A VARIABLE FUTURE-TIME-STEPS METHOD FOR SOLVING NONLINEAR UNSTEADY INVERSE HEAT CONDUCTION PROBLEMS

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Abstract: *In some nonlinear unsteady inverse problems, the inverse solution will oscillate violently in the whole time domain due to the sharp change of the sensitivity coefficients. To deal with this problem, a new sequential function specification method with variable future time steps is proposed in this paper. The future time steps are adjusted by the error amplification coefficients which are defined as the reciprocal of the square sum of the sensitivity coefficients. When the error amplification coefficients are small, a small number of future time steps is used to reduce the deterministic error; while in the period with large error amplification coefficient, a large number of future time steps is used to reduce stochastic error. Finally, the total error of estimated heat flux is reduced. Avoid the sharp fluctuation of estimated heat flux in time domain due to the sharp change of sensitivity coefficients. The variable future-time-steps method is applied to the estimation of one-dimensional nonlinear unsteady heat flux without and with ablation through numerical experiments. Numerical experiments show that the proposed method can not only estimate various forms of heat flux, but also its inversion results are significantly better than those of the fixed future time steps method based on the discrepancy principle, and also better than those of the fixed future time step method based on the minimum relative error of heat flux.*

Key words: *Inverse heat conduction problem; sequential function specification method; variable future time steps; ablation*

1. Introduction

Inverse heat conduction problem (IHCP) refers to the estimation problem of some unknown characteristic parameters of the heat conduction system, such as boundary conditions, thermophysical parameters, geometric shapes, initial conditions and source terms, according to some temperature information inside or/and on the surface of the heat conduction system. IHCPs are widely used in scientific research and many technical fields, such as power engineering [1], aerospace [2], material processing [3], microelectronics [4], metallurgical engineering [5], nondestructive testing [6], biological engineering [7; 8].

In recent years, the research on inverse heat conduction problem is very active. Some important achievements have been made in inverse heat conduction problem and its application, and some valuable mathematical methods have been developed to solve inverse heat conduction problems. According to the different time domain involved in the measurement information required by inverse methods, the research methods of inverse heat conduction problems can be divided into two types: whole-domain inverse algorithm and sequential inverse algorithm. whole-domain inverse algorithms mainly include Tikhonov regularization [9], iterative regularization[10; 11], intelligent optimization algorithm [12]; sequential inverse algorithms mainly include sequential function specification method (SFSM) [13], model prediction inverse method [14], Kalman filter technology [15], and PID inverse method [16; 17], etc. The whole-domain inverse method generally has higher computational accuracy. The sequential inverse method is a real-time or near real-time estimation method, which has the advantages of low storage requirements of temperature information and higher calculation efficiency. Among the sequential inverse algorithms, the SFSM proposed by J. V. Beck has been widely applied in many fields due to its simplicity and effectiveness.

The inverse heat conduction problem is ill-conditioned, namely, the existence, uniqueness and stability of the Hadamard solution of the inverse heat conduction problem cannot be satisfied at the same time. Therefore, any efficient inverse algorithm needs to compromise between fidelity and stability, the compromise is called regularization technology. SFSM realizes regularization by selecting the number of future time steps. The selection of the number of future time steps has a significant impact on the inversion results of SFSM, and must be carefully selected.

In Refs. [18; 19], for the one-dimensional linear unsteady inverse heat conduction problem, the influence of measurement error, measurement point position, sampling time, heat flux form on the optimal number of future time steps is examined by applying the discrepancy principle in the whole-time domain. Zhang et al. [20] optimized the fixed number of future time steps by using the discrepancy principle in the whole-time domain and estimated the heat flux distribution of two-dimensional billet.

For the above linear problems or weakly nonlinear problems, using the fixed future time steps which is determined by the discrepancy principle in the whole-time domain can obtain relatively satisfactory inversion results. However, for strongly nonlinear problems, it is difficult to obtain satisfactory inversion results by using the fixed number of future time steps. For example, in the bone grinding heat source identification problem, the grinding heat source moves relative to the measurement point, resulting in this moving heat source identification problem being a kind of strongly nonlinear problem. Therefore, in the whole movement process of heat source, using the fixed number of future time steps will inevitably lead to too large inversion error in the time period when the heat source is far from the measurement point, and even the inversion result has no practical significance [21]. A more effective strategy is to adopt different number of future time steps in each local time domain. In this way, the balance between fidelity and stability can be achieved in the local time domain, and the optimal inversion result in the whole time domain can be obtained.

