NEW APPROXIMATE SOLUTIONS TO TIME FRACTIONAL ORDER PARTIAL DIFFERENTIAL EQUATIONS
Optimal Auxilary Function Method

by

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In this article, approximate solutions of some PDE of fractional order are investigated with the help of a new semi-analytical method called the optimal auxiliary function method. The proposed method was tested upon the time-fractional Fisher equation, the time-fractional Fornberg-Whitham equation, and the time-fractional Inviscid Burger equation. The beauty of this method is that there is no need for discretization and assumptions of small or large parameters and provides an approximate solution after only one iteration. The numerical results obtained by the proposed method compared with the other existing methods used in the literature. From the numerical and graphical results, it is clear that the proposed method gives a better solution than existing methods. The MATHEMATICA software package has been used for the huge computational work.

Key words: *optimal auxiliary function method, approximate solution, fractional order differential equations, Caputo's derivative*

Introduction

Most of the problems arise in nature such as in liquid mechanics, biology, and thermodynamics, *etc*. are the models which can be set to mathematical form by the use of differential equation (DE). The DE may be linear or non-linear depends upon the nature of the problems arising in different areas of sciences. Linear models of DE may be solved by the use of simple analytical methods and most of the problems have an exact solution but when these models are in the form of non-linear DE then it is difficult to solve them easily by simple approaches. That's why we different methods to solve such non-linear models of DE. Nowadays the fractional-order DE are a great focus of researches. The fractional calculus is the modification of classical calculus. To solve the non-linear PDE of fractional order (FPDE), a variety of numerical and analytical techniques are used. Many researchers are using computational and analytical techniques to solve FPDE *i.e.*, Yapez-Martinez *et al.,* see in [1] obtained the numerical solution

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of some non-linear fractional-order equations. Atangana *et al*. [2] solved the non-linear Fisher's equations, reaction-diffusion equation with help of a new fractional operator. Atangana *et al*. [3] analyzed the Keller' Segel model with a fractional derivative without a singular kernel. A lot of work has been done and many applications of fractional order PDE can be seen in series of papers [4-11].

We have applied the semi-analytical method called the optimal auxiliary functions method (OAFM) to fractional order PDE. The proposed method was introduced by Marinca *et al*. [12, 13] and has been used for different fluid problems. Later on, many other researchers applied OAFM different PDE of integer order and for fractional-order Zada *et al*. [14]. The OAFM works without assuming any small or large parameter. The suggested technique has the advantage of being able to deal with both linear and non-linear problems effectively and without reducing generality.

Basic definitions

Definition 1. The fractional integral operator I^α of order $\alpha \geq 0$ in the Riemann-Liouville sense of a function, is defined:

$$I^\alpha f(\chi) = \frac{1}{\Gamma(\alpha)}\int_0^\chi (\chi - s)^{\alpha-1} f(s)\mathrm{d}s,\ \ \alpha, \chi > 0 \tag{1}$$

where $I^0 f(\chi) = f(\chi)$ and Γ is the well-known function.

Definition 2. Riemann-Liouville fractional derivative can be defined, $If(r) \in C[a, b]$ then:

$$I_a^\alpha = \frac{1}{\Gamma(\alpha)}\int_a^r \frac{g(\chi)}{(r-\chi)^{1-\alpha}}\mathrm{d}\chi \tag{2}$$

Some properties of fractional derivative and integral are given as $f \in C_\mu, \mu \geq 1, \alpha, \beta \geq 0$, and $\lambda > -1$ then:

$$\bullet I^\alpha I^\beta = I^{\alpha+\beta} f(\chi)$$
$$\bullet I^\beta I^\alpha = I^{\alpha+\beta} f(\chi)$$
$$\bullet I^\alpha \chi^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1+\alpha)}\chi^{\lambda+\alpha}$$

