POROUS MEDIUM EQUATION WITH ELZAKI TRANSFORM HOMOTOPY PERTURBATION

by

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We have shown how to solve 1-D fourth order parabolic linear PDE with varable coefficients in this article. We have applied the Elzaki transform homotopy perturbation method. It is used in conjunction with the homotopy perturbation approach in this method. To further showcase the Elzaki transform-homotopy perturbation technique's competence and reliability, we have provided solution the parabolic linear PDE. It was found that both HPM and Laplace decomposition method were similar when compared to resulting analytical solutions. Results suggest that the Elzaki transform in conjunction with the homotopy perturbation technique is capable and practicable for use in these cases.

Key words: *homotopy perturbation method, linear PDE, Elzaki transform*

Introduction

In a wide range of physical settings, PDE with beginning and end states have been utilized to simulate. Heat transfer analysis and wave propagations, potential analysis*, etc.* are modeled by the use of linear PDE. Also in mathematical biology, age-structure models have been investigated for various results by using PDE. As a result of these applications, researchers have attributed considerable attention in the recent several decades to the study of the aforementioned domain for various aspects, including qualitative analysis, numerical and stability analysis, and optimization, among others. For the aforementioned aspects, several sophisticated tools, theories and methods have been established in literature by various researchers. Treating linear problems of the aforementioned area by using various integral transform coupled with decomposition and perturbation techniques have gotten considerable attention from researcher. In recent years, several researchers have used a variety of strategies to try to find a solution these equations. Some of these methods used by the researchers are being mentioned below:

– Homotopy perturbation method (HPM) [1-4].
– Variational iteration method (VIM) [5-7].
– Homotopy analysis nethod.
– Laplace decomposition algorithm.
– Adomian decomposition method (ADM) [8, 9].
– Integral transform method, Elzaki transform, and natural transform [10-12]

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In our current research, we used the Elzaki transform and HPM to address various differential equations which are both linear and non-linear. Elzaki transform HPM's effectiveness to determine higher order linear PDE with different coefficients is primary objective of this research. With this approach, the explanation is produced in the form of a fast convergent series that eventually leads to a closed solution. The most important advantage of the considered scheme is that it needs no prior discretization of data neither any collocation of function. The implementation is easy and understandable.

Background materials

Here we recollect some fundamental results from [1, 5, 8, 9].

The history of the Elzaki transform

In order to understand the modified Elzaki transform, we need to know the following. Elzaki transform for the function $f(t)$:

$$E\left[f(t)\right]=v\int_0^\infty e^{-t/v}f(t)\,\mathrm{d}t,\ \ t>0$$

The Elzaki transformation was demonstrated [2, 10] and being used to determine different types of mathematical equations such as integral equations, PDE, ordinary PDE*, etc.* Some of the differential equations, for which the *Sumudu transform* had failed to provide a conclusion, can now be solved using the Elzaki transform [2]. Elzaki transform and HPM is taken into consideration provide solution various non-linear PDE. By integrating various factors, we can obtain the Elzaki transform of partial derivative.

Thus, we have:

$$E\left[\frac{\partial f(t)}{\partial t}\right]=\frac{1}{v}T(x,v)-vf(x,0)$$

$$E\left[\frac{\partial^2 f(t)}{\partial t^2}\right]=\frac{1}{v^2}T(x,v)-f(x,0)-v\frac{\partial f(x,0)}{\partial t}$$

$$E\left[\frac{\partial^2 f(t)}{\partial t^2}\right]=\frac{1}{v^2}T(x,v)-f(x,0)-v\frac{\partial f(x,0)}{\partial t}$$

$$E\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=\frac{\mathrm{d}^2}{\mathrm{d}x^2}T(x,v)$$

Proof:

Integration by parts is used to generate the ELzaki transform of partial derivatives:

$$E\left[\frac{\partial f(x,t)}{\partial t}\right]=\int_0^\infty v\frac{\partial f(x,\,t)}{\partial t}e^{-1/v}\mathrm{d}t=\lim_{p\to\infty}v\left\{\left[e^{-1/v}f(x,\,t)\right]\right\}_0^p-\int_0^p\left[e^{-1/v}f(x,t)\mathrm{d}t\right]$$

We estimate that f is piecewise continuous and it is of exponential order:

$$E\left[\frac{\partial f(x,t)}{\partial x}\right]=\int_0^\infty v\frac{\partial f(x,t)}{\partial x}e^{-t/v}\mathrm{d}t=\frac{\partial}{\partial x}\int_0^\infty vf(x,t)e^{-t/v}\mathrm{d}t$$