Blanc et al. [19] solved the one-dimensional linear unsteady heat conduction inverse problem by automatically adjusting the number of future time steps of SFSM using the discrepancy principle, obtained the optimal solution at each estimated time, and realized the balance between fidelity and stability in the local time domain. This method can obtain satisfactory inversion results for linear or weakly nonlinear inverse problems. However, for the strongly nonlinear inverse problem, using Blanc's method directly cannot get a satisfactory solution. There are two main reasons: on the one hand, strong nonlinearity leads to the sharp change of sensitivity coefficient with time, which leads to the sharp change of estimated heat flux error with time. Therefore, the inversion results fluctuate violently in the whole-time domain. Blanc's method does not consider the problem that the inversion solution fluctuates violently in the whole-time domain due to the drastic change of the sensitivity coefficients in the time domain. On the other hand, for the linear or quasi-linear direct problem, an accurate temperature prediction model at the measurement point can be established through the first-order approximation of Taylor series. A reasonable number of future time steps can be obtained by automatically adjusting the number of future time steps based on the weighted discrepancy principle. However, for the seriously nonlinear problem, the prediction model established by using the first-order approximation of Taylor series has low accuracy. At this time, Blanc's method tends to use small future time steps to meet the weighted discrepancy principle, and small future time steps will inevitably lead to unstable inversion results. Based on these two reasons, it is inappropriate to apply Blanc's method directly to the seriously nonlinear problems. For the strongly nonlinear estimation problem of ablation heat flux of composite materials, Mohammadiun et al. [22] adopted the variable future time steps strategy by the trial and error method. Different numbers of future time steps in each local time domain of ablation are adopted, and good inversion results are obtained. This idea is of great significance for solving strongly nonlinear inverse heat transfer problems.

In this paper, we attempt to establish a new sequential function specification method with variable future time steps. The future time steps of this method are adjusted by the error amplification coefficients which are defined as the reciprocal of the square sum of the sensitivity coefficients. When the error amplification coefficients are small, a small number of future time steps is used to reduce the deterministic error; while in the period with large error amplification coefficient, a large number of future time steps is used to reduce stochastic error. Finally, the total error of estimated heat flux is reduced. Avoid the sharp fluctuation of estimated heat flux in time domain due to the sharp change of sensitivity coefficients.

The present paper is organized in the following way. First, the direct and inverse heat conduction problems with variable geometry domain due to ablation and the numerical solution of the direct problem are presented. Then, the sequential function specification method is briefly reviewed, and then, a new variable future-time-steps method is proposed. Finally, the new variable future-time-steps method is applied to the heat flux estimation with and without ablation, and the new method is compared with the fixed future time step method in detail.

2. Direct and inverse heat conduction problems with variable geometry domain

To illustrate the variable future-time-steps method for solving the nonlinear inverse heat conduction problem, we take the one-dimensional heat conduction problem of a flat plate with variable geometry domain (as shown in Fig. 1) as an example for analysis. Suppose that the right side is insulated and the position is fixed; the left side is the heating surface, and its position changes with time due to some reasons (such as material pyrolysis and ablation [23]). The governing equation, the initial condition and boundary conditions are as follows:

$$\rho c_p(T)\frac{\partial T(x,t)}{\partial t}=\frac{\partial}{\partial x}\left[\lambda(T)\frac{\partial T(x,t)}{\partial x}\right]+\rho c_p(T)\dot{s}\frac{\partial T(x,t)}{\partial x},\ 0<x<L(t), \tag{1a}$$

$$T(x,0)=T_0(x),\ 0\le x\le L_0, \tag{1b}$$

$$-\lambda(T)\frac{\partial T}{\partial x}\bigg|_{x=L(t)}=0, \tag{1c}$$

$$-\lambda(T)\frac{\partial T}{\partial x}\bigg|_{x=0}=q(t)\ , \tag{1d}$$

where x represents the coordinate which origin always attached to the changing ablating surface, as shown in Fig.1; $T_0(x)$ is the initial temperature; $L(t)$ is the thickness of the plate, which can be recorded by the displacement measurement instrument; L_0 is the original thickness of the plate; λ, c_p are thermal conductivity and specific heat at constant pressure, respectively, and they are function of temperature; ρ is density, and it is constant; $\dot{s}=\mathrm{d}s/\mathrm{d}t$ and it indicates the ablating rate. The direct heat conduction problem considered here is concerned with calculating the temperature distribution of the ablator when the governing equation, boundary conditions, initial condition, thermophysical parameters and physical dimension are known. On the contrary, the inverse heat conduction problem is to estimate the unknown surface heat flux by additional temperature measurement information, ablation displacement information and Eqs. (1a) - (1c).

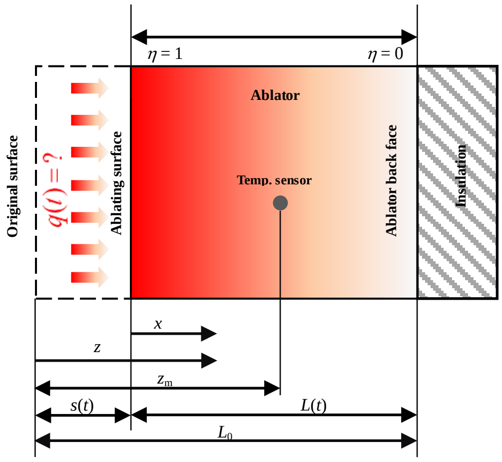


Fig. 1. Schematic of coordinate system for one-dimensional transient heat conduction slab with variable geometry domain.