Analysis of OAFM

In this section, the OAFM is discussed for the fractional order PDE with convergence analysis. For this we consider the most general form of a non-linear fractional order differential equation:

$$\frac{\partial^\alpha w(\phi,\tau)}{\partial \tau^\alpha} = \wp(\phi,\tau) + N\left[w(\phi,\tau)\right] = 0 \tag{3}$$

subject to the boundary condition:

$$\begin{aligned} &\frac{\partial^{(\alpha-k)} w(\phi,\tau)}{\partial \tau^{(k-1)}} = h_k(\phi)(k = 0,1,2...,n-1),\ \frac{\partial^{(\alpha-n)} w(\phi,0)}{\partial \tau^{(\alpha-n)}} = 0,\ \ n = \alpha \\ &\frac{\partial^k w(\phi,\tau)}{\partial \tau^k} = g_k(\phi)(k = 0,1,2...,n-1),\ \frac{\partial^n w(\phi,0)}{\partial \tau^n} = 0,\ \ n = \alpha \end{aligned} \tag{4}$$

In eq. (3) $\partial^{\alpha}/\partial\tau^{\alpha}$ is the Riemann-Liouville fractional derivative operator, $w(\phi,\tau)$ – the function be approximate, N – the non-linear operator, and $\wp(\phi,\tau)$ – the known analytic function.

Step 1: To find an approximate solution (4), we consider the solution in two-component:

$$w(\phi,\tau) = w_0(\phi,\tau) + w_1(\phi,\tau,C_i), \; i = 1,2,3,\cdots n \tag{5}$$

Step 2: For obtaining the zero and first-order solution, put the (5) in (3) we obtain:

$$\frac{\partial^{\alpha} w_0(\phi,\tau)}{\partial\tau^{\alpha}} + \frac{\partial^{\alpha} w_1(\phi,\tau)}{\partial\tau^{\alpha}} + \wp(\phi,\tau) + N\left[\frac{\partial^{\alpha} w_0(\phi,\tau)}{\partial\tau^{\alpha}} + \frac{\partial^{\alpha} w_1(\phi,\tau),C_i}{\partial\tau^{\alpha}}\right] = 0 \tag{6}$$

Step 3: The initial approximation $w_0(\phi,\tau)$ can get from the linear equation:

$$\frac{\partial^{\alpha} w_0(\phi,\tau)}{\partial\tau^{\alpha}} + \wp(\phi,\tau) = 0 \tag{7}$$

applying inverse operator, we get $w_0(\phi,\chi)$:

$$w_0(\phi,\tau) = \wp(\phi,\tau) \tag{8}$$

Step 4: Using (8) the non-linear term can be expended:

$$N\left[\frac{\partial^{\alpha} w_0(\phi,\tau)}{\partial\tau^{\alpha}} + \frac{\partial^{\alpha} w_1(\phi,\tau,C_i)}{\partial\tau^{\alpha}}\right] = N\left[w_0(\phi,\tau)\right] + \sum_{k=1}^{\infty}\frac{w_1^k}{k!}N^{(k)}\left[w_0(\phi,\tau)\right] \tag{9}$$

Step 5: For the first order approximation, we consider:

$$\frac{\partial^{\alpha} w_1(\phi,\tau,C_i)}{\partial\tau^{\alpha}} = -A_1\left[w_0(\phi,\tau)\right]N\left[w_0(\phi,\tau)\right] - A_2\left[w_0(\phi,\tau),C_j\right] \tag{10}$$

Remark 1. Where A_1 and A_2 are auxiliary functions which dependent upon $w_0(\phi,\tau)$ and the convergence control parameter C_i and C_j, $i = 1, 2, 3,\ldots$, $j = s + 1, s + 2,\ldots \rho$.

Remark 2. The A_1 and A_2 are in the form of $w_0(\phi,\tau)$ or $N[w_0(\phi,\tau)]$ in the combination of both $w_0(\phi,\tau)$ and$N[w_0(\phi,\tau)]$ but they are not unique.

Remark 3.

– If $w_0(\phi,\tau)$ or $N[w_0(\phi,\tau)]$ are polynomial functions then A_1 and A_2 are taken as the sum of polynomial functions.
– If $w_0(\phi,\tau)$ or $N[w_0(\phi,\tau)]$ are exponential functions then A_1 and A_2 and are taken as the sum of exponential functions.
– If $w_0(\phi,\tau)$ or $N[w_0(\phi,\tau)]$ are trigonometric functions then A_1 and A_2 and are taken as the sum of trigonometric functions.

Step 6: By the use of inverse operator and substitution of auxiliary function in (10), we obtain the first-order solution $w_1(\phi,\tau)$ by OAFM.

