw^2},\ m_2 = m_2,\ m_3 = 0 \tag{14}$$

$$n_0 = -\frac{2(w+k)}{k},\ n_1 = 0,\ n_2 = -12w^2 m_2,\ m_1 = \frac{1}{4}\frac{w+k}{kw^2},\ m_2 = m_2,\ m_3 = 0 \tag{15}$$

Substituting eq. (14) with the solution given in [28] into eq. (12) to recover the solutions:

$$h_1(x,t) = -\frac{3(w+k)}{k}\operatorname{sech}\left(\frac{1}{2w}\sqrt{-\frac{w+k}{k}}(kx+wt)\right) \tag{16}$$

$$h_2(x,t) = \frac{3(w+k)}{k}\operatorname{csch}\left(\frac{1}{2w}\sqrt{-\frac{w+k}{k}}(kx+wt)\right) \tag{17}$$

$$h_3^{\pm}(x,t) = \frac{6(w+k)}{k\left(\pm\cosh\left(\frac{1}{w}\sqrt{-\frac{w+k}{k}}(kx+wt)\right)-1\right)} \tag{18}$$

$$h_4^{\pm}(x,t) = \frac{48m_2(w+k)\exp\left(\pm\frac{1}{w}\sqrt{-\frac{w+k}{k}}(kx+wt)\right)}{k\left(\exp\left(\frac{1}{w}\sqrt{-\frac{w+k}{k}}(kx+wt)\right)-4m_2\right)^2} \tag{19}$$

$$h_5(x,t) = -\frac{3(w+k)}{k}\sec\left(\frac{1}{2w}\sqrt{\frac{w+k}{k}}(kx+wt)\right)^2 \tag{20}$$

$$h_6(x,t) = \frac{3(w+k)}{k}\csc\left(\frac{1}{2w}\sqrt{\frac{w+k}{k}}(kx+wt)\right)^2 \tag{21}$$

$$h_7^{\pm}(x,t) = \frac{6(w+k)}{k\left(\pm\cos\left(\frac{1}{w}\sqrt{\frac{w+k}{k}}(kx+wt)\right)-1\right)} \tag{22}$$

$$h_8^{\pm}(x,t) = \frac{6(w+k)}{k\left(\pm\sin\left(\frac{1}{w}\sqrt{\frac{w+k}{k}}(kx+wt)\right)-1\right)} \tag{23}$$

Substituing eq. (15) with the solution given in [28] into eq. (12) to recoverthe following solutions:

$$h_9(x,t) = -\frac{w+k}{k}\left(2-3\operatorname{sech}\left(\frac{1}{2w}\sqrt{\frac{w+k}{k}}(kx+wt)\right)\right) \tag{24}$$

$$h_{10}(x,t) = -\frac{(w+k)}{k}\left(2+3\operatorname{csch}\left(\frac{1}{2w}\sqrt{\frac{w+k}{k}}(kx+wt)\right)^2\right) \tag{25}$$

$$h_{11}^{\pm}(x,t) = -\frac{2(w+k)}{k} - \frac{6(w+k)}{k\left(\pm\cosh\left(\frac{1}{w}\sqrt{\frac{w+k}{k}}(kx+wt)\right)-1\right)} \tag{26}$$

$$h_{12}^{\pm}(x,t) = -\frac{2(w+k)}{k} - \frac{48m_2(w+k)\exp\pm\left(\frac{1}{w}\sqrt{\frac{w+k}{k}}(kx+wt)\right)}{k\left(\exp\left(\frac{1}{w}\sqrt{\frac{w+k}{k}}(kx+wt)\right)-4m_2\right)^2} \tag{27}$$

$$h_{13}(x,t) = -\frac{(w+k)}{k}\left(2-3\sec\left(\frac{1}{2w}\sqrt{-\frac{w+k}{k}}(kx+wt)\right)^2\right) \tag{28}$$

$$h_{14}(x,t) = -\frac{(w+k)}{k}\left(2-3\csc\left(\frac{1}{2w}\sqrt{-\frac{w+k}{k}}(kx+wt)\right)^2\right) \tag{29}$$

$$h_{15}^{\pm}(x,t) = -\frac{2(w+k)}{k} - \frac{6(w+k)}{k\left(\pm\cos\left(\frac{1}{w}\sqrt{-\frac{w+k}{k}}(kx+wt)\right)-1\right)} \tag{30}$$

$$h_{16}^{\pm}(x,t) = -\frac{2(w+k)}{k} - \frac{6(w+k)}{k\left(\pm\sin\left(\frac{1}{w}\sqrt{-\frac{w+k}{k}}(kx+wt)\right)-1\right)} \tag{31}$$

## The 3-D and contour plots of some traveling wave solutions

We have plotted both the 3-D for some of the recovered solutions using some appropriate values to exhibit some of the characteristics of the solitary solutions of eqs. (16), (19), (21)-(25) in figs. 2(a)-2(f).