3. Numerical solution of direct heat conduction problems with variable geometry domain

Because the space domain of numerical solution changes with time, a moving grid will be generated. In this paper, Landau coordinate system is used to deal with the variable geometry domain problem [24]. The Landau coordinate system makes the Landau coordinates of any node constant in the whole calculation process by defining new dimensionless coordinates. As shown in Fig.1, the origin of the z coordinate is in the original ablation surface, and the origin of the x coordinate is always attached to the ablating surface. Landau coordinates at any location are defined as:

$$\eta=\frac{L(t)-x}{L(t)}=\frac{L_0-z}{L_0-s(t)}, \tag{2}$$

where $s(t)$ is the surface recession. Therefore, for a given time step, any nodes have $\Delta z_i^{k+1}=z_i^{k+1}-z_i^k=\eta_i\dot{s}^{k+1}\Delta t$, and the relationship between the velocity of any nodes and the ablating rate, $v_i^{k+1}=\eta_i\dot{s}^{k+1}$, can be obtained. $\dot{s}^{k+1}=(s^{k+1}-s^k)/\Delta t$, which is the discrete format of the ablating rate, $\dot{s}$.

Because the direct heat conduction problem with variable geometry domain is nonlinear, it is linearized at each time step by using thermophysical properties and surface recession at previous time step. The implicit discrete format of Eq. (1) can be obtained by using the finite difference method [25]. The tridiagonal matrix algorithm (TDMA) is used to solve the algebraic equations, and the dynamic temperature distribution, T_i^k, can be obtained. $T_i^k=T(x_i,t_k)$; $x_i=(i-1)\Delta x$; $t_k=k\Delta t$; $\Delta x=L(t)/(I-1)$; I

is the number of nodes; Δt is size of time step; Δx_i is the width of the ith control element; $k = 1, 2, \ldots, K$; $i = 1, 2, \ldots, I$.

4. Sequential function specification method for inverse heat conduction problems with variable geometry domain

Sequential function specification method (SFSM) sequentially estimates the unknown heat flux in a step-by-step fashion. It first temporarily assumes that the heat flux components at r future time steps are equal, and then estimates the assumed heat flux by optimizing the least square objective function between the measured temperature and the calculated temperature at r future time steps. Finally, the temporarily assumed heat flux is brought into the direct heat transfer problem for correction. Repeat the above steps at the next time until the entire estimation process is completed.

SFSM temporarily assumes that the heat flux components at time t_k, t_{k+1}, …, t_{k+r^k-1} are equal, namely, $\hat{q}_{k+1}=\hat{q}_{k+2}=\cdots\hat{q}_{k+r^k-1}=\hat{q}_k$.

Define objective function $J(\hat{q}_k)$

$$J(\hat{q}_k)=\sum_{i=1}^{r^k}[Y^{k+i-1}-T^{k+i-1}(\hat{q}_k)]^2 , \tag{3}$$

where Y^{k+i-1} and $T^{k+i-1}(\hat{q}_k)$ are the measured temperature and calculated temperature at the measurement point and time t_{k+i-1}, respectively.

Set the guess value of the heat flux to be estimated q_k^0. The calculated temperature at the measurement point can be calculate by first order Taylor approximation at q_k^0.

$$T^{k+i-1}(\hat{q}_k)\approx\hat{T}^{k+i-1}(q_k^0)+Z^{k,i-1}(\hat{q}_k-q_k^0) , \tag{4}$$

where $Z^{k,i-1}$ is the step function sensitivity coefficient, which is calculated by the following formula：

$$Z^{k,i-1}=\frac{\partial T^{k+i-1}}{\partial\hat{q}_k}\approx\frac{T^{k+i-1}(q_k^0+\Delta q)-T^{k+i-1}(q_k^0)}{\Delta q} , \tag{5}$$

where Δq is a relatively small quantity. In this article, Δq=1×10^{-5} W/m^2 is used by the trial-and-error method.

Inserting Eq. (4) into Eq. (3) and setting $\partial J(\hat{q}_k)/\partial\hat{q}_k=0$ yields

$$\hat{q}_k=q_k^0+\sum_{i=1}^{r^k}\left[Y^{k+i-1}-\hat{T}^{k+i-1}(q_k^0)\right]Z^{k,i-1}\Bigg/\sum_{i=1}^{r^k}(Z^{k,i-1})^2 . \tag{6}$$

This is the classical sequential function specification method. Usually, r^k is a constant in this method, it is called the fixed future-time-steps method in this work. The optimal future time steps of fixed future-time-steps method, r_{opt}, is usually determined by the discrepancy principle or the criterion of minimum relative error of heat flux.