Step 7: To find the residual we calculate the values of C_i and C_j by a different method such as the Collocation, or Least Square, Galerkin's, Ritz method, *etc.*:

$$H\left(C_\iota,C_j\right) = \int_0^{\delta}\int_{\Omega} R^2\left(\phi,\tau;C_\iota,C_j\right)\mathrm{d}\phi\mathrm{d}\tau \tag{11}$$

where R is the residual:

$$R\left(\phi,\tau,C_i,C_j\right) = \frac{\partial^{\alpha} w(\phi,\tau,C_i,C_J)}{\partial\tau^{\alpha}} + \wp(\phi,\tau) + N\left[w\left(\phi,\tau,C_i,C_j\right)\right] \tag{12}$$

Applications

In this section, the extended form of OAFM is implemented to fractional order PDE. Most of the computation work has been carried out through MATHEMATICA 10.

Example 1. Consider Inviscid Burger's non-linear non-homogeneous time-fractional eq. [15]:

$$\frac{\partial^\alpha w(\phi,\tau)}{\partial\tau^\alpha}+w(\phi,\tau)\frac{\partial w(\phi,\tau)}{\partial\phi}=1+\phi+\tau,\ \ 0<\alpha\le 1 \tag{13}$$

with the initial condition:

$$w(\phi,0)=\phi \tag{14}$$

For $\alpha = 0.1$ the exact solution (14):

$$w(\phi,\tau)=\phi+\tau \tag{15}$$

we have linear and non-linear parts form (14):

$$L\left[w(\phi,\tau)\right]=\frac{\partial^\alpha w(\phi,\tau)}{\partial\tau^\alpha}$$
$$N\left[w(\phi,\tau)\right]=w(\phi,\tau)\frac{\partial w(\phi,\tau)}{\partial\phi}-1-\phi-\tau \tag{16}$$

using the basic idea of OAFM, to get the zero-order problem from (8):

$$\frac{\partial^\alpha w_0(\phi,\tau)}{\partial\tau^\alpha}=0,\ \ w_0(\phi,\tau)=\phi \tag{17}$$

Using the inverse operator to (19) we get the zero order solution:

$$w_0(\phi,\tau)=\phi \tag{18}$$

Using (20) into (18), the non-linear operator:

$$N\left[w_0(\phi,\tau)\right]=w_0(\phi,\tau)\frac{\partial w_0(\phi,\tau)}{\partial\phi}-1-\phi-\tau \tag{19}$$

The first approximation as $w_1(\phi,\tau)$ is obtained:

$$\frac{\partial^\alpha w_1(\phi,\tau)}{\partial\tau^\alpha}+\zeta_1\left[w_0(\phi,\tau),C_i\right]N\left[w_0(\phi,\tau)\right]+\zeta_2\left[w_0(\phi,\tau),C_j\right]=0 \tag{20}$$

Now we select A_1 and A_2 for the non-linear operator accordingly:

$$A_1=-C_1$$
$$A_2=C_2\left(2\tau\right)+C_3\left(3\tau\right)^2+C_4\left(4\tau\right)^3 \tag{21}$$

using eqs. (20) and (21) in eq. (22), and applying the inverse operator, the approximation solution is obtained:

$$w_1\left(\phi,\tau\right)=\frac{\tau^\alpha\left\{C_1+\tau\left(1+\tau\right)\left[2C_2+\tau\left(9C_3+64C_4\tau\right)\right]\right\}}{\Gamma(1+\alpha)} \tag{22}$$

Adding eqs. (20) and (24) we get the first order approximate solution:

$$w(\phi,\tau)=\phi+\frac{\tau^\alpha\left\{C_1+\tau\left(1+\tau\right)\left[2C_2+\tau\left(9C_3+64C_4\tau\right)\right]\right\}}{\Gamma(1+\alpha)} \tag{23}$$

For finding the convergence control parameters present in eq. (23), we use the collocation method. The numerical values are tabulated in tabs. 1 and 2. Using these values in eq. (23), we get the first-order approximate solution of *Problem 1*, figs. 1 and 2.