Leibnitz's rule is being applied get to the conclusion:

$$E\left[\frac{\partial f(t)}{\partial x}\right] = \frac{\mathrm{d}}{\mathrm{d}x} T(x,v)$$

Using similar approach:

$$E\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = \frac{\mathrm{d}^2}{\mathrm{d}x^2} T(x,v)$$

Let $\partial f/\partial t = g$, then we have:

$$E\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right] = E\frac{\partial g(x,t)}{\partial t} = E\frac{[g(x,t)]}{v} - vg(x,0)$$

Mathematical induction allows us to further broaden this outcome with minimal fuss.

Homotopy perturbation method

Let us consider an equation understand the HPM:

$$L(v) = 0 \tag{1}$$

For any differential operator L, we would consider $H(v, p)$ as a convex homotopy:

$$H(v,p) = (1-p)F(v) + pL(v) \tag{2}$$

There are functional operators with known solutions, such as $F(v)$, that is easily captured.

Therefore:

$$H(v,p) = 0 \tag{3}$$

We have:

$$H(v, 0) = F(v),\ H(v,1) = L(v)$$

We are considering power series as a solution:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \tag{4}$$

if $p \to 1$, then eq. (4) substitutes to eq. (2) and hence, we have the solution:

$$f = \lim_{p\to 1} \sum_{i=0}^{\infty} v_i \tag{5}$$

Assuming that eq. (6) has an uncommon explanation, the assessment of similar power of p furnishes different order solutions.

Homotopy perturbation Elzaki transform method:

The PDE with variable coefficients in a 1-D linear non-homogeneous PDE is now being considered:

$$Dv(y,T) + Sv(y,T) + Nv(y,T) = p(y,T),\ \ v(y,0) = h(y,0),\ \ v_t(y,0) = f(y) \tag{6}$$

where D is the linear differential operator, S – the linear differential operator of less order than D, N – the general non-linear differential operator, and $p(y, t)$ – the source term.

Applying Elzaki transform to eq. (6), to get:

$$E[v(y, T)] = u^2 E[p(y, t)] + u^2 h(y) + u^3 f(y) - u^2 E[Sv(y, T) + Nv(y, T)]$$

By inverse Elzaki transform, we get:

$$v(y,T) = P(y,T) - E^{-1}\left\{u^2 E\left[Sv(y,T) + Nv(y,T)\right]\right\}$$

the HPM is now being used

$$N\left[V(y,T)\right] = \sum_{n=0}^{\infty} q^n H_n(y,T)$$

$$H_n(v_0, v_1, ..., v_n) = \frac{1}{n!}\frac{\partial}{\partial q^n}\left[N\left(\sum_{i=0}^{\infty} q^i v_i\right)\right]_{q=o}$$

$$\sum_{i=0}^{\infty} q^n v_n(y,T) = P(y,T) - q\left[\frac{1}{E}\left[u^2 E\left(S\sum_{i=0}^{\infty} Q^n v_n(y,T) + \sum_{i=0}^{\infty} q^n H_n(v)\right]\right]\right]$$

Coefficient of power of p is compared, which now gives the result:

$$V(y,T) = v_0 + v_1 + v_2.....$$

Theorem 1: Let us consider X as Banach space and $T: X \to X$ as contraction operator, U, $\bar{U} \in X$, we have $\|T(U) - T(\bar{U})\|_X \leq l\|U - \bar{U}\|_X$, $0 \leq l < 1$. Thus the proposed problem has a unique solution. Further, we can write the truncated series:

$$U_n = T(U_{n-1}),\ U_{n-1} = \sum_{k=0}^{n-1} U_k,\ n = 1,2,3...$$

Also, if $U_0 \in B_r(U) = \{U \in X: \|U - \bar{U}\|_X < r\}$, then we need to prove that: $U_n \in B_r(U)$ and $\lim_{n\to\infty} U_n = U$.