The idea of the discrepancy principle is that the root mean square error of the measured temperature and the reconstructed temperature should be consistent with the standard deviation of the measurement error. Therefore, r_{opt} can be determined according to the following formula:

$$r_{\text{opt}}=\min_{r\in\mathbf{N}^+}\left\{\sqrt{\frac{1}{K-1}\sum_{k=1}^{K}(\hat{T}_k(r)-Y_k)^2}\geq\sigma\right\}, \tag{7}$$

where $\hat{T}_k(r)$ is the reconstructed temperature at the measurement location based on the estimated heat flux, σ is the standard deviation of measurement temperature error.

The r_{opt} determined by the criterion of minimum relative error of heat flux is the future time steps corresponding to the minimum mean square error of the retrieved heat flux and the actual heat flow. It can be expressed as

$$r_{\mathrm{opt}} = \underset{r \in \mathbf{N}^+}{\arg\min} \left[\frac{1}{K-1} \sum_{k=1}^{K} (\hat{q}_k(r) - q_k)^2 \right]. \tag{8}$$

5. Sequential function specification method with variable future time steps

Different error statistical assumptions will lead to different estimation methods. Therefore, reasonable statistical assumptions must be made. The statistical assumptions about temperature measurement errors in this work are referred to Ref. [19].

In order to measure the estimation error, we use the mean squared error of the estimated heat flux to measure:

$$\mathscr{S}_k^2 = E[(\hat{q}_k - q_k)^2], \tag{9}$$

where q_k is the exact but usually unknown heat flux.

The mean squared error of heat flux retrieved by SFSM consists of deterministic error and random error [26]. Adding and subtracting the expected value of estimated heat flux on the right side of Eq. (9) gives:

$$\mathscr{S}_k^2 = E\left(\left\{ \left[\hat{q}_k - E(\hat{q}_k)\right] - \left[q_k - E(\hat{q}_k)\right] \right\}^2 \right) \tag{10a}$$

$$\mathscr{S}_k^2 = E\left\{ \left[\hat{q}_k - E(\hat{q}_k)\right]^2 \right\} - 2E\left\{ \left[\hat{q}_k - E(\hat{q}_k)\right]\left[q_k - E(\hat{q}_k)\right] \right\} + E\left\{ \left[q_k - E(\hat{q}_k)\right]^2 \right\}, \tag{10b}$$

where the first term on the right of Eq. (10b) is the variance of the estimator, $\hat{q}_k$,

$$V(\hat{q}_k) = E\left\{ \left[\hat{q}_k - E(\hat{q}_k)\right]^2 \right\} = E\left(\left\{ q_k^0 + \frac{\sum_{i=1}^{r^k} \left[Y^{k+i-1} - \hat{T}^{k+i-1}(q_k^0) \right] Z^{k,i-1}}{\sum_{i=1}^{r^k} (Z^{k,i-1})^2} - q_k^0 - \frac{\sum_{i=1}^{r^k} \left[E(Y^{k+i-1}) - \hat{T}^{k+i-1}(q_k^0) \right] Z^{k,i-1}}{\sum_{i=1}^{r^k} (Z^{k,i-1})^2} \right\}^2 \right) = E\left\{ \left[\frac{\sum_{i=1}^{r^k} \varepsilon^{k+i-1} Z^{k,i-1}}{\sum_{i=1}^{r^k} (Z^{k,i-1})^2} \right]^2 \right\}. \tag{11a}$$

Based on statistical assumptions in Ref. [19], the following formula can be derived.

$$V(\hat{q}_k) = \sigma^2 \sum_{i=1}^{r^k} \left[\frac{Z^{k,i-1}}{\sum_{i=1}^{r^k} (Z^{k,i-1})^2} \right]^2 = \sigma^2 \Big/ \sum_{i=1}^{r^k} (Z^{k,i-1})^2 . \tag{11b}$$

It can be seen from Eq. (11b) that the variation of sensitivity coefficients has a great influence on the variance of the estimator. If the sensitivity coefficients change dramatically with time, the inversion result will oscillate violently with time, even it will lead to unstable inversion results. Therefore, measures must be taken to avoid this situation.

The second term on the right side of Eq. (10b) can be proved to be equal to zero. $\hat{q}_k - E(\hat{q}_k)$ is not a random variable, therefore, $E\left[\hat{q}_k - E(\hat{q}_k)\right] = E(\hat{q}_k) - E(\hat{q}_k) = 0$.

The last term on the right side of Eq. (10b) is the square of a bias, and the outer expected value symbol can be dropped. It can be represented by $\mathscr{D}_k^2 = \left[q_k - E(\hat{q}_k)\right]^2$.