Table 1. Convergence control parameters for different values of α for *Example 1*

C_i	$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.7$
C_1	0.999999999999999480	1.02464231914095	1.0695878711393512200
C_2	$7.09140491455450 \cdot 10^{-15}$	–0.06934160441034	–0.1086172198564621110
C_3	$-2.33206336671750 \cdot 10^{-15}$	0.02099652688745	0.0328316457878370660
C_4	$1.438241672916320 \cdot 10^{-16}$	-0.00113677928872	–0.0017862114301269220

Table 2. Comparison of absolute error obtained by OAFM solution and HPTM solution for *Example 1*, when α = 1.0

ϕ	$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.7$	Absolute error HPTM	Absolute error OAFM
0.2	0.25	0.5	0.5	$1.62760\ 2.604167 \cdot 10^{-4}$	$5.55112 \cdot 10^{-16}$
0.25	0.5	0.75	0.75	$2.604167\ 2.604167 \cdot 10^{-3}$	$4.44089 \cdot 10^{-16}$
0.25	0.75	1.0	1.0	$1.318359\ 2.604167 \cdot 10^{-2}$	$3.33067 \cdot 10^{-16}$
0.25	1.0	1.25	1.25	$4.166666\ 2.604167 \cdot 10^{-2}$	$4.44089 \cdot 10^{-16}$
0.5	0.25	0.75	0.75	$1.62760\ 2.604167 \cdot 10^{-4}$	$5.55112 \cdot 10^{-16}$
0.5	0.5	1.0	1.0	$2.604167 \cdot 10^{-3}$	$4.44089 \cdot 10^{-16}$
0.5	0.75	1.25	1.25	$1.318359\ 2.604167 \cdot 10^{-2}$	$3.33067 \cdot 10^{-16}$
0.5	1.0	1.5	1.5	$4.166666\ 2.604167 \cdot 10^{-2}$	$4.44089 \cdot 10^{-16}$

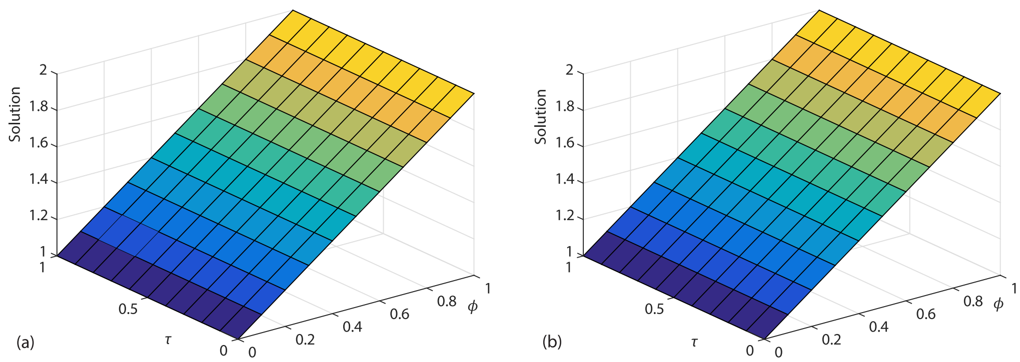

Figure 1. The OAFM solution (a) and exact solution (b) for *Problem 1*

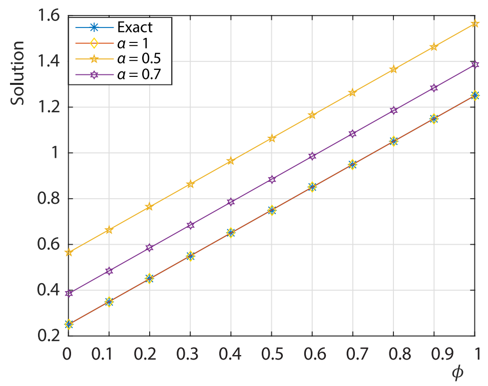

Figure 2. Impact of α on OAFM solution for *Problem 1*

Example 2. Consider non-linear time-fractional Fisher's equation [15]:

$$\frac{\partial^\alpha w(\phi,\tau)}{\partial\tau^\alpha}=\frac{\partial^2 w(\phi,\tau)}{\partial\phi^2}+6w(\phi,\tau)\left[1-w(\phi,\tau)\right],\ \ 0<\alpha\le 1 \tag{24}$$