Proof: On using mathematical induction on the operator equation $U_n = T(U_{n-1})$, by using Banach contraction result as for $n = 1$, one has:

$$\| U_1 - U \|_X = \| T(U_0) - T(U) \|_X \leq l \| U_0 - U \|_X$$

$$\| U_2 - U \|_X = T(U_1) - T(U) \|_X \leq l^2 \| U_0 - U \|_X$$

Let the induction is true at $n - 1$ say, then we have:

$$\| U_{n-1} - U \|_X = \| T(U_{n-2}) - T(U) \|_X \leq l^{n-1} \| U_0 - U \|_X$$

Hence using the aforementioned result for $n - 1$ and for $n = 1$, we have:

$$\| U_n - U \|_X = \| T(U_{n-1}) - T(U) \|_X \leq l^n \| U_0 - U \|_X < l^n r < r$$

Hence, we have $\|U_n - U\|_X < r$. Thus $U_n \in B_r(U)$ has been proved. Also, we see that $\lim_{n\to\infty} \| U_n - U\|_{X\to 0}$.

Thus $\lim_{n\to\infty} U_n = U$. For further detail proof, reader should see Shah *et al*. [12].

Examples

To support our aforementioned methodology, we give some examples.

Example 1: Consider the non-linear gas dynamic equation:

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial y}\left(u^{-2/3}\frac{\partial u}{\partial y}\right) \tag{7}$$

with the following initial condition:

$$U(y,0)=(2y)^{-3/2} \tag{8}$$

Elzaki transform is being used on eq. (7):

$$E\left[\frac{\partial u}{\partial t}\right]=E\left[\frac{\partial u}{\partial y}\left(u^{-2/3}\frac{\partial u}{\partial y}\right)\right] \tag{9}$$

We can re-write the eq. (7):

$$\frac{1}{v}E\left[u\left(y,t\right)\right]-vu\left(y,0\right)=E\left[\frac{\partial u}{\partial y}\left(u^{-2/3}\frac{\partial u}{\partial y}\right)\right] \tag{10}$$

using initial condition eq. (8) can be written:

$$E\left[u\left(y,t\right)\right]=v^2(2y)^{-3/2}+vE\left[\frac{\partial u}{\partial y}(u^{-\frac{2}{3}}\frac{\partial u}{\partial y})\right]... \tag{11}$$

Appying inverse E-transform to eq. (11):

$$\left[u\left(y,t\right)\right]=v^2\left(2y\right)^{-3/2}+E^{-1}\left(vE\left[\frac{\partial u}{\partial y}\left(u^{-2/3}\frac{\partial u}{\partial y}\right)\right]\right)$$

the HPM is now being applied

$$u\left(y,t\right)=\sum_{n=0}^{\infty}p^n u_n\left(y,t\right)$$

non-linear term is now further reduced to

$$N\left[u\left(y,t\right)\right]=\sum_{n=0}^{\infty}p^n H_n\left(u\right)$$

$$\sum_{n=0}^{\infty}p^n u_n\left(y,t\right)=v^2\left(2y\right)^{-3/2}+pE^{-1}\left(vE\left[\sum_{n=0}^{\infty}p^n u_n\left(y,t\right)-\sum_{n=0}^{\infty}p^n H_n\left(u\right)\right]\right)$$

He’s polynomials is being denoted by $H_n(u)$.

The initial two-components of He’s polynomials:

$$H_0(u)=u_0^{-2/3}\frac{\partial^2 u_0}{\partial y^2}-\frac{2}{3}u_0^{-5/3}\left(\frac{\partial u_0}{\partial y}\right)^2$$

$$H_1(u)=u_0^{-2/3}\left[\frac{\partial^2 u_0}{\partial y^2}-\frac{2}{3}\frac{\partial^{2u_0}}{\partial y^2}\left(\frac{u_1}{u_0}\right)\right]-\frac{2}{3}u_0^{-5/3}\left[2\left(\frac{\partial u_0}{\partial y}\right)\left(\frac{\partial u_1}{\partial y}\right)-\frac{5}{3}\left(\frac{\partial u_0}{\partial y}\right)^2\left(\frac{u_1}{u_0}\right)\right]$$

Multiple coefficient powers of p are compared, we get:

$$p^0 : u_0\left(y,t\right) = \left(2y\right)^{-3/2}$$

$$H_0(u) = u_0^{-2/3}\frac{\partial^2 u_0}{\partial y^2} - \frac{2}{3}u_0^{-\frac{5}{3}}\left(\frac{\partial u_0}{\partial y}\right)^2 = 9(2y)^{-5/2}$$

$$p^1 : u_1\left(y,t\right) = E^{-1}\left(vE\left[u_0 - H_0\left(u\right)\right]\right) = E^{-1}\left(vE\left[(9(2y)^{-\frac{5}{2}}\right]\right) = 9\left(2y\right)^{-5/2} t$$

In the same way, we can derive more values:

$$p^2 : u_2\left(y,t\right) = \frac{135}{2}\left(2y\right)^{-7/2} t$$

The resultant solution is now being obtained:

$$u(y,t) = u_0(y,t) + u_1(y,t) + ... = \left(2y\right)^{-3/2} + 9\left(2y\right)^{-5/2} t + \frac{135}{2}\left(2y\right)^{-7/2} t + ...$$

The solution is consolidating to the exact solution the problem.