## Conclusion

This paper extensively dealt with the JEE using the bifurcation method and the EAET. Quite a several new and effective exact traveling wave solutions were recovered for the JEE that has not been reported before. Therefore, the EAET has provided the robust and efficient for the JEE. The acquired results are shown in figs. 1 and 2 with the different values.

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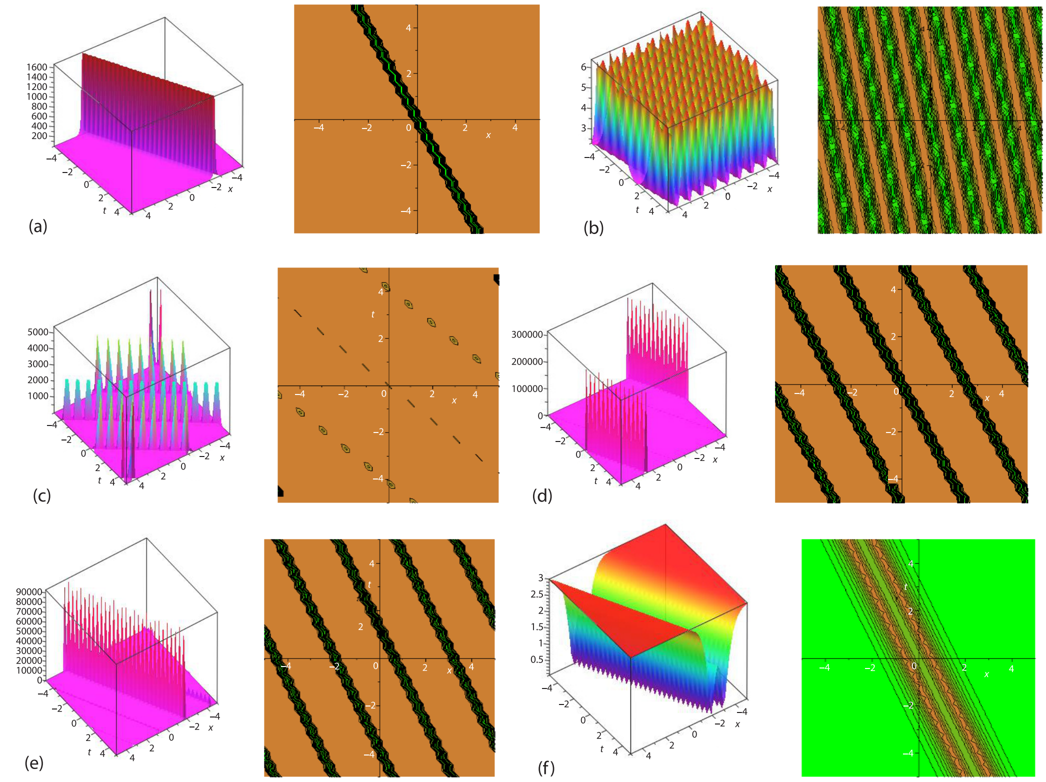

**Figure 2. The 3-D and counter plots of some acquired solutions for the different values; (a) $|h_{10}|$ for $m_2 = 1$, $k = 1$, $w = 1.5$, (b) $|h_4|$ for $m_2 = 1$, $k = 2.5$, $w = 0.5$, (c) $|h_6|$ for $m_2 = 1$, $k = 1.5$, $w = 2$, (d) $|h_7|$ for $m_2 = 1$, $k = 2$, $w = 1$, (e) 3-D and contour plots of $|h_8|$ for $m_2 = 1$, $k = 2$, $w = 1$, and (f) $|h_9|$ for $m_2 = 1$, $k = 2$, $w = 1$**

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