The initial condition of the non-linear time-fractional Fisher equation:

$$w\left(\phi,0\right)=\frac{1}{\left(1+\mathrm{e}^{\phi}\right)^2} \tag{25}$$

For special case α = 1.0, the exact solution for eq. (26) is given:

$$w(\phi,\tau)=\frac{1}{\left(1+\mathrm{e}^{\phi-5\tau}\right)^2} \tag{26}$$

The linear and non-linear terms can be chosen from eq. (29):

$$L\left[w(\phi,\tau)\right]=\frac{\partial^\alpha w(\phi,\tau)}{\partial\tau^\alpha}$$
$$N\left[w(\phi,\tau)\right]=-\frac{\partial^2 w(\phi,\tau)}{\partial\phi^2}-6w(\phi,\tau)\left[1-w(\phi,\tau)\right] \tag{27}$$

Using the basic idea of OAFM with the inverse operator, the zero-order problem can be gotten:

$$w_0(\phi,\tau)=\frac{1}{\left(1+\mathrm{e}^{\phi}\right)^2} \tag{28}$$

Here we select A_1, A_2 as according to the non-linear operator for the first operator as, $A_1 = -C_1$, $A_2 = 0$. Using the same procedure of OAFM as discussed for *Problem 1*, we get the first-order solution:

$$w_1(\phi,\tau)=-\frac{10C_1\mathrm{e}^{\phi}\tau^\alpha}{\alpha\left(1+\mathrm{e}^{\phi}\right)^3\Gamma(1+\alpha)} \tag{29}$$

To obtain the first order OAFM solution, we add eqs. (31) and (33):

$$w(\phi,\tau)=\frac{1}{\left(1+\mathrm{e}^{\phi}\right)^2}-\frac{10C_1\mathrm{e}^{\phi}\tau^\alpha}{\alpha\left(1+\mathrm{e}^{\phi}\right)^3\Gamma(\alpha)} \tag{30}$$

For finding the convergence control parameters present in eq. (30), we use the collocation method. The numerical values are tabulated in tabs. 3 and 4. Using these values in eq. (30), we get the first-order approximate solution of *Example 2*, fig. 3.

Table 3. Parameters for different values of α for *Example 2*

Parameters	α = 1.0	α = 0.8	α = 0.7
C_1	–1.0013046523143574	–1.07601806643966	–1.183040098101294

Table 4. Comparison between OAFM solutions and abs. error with exact, ADM for on-linear time-fractional Fisher's equation when α = 1.0 and τ = 0.001

ϕ	$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.7$	Absolute errors ADM	OAFM
0.01	0.248752	0.248752	$1.480 \cdot 10^{-3}$	$-1.453 \cdot 10^{-6}$	$4.7211 \cdot 10^{-8}$
0.02	0.246264	0.246264	$1.434 \cdot 10^{-3}$	$-1.825 \cdot 10^{-6}$	$2.3731 \cdot 10^{-8}$
0.03	0.243789	0.243789	$1.395 \cdot 10^{-2}$	$-1.798 \cdot 10^{-6}$	$4.9084 \cdot 10^{-8}$
.04	0.241327	0.241327	$1.361 \cdot 10^{-2}$	$-1.771 \cdot 10^{-6}$	$2.2505 \cdot 10^{-8}$
0.05	0.238878	0.238878	$1.331 \cdot 10^{-2}$	$-1.743 \cdot 10^{-6}$	$4.5252 \cdot 10^{-8}$

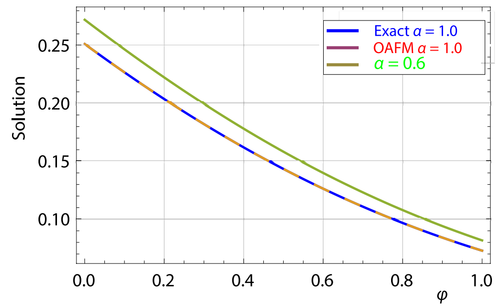

Figure 3. Behaviour of α on the solution of OAFM for *Example 2*

Example 3. Consider the time-fractional Fornberg-Whitham equation [15]:

$$\frac{\partial^\alpha w(\phi,\tau)}{\partial\tau^\alpha} = \frac{\partial^3 w(\phi,\tau)}{\partial\phi^2\partial\tau} - \frac{\partial w(\phi,\tau)}{\partial\phi} + w(\phi,\tau)\frac{\partial^3 w(\phi,\tau)}{\partial\phi^3} - w(\phi,\tau)\frac{\partial w(\phi,\tau)}{\partial\phi} + 3\frac{\partial w(\phi,\tau)}{\partial\phi}\frac{\partial^2 w(\phi,\tau)}{\partial\phi^2} \tag{31}$$