Example 2:

Let's take into consideration the following non-linear gas dynamic equation with non-homogeneous non-linearity:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right) \tag{12}$$

by the corresponding primary condition:

$$U\left(x,0\right) = \frac{1}{x^2} \tag{13}$$

Elzaki transform is being applied to eq. (12) we have:

$$E\left[\frac{\partial u}{\partial t}\right] = E\left[\frac{\partial u}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right)\right] \tag{14}$$

Using differential properties of Elzaki transform is then used, and eq. (1) is now introduced:

$$\frac{1}{v}E\left[u\left(x,t\right)\right] - vu\left(x,0\right) = E\left[\frac{\partial u}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right)\right] \tag{15}$$

Equation (12) can be re-written from eq. (13):

$$E\left[u\left(x,t\right)\right] = v^2\frac{1}{x^2} + vE\left[\frac{\partial u}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right)\right] \tag{16}$$

Applying inverse E to eq. (16):

$$\left[u\left(x,t\right)\right] = \frac{1}{x^2} + E^{-1}\left(vE\left[\frac{\partial u}{\partial x}\left(u^{-2}\frac{\partial u}{\partial x}\right)\right]\right)$$

From HPM:

$$u(x,t)=\sum_{n=0}^{\infty}p^{n}u_{n}(x,t)$$

$$N\left[u(x,t)\right]=\sum_{n=0}^{\infty}p^{n}H_{n}(u)$$

$$\sum_{n=0}^{\infty}p^{n}u_{n}(x,t)=\frac{1}{x^{2}}+pE^{-1}\left(vE\left[\sum_{n=0}^{\infty}p^{n}u_{n}(x,t)-\sum_{n=0}^{\infty}p^{n}H_{n}(u)\right]\right)$$

He's polynomials is being denoted by $H_n(u)$ and the initial three components of He's polynomials are as follows, where $H_n(u)$ are He's polynomials. The first three components of He's polynomials are given:

$$H_0(u)=u_0^{-2}-2u_0^{-3}\left(\frac{\partial u}{\partial x}\right)^2$$

$$H_1(u)=u_0^{-2}\left[\left(\frac{u_1}{u_0}\right)\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_1}{\partial x^2}\right]-u_0^{-3}\left[2\left(\frac{\partial u_0}{\partial x}\right)\left(\frac{\partial u_1}{\partial x}\right)-3\left(\frac{\partial u_0}{\partial x}\right)^2\left(\frac{u_1}{u_0}\right)\right]$$

The coefficients of various powers of p are compared.

$$p^0: u_0(x,t)=\frac{1}{x^2}$$

$$H_0(u)=u_0^2+u_0\frac{\partial u_0}{\partial x}=\left(\frac{1}{x^2}\right)^{-2}-2\frac{1}{x^2}\left(\frac{-2}{x^3}\right)=-2$$

$$p^1: u_1(x,t)=E^{-1}\left(vE\left[u_0-H_0(u)\right]\right)$$

$$p^1: u_1(x,t)=E^{-1}\left(vE\left[-2\right]\right)=-2t$$

In the same way, multiple values are being obtained:

$$p^2: u_2(x,t)=-t^2$$

The remaining terms can be calculated in the same way. As a result, the answer can be expressed:

$$u(x,t)=u_0(x,t)+u_1(x,t)...=-2\left[1+t+t^2+\ldots\right]=\frac{-2}{1-t}$$

The resultant solution is obtained, which depicts the exact solution for the problem mentioned previously.

Conclusion

An analytical solution for the gas dynamics equation can be constructed using an integral transform known as the *Elzaki transform* and a homotopy perturbation method. When these two approaches were combined, they produced answers to the equation that were both dependable and precise. In a quickly convergent sequence, analytical approximation is supplied with terms computed exclusively. Using this method, non-linear partial differential equations can be solved more quickly and accurately than previously possible.

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