It can be seen from Ref. [27] that the heat flux estimated by the SFSM is the weighted average of the heat flux at r future time steps. Therefore, the square of the deterministic error can be expressed as:

$$\mathscr{D}_k^2 = \left[q_k - E\left(\sum_{i=1}^{r^k} w_{k,i-1} \tilde{q}_{k+i-1} \right) \right]^2 , \tag{12}$$

where $w_{k,i-1} = \sum_{j=1}^{r^k} (Z^{k,j-1} X^{k,j-i+1}) \Big/ \sum_{j=1}^{r^k} (Z^{k,j-1})^2$, $X^{k,j-i+1}$ is the pulse sensitivity coefficient. $\tilde{q}_{k+i-1}(i=1,2,...,r^k)$ are r^k future heat flux values corresponding to the exact matching of the measured temperature. $\sum_{i=1}^{r^k} w_{k,i-1} = 1$. It can be seen from Eq. (12) that for the case where the heat flux changes slowly with time, the change of weighting coefficients (or the variation of sensitivity coefficients) almost does not affect the deterministic error.

Therefore, the mean squared error of the estimator, $\hat{q}_k$, is composed of the square of the variance and the deterministic error, and their relationship is as follows:

$$\mathscr{S}_k^2 = V(\hat{q}_k) + \mathscr{D}_k^2 = \sigma^2 \Big/ \sum_{i=1}^{r^k} (Z^{k+i-1})^2 + \left[q_k - E\left(\sum_{i=1}^{r^k} w_{k,i-1} \tilde{q}_{k+i-1} \right) \right]^2 . \tag{13}$$

For the slowly changing heat flux, the variation of sensitivity coefficients mainly affects the variance of estimated heat flux, and has little influence on the deterministic error. Therefore, when the sensitivity coefficients change significantly, the variance mainly affects the inversion error. Effective means must be used to control the rapid change of variance, otherwise the inversion result will oscillate violently, and the inversion solution will be unstable.

In this paper, we attempt to overcome the instability of inversion result caused by the drastic variation of the sensitivity coefficients by adopting the variable future-time-steps method. The error amplification coefficient at the *k*th moment, A_k , is introduced, namely,

$$A^k = 1 \Big/ \sum_{i=1}^{r^k} (Z^{k,i-1})^2 . \tag{14}$$

The error amplification coefficient represents the degree to which the random error of temperature measurement in the inversion result is amplified at the *k*th time step. Its unit is $W^2 m^{-4} K^{-2}$. The larger A^k is, the larger the random error is amplified. If A^k oscillates violently with time, the temperature measurement error will be amplified differently at different times, and the inversion results will oscillate violently in time domain. Therefore, the occurrence of such oscillation must be reasonably limited. The minimum error amplification coefficient under the optimal future time steps is defined as the benchmark error amplification coefficient:

$$A_b = \min_{k=1}^{K} A_{opt}^k = 1 \Big/ \max_{k=1}^{K} \left[\sum_{i=1}^{r_{opt}} (Z^{k,i-1})^2 \right] , \tag{15}$$

where A_{opt}^k denotes the error amplification coefficient under the optimal future time steps, r_{opt} . A_b represents the minimum limit of the amplification of the measurement random error in the inversion results under the condition that the deterministic error is not over amplified. Therefore, we use it to adjust the number of future time steps.

In the calculation, the benchmark error amplification coefficient must be found first. Then, by adjusting the number of future time steps, r^k , the error of inversion results in the time with small sensitivity coefficient is roughly consistent with that in the time with big sensitivity coefficient. The adjustment of the number of future time steps is usually to increase the number of future time steps, so as to make inversion results more stable in the time with small sensitivity coefficient. By changing the number of future time steps, it can effectively avoid the unreasonable situation that the inversion error oscillates violently in the whole time domain.

6. Estimation process of sequential function specification method with variable future time steps

The calculation flowchart of the SFSM with variable future time steps for solving the nonlinear inverse heat conduction problem is shown in Fig. 2. The calculation process includes the calculation of the benchmark error amplification coefficient and the estimation process of variable future-time-steps method. It can be seen from the calculation flowchart that compared with the classical SFSM, the new method needs to determine the benchmark error amplification coefficient in advance. However, the determination of the benchmark error amplification coefficient in advance requires the estimation of the boundary heat flux in the whole time domain. Overall, the new method is a whole-domain inverse method. Considering that the future time steps of the new method at each time step are adjusted based on the benchmark error amplification coefficient which needs the information in whole time domain, the new method has an insight of whole evolution process.