with initial condition, $w(\phi, 0) = e^{\phi/2}$. For special case when α = 1.0 then the exact solution for (31):

$$w(\phi,\tau) = e^{\frac{1}{2}\left(\phi - \frac{4\tau}{3}\right)}$$

The auxiliary functions can be choose for *Example 3*:

$$A_1 = -\left[C_1 + 2C_2(\tau) + 2C_3(\tau)^2 + 2C_4(\tau)^3 + 2C_5(\tau)^4 + 2C_6(\tau)^5\right]$$
$$A_2 = 0 \tag{32}$$

then using the same procedure as discussed in *Example 2*, we get zero-order and the first order AOFM solution for *Example 3*:

$$w_0(\phi,\tau) = e^{\phi/2} \tag{33}$$

$$w_1(\phi,\tau) = \frac{e^{\phi/2}\tau^\alpha\left[C_1 + 2\tau\left(C_2 + \tau\left\{C_3 + \tau\left[C_4 + \tau(C_5 + C_6\tau)\right]\right\}\right)\right]}{2\Gamma(1+\alpha)} \tag{34}$$

To obtain the first order OAFM solution, we add eqs. (46) and (47):

$$w(\phi,\tau) = e^{\phi/2} + \frac{e^{\phi/2}\tau^\alpha\left[C_1 + 2\tau\left(C_2 + \tau\left\{C_3 + \tau\left[C_4 + \tau(C_5 + C_6\tau)\right]\right\}\right)\right]}{2\Gamma(1+\alpha)} \tag{35}$$

For finding the convergence control parameters present in eq. (35), we use the collocation method. the numerical values are tabulated in tabs. 5 and 6. Using these values in eq. (35), we get the first-order approximate solution of *Example 3*, fig. 4.

Table 5. Convergence control parameters for different values of α for *Example 3*

C_i	$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.7$
C_1	–1.3333328680180432	–2.015914169868006	–3.214654344418841
C_2	0.2222193527869724	1.6609111229542566	4.560007833350679
C_3	–0.0493681190158767	–3.409779522526362	–10.980886563771994
C_4	0.00819312144303661	4.4012448967414874	14.967519893198872
C_5	–0.00104635037495981	–2.976078223819042	–10.405871287220586

Table 6. Comparison of numerical results of Fornberg-Whitham equation via OAFM is compared with HPTM when $\alpha = 1.0$

C_i	$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.7$	Absolute errors HPTM	Absolute error OAFM
0.1	1	1.54239	1.54239	$1.19 \cdot 10^{-3}$	$9.74481 \cdot 10^{-9}$
0.1	2	2.54297	2.54297	$1.96 \cdot 10^{-3}$	$1.60665 \cdot 10^{-8}$
0.1	3	4.19265	4.19265	$3.24 \cdot 10^{-3}$	$2.64891 \cdot 10^{-8}$
0.1	4	6.91251	6.91251	$5.34 \cdot 10^{-3}$	$4.36732 \cdot 10^{-8}$
0.1	5	11.3968	11.3968	$8.80 \cdot 10^{-3}$	$7.20049 \cdot 10^{-8}$

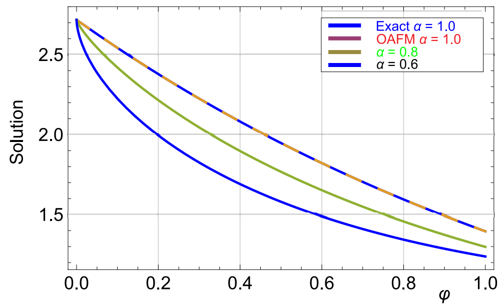

Figure 4. Behaviour of α on the solution of OAFM for *Example 3*

Conclusion

The OAFM successfully applied for different non-linear fractional order DE. It is observed that the proposed method gives a very powerful approximate solution after only one iteration. There is no need of high computational work in method. Similarly, if we want to increase the accuracy of approximate solution, only convergence control parters should be increase in the auxiliary function. The proposed method is very effective and easy to implement for fractional order non-linear problems.

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