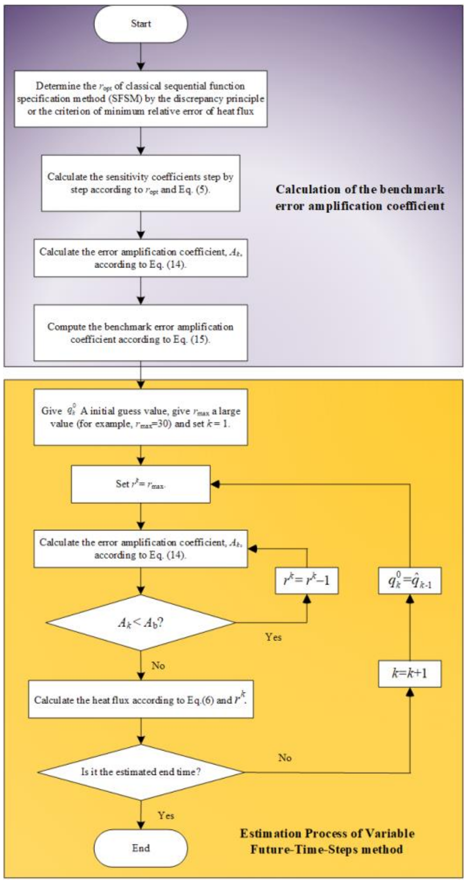


Fig. 2 Flowchart of the sequential function specification method with variable future time steps.

Another notable feature of the new method is that although the new method adds an inner loop on the basis of the classical SFSM (see the estimation process of variable future-time-steps method in Fig. 2), since the inner loop does not involve the repeated solution of the direct and inverse problems, the amount of inverse calculation has hardly increased. The new variable future-time-steps method has high computational efficiency.

7. Numerical tests and discussions

7.1 Solution conditions of inverse problem

In this section, we attempt to verify the effectiveness of the variable future-time-steps method by numerical simulation. Numerical experiments are carried out on a one-dimensional carbon-carbon composite plate which has the initial thickness L_0=0.05 m and initial temperature $T_0(x)$ =100 K. The thermophysical properties of carbon-carbon composites are shown in Fig. 3. The density of carbon-carbon composites is 1900 kg/m^3. In numerical simulation, the number of nodes I = 21, the size of time step Δt = 0.1 s.

In practical engineering applications, due to human or non-human factors, there are inevitable measurement errors in the actual temperature (and ablation surface position) measurement results. In numerical simulation, the actual measurement temperature or the measurement position of the ablation surface are usually simulated by the virtual measurement temperature or location of the ablation surface, that is, they are generated by exact values plus random disturbances. They can be expressed by the following formula:

$$M = M_{\text{exa}} + \sigma\omega\,, \tag{16}$$

where M_{exa} and M represent exact measurement and "actual" measurement, respectively. σ is the standard deviation of the measurement error, and ω is a random number of the standard normal distribution with zero mean and unit standard deviation in the confidence interval [–2.576, 2.576] for a 99% confidence coefficient.

To facilitate the comparison of inversion results, the relative error of estimated heat flux, S_{q}, is defined as $S_{\text{q}} = \int_0^{t_{\text{f}}} [q(t) - \hat{q}(t)]^2 \text{d}t \Big/ \int_0^{t_{\text{f}}} q(t)^2 \text{d}t \times 100\%$.

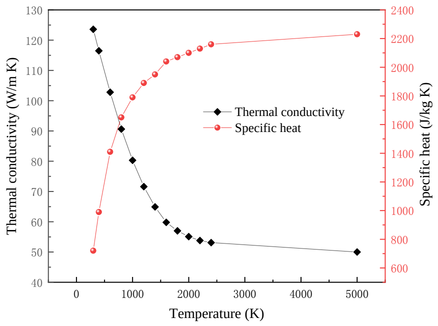


Fig. 3 Thermophysical properties of carbon-carbon composites.

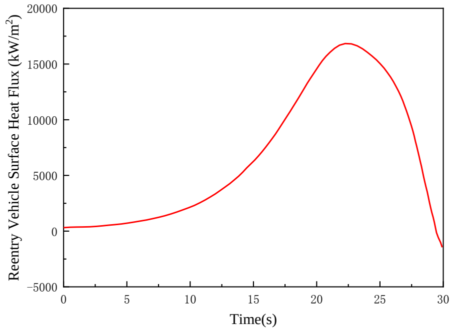


Fig.4 Typical reentry vehicle surface heat flux [25; 28]**.**

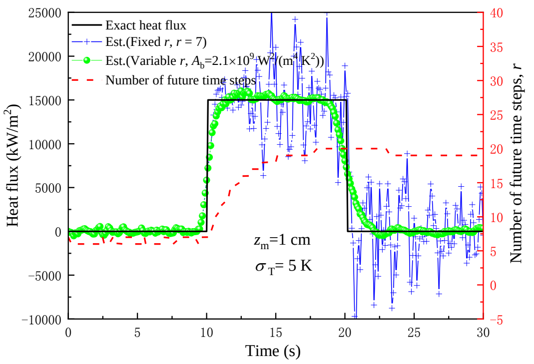
7.2 Estimation of heat flux with temperature-dependent physical properties and no ablation

In this section, the inversion results of rectangular wave, triangular wave and typical reentry vehicle surface heat flux [25; 28] retrieved by variable future-time-steps method and traditional fixed future-time-steps method are compared through numerical tests. In this section, it is assumed that the material is highly heat-resistant, the change of sensitivity coefficients caused by surface ablation is not considered temporarily, that is, the influence of sensitivity coefficients caused by the change of thermophysical properties on the inversion result is only considered. The surface heat flux of a typical reentry vehicle is shown in Fig.4.

In this section, the number of future time steps in traditional fixed future-time-steps method is determined by the discrepancy principle, and the determination method is referred to Ref. [19]. The benchmark error amplification coefficient adopts the error amplification coefficient at the first moment.

For the above three different heat flux forms, the inversion results of variable future-time-steps method and traditional fixed future-time-steps method under different measurement point positions and measurement errors are compared and shown in Tab.1. Figs. 5 - 7 shows the inversion results of three different heat flux forms retrieved by the two methods when the temperature measurement error σ_{T} = 5 K and the measurement point location z_{m}=1 cm. The dotted line in the figures is the corresponding future time steps when the future time steps are variable.

Tab. 1 and Figs. 5 - 7 show that the variable future-time-steps method is significantly better than the fixed future-time-steps method. For z_{m}=1 cm, the estimation errors of variable future-time-steps method are 0.73 ~ 3.8 times lower than that of the fixed future-time-steps method. For z_{m}=1.5 cm, the estimation errors of variable future-time-steps method are 3.1 ~ 37.0 times lower than that of the fixed future-time-steps method. The heat flux estimation error of fixed future-time-steps method in the initial phase is small, but the heat flux estimation error in the final phase is too large. The variable future-time-steps method uses small future time steps in the initial phase and large future time steps in the final phase, so that the heat flux estimation error is small in the entire time domain. The reason can be observed through the variation rule of measurement error amplification coefficient with time. Fig. 8 shows the variation of measurement error amplification coefficient with time during rectangular wave estimation when z_m=1 cm. When the number of future time steps are equal to 1, the error amplification coefficient is equal to the reciprocal of the square of the sensitivity coefficient. Since the sensitivity coefficient decreases with time, the error amplification coefficient increases with time. When the measurement errors are the same, the error amplification is small in the initial phase, and large in the final phase, using the fixed future-time-steps method will cause the inversion results to be unstable in the time domain, and will lead to excessive relative error of estimated heat flux. It can also be seen from Fig. 10 that the measurement error amplification coefficient decreases with the increase of future time steps. This feature shows that the error amplification coefficient can be reduced by increasing the future time steps to overcome the instability of the solution caused by the variation of sensitivity coefficients. The solid line in Fig.8 shows when the benchmark error amplification coefficient A_b=2.10×10^9 W^{2}m^{-4} K^{-2}, the variation of error amplification coefficient with time in variable future-time-steps method. The observation shows that the error amplification coefficient line is basically stable at different time, that is, the error amplification coefficient is approximately equal to a constant. Therefore, the variable future-time-steps method can overcome the instability of the solution caused by the variation of sensitivity coefficients. The inversion result of rectangular wave in Fig. 5 also confirms the above conclusions. In addition, it should be noted that if the too large future time steps in variable future-time-steps method will increase the error of the downward step phase, that is, it increases the deterministic error of estimated heat flux, but it brings greater stability benefits.



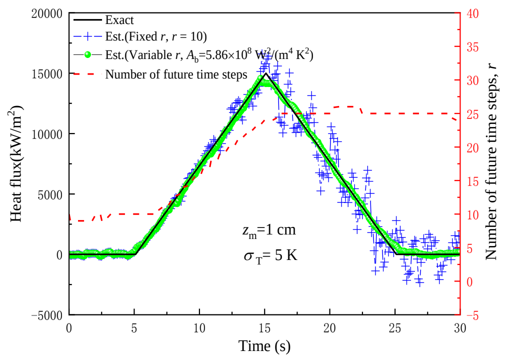
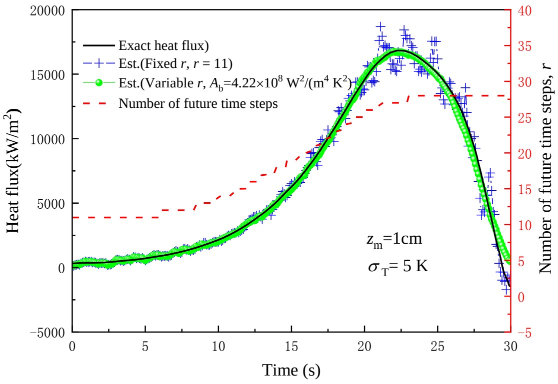


Fig. 5 Estimated rectangular heat flux by variable and fixed future-time-steps method.

Fig. 6 Estimated triangular heat flux by variable and fixed future-time-steps method.



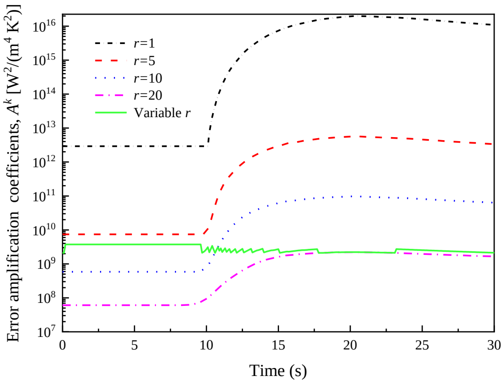


Fig. 7 Estimated typical reentry vehicles surface heat flux by variable and fixed future-time-steps method.

Fig. 8 Error amplification coefficients during rectangular heat flux estimation under different future time steps.

From Tab. 1, it can also be observed that the estimation results of triangular and typical reentry vehicles surface heat flux have smaller errors compared with rectangular heat flux, and they are easier to be estimated, regardless of the variable or fixed future-time-steps method. This is because the rectangular heat flux contains more high-frequency signals. Under the same conditions, the greater the measurement error, the greater the inversion error. Under the same conditions, the farther the measurement point is, the greater the inversion error is. Because the farther the measurement point is, the smaller the sensitivity coefficients are, and the greater the error amplification effect is.

Tab.1 Comparison of inversion results of variable and fixed future-time-steps method.

Item		σ=5 K			σ=3 K		
		q_{squ}	q_{tri}	q_{re}	q_{squ}	q_{tri}	q_{re}
z_m=1cm	S_q of variable r (%)	17.59	3.01	5.06	14.67	2.45	3.80
	S_q of fixed r(%)	41.13	14.08	9.44	40.21	11.75	6.58
	r for fixed r	7	10	11	6	8	10
z_m=1.5cm	S_q of variable r(%)	21.02	4.11	6.39	19.88	3.29	5.26
	S_q of fixed r(%)	171.54	156.30	40.32	121.04	30.67	21.50
	r for fixed r	9	12	14	8	10	12

7.3 Estimation of heat flux with temperature-dependent physical properties and ablation

In this section, the variable future-time-steps method is applied to the heat flux estimation of composite materials with ablation. This case is taken from Ref. [28]. The problem solved in this case is basically the same as that in Section 7.2, except that the surface ablation recession shown in Fig.9 is generated on the surface of the ablator due to the thermochemical effect. The temperature thermocouple is arranged at the position where z_m = 1 cm, and the ablation surface position is recorded by the position sensor. Because both temperature and position measurements have measurement errors. Therefore, we will discuss three cases: the temperature measurement error is 5 K, and the position measurement error is 0 mm (Case #1); the temperature measurement error is 0 K, and the position measurement error is 3.88×10^{-5} mm, as shown in Fig.9 (Case #2); the temperature measurement error is 5 K, and the position measurement error is 3.88×10^{-5} mm (Case #3).

Because the discrepancy principle cannot be used in the case with position measurement error. Therefore, the criterion of minimum relative error of heat flux is adopted in this section to determine the number of future time steps in the fixed future-time-steps method. For Case 1#, both the discrepancy principle and the criterion of minimum relative error of heat flux are adopted. The benchmark error amplification coefficient is also determined by the criterion of minimum relative error of heat flux.

Tab.2 lists the relative errors of estimated heat flux by the variable and fixed future-time-steps method for the above three cases. The second row of Tab. 2 lists the benchmark error amplification coefficient in variable future-time-steps method; the fourth row of Tab. 2 lists the number of future time steps in fixed future-time-steps method. Fig. 10 shows the inversion results of fixed and variable future-time-steps method for Case #1 based on the discrepancy principle. It can be seen from the Fig. 10 that the inversion error of the new method is obviously smaller than that of the traditional fixed future-time-steps method, whose future time steps are determined through the discrepancy principle. The inversion error of the new method is 1.9 times smaller than that of the traditional fixed future-time-steps method in this case.

Figs. 11 and 12 show the inversion results of fixed and variable future-time-steps method based on the criterion of minimum relative error of heat flux for Cases # 2 and # 3, respectively. By comparing the relative error of heat flux of the two methods, it can be seen that when the hyperparameters, A_b and r, are determined by using the criterion of minimum relative error of heat flux, the inversion result of the variable future-time-steps method is slightly better than that of fixed future-time-steps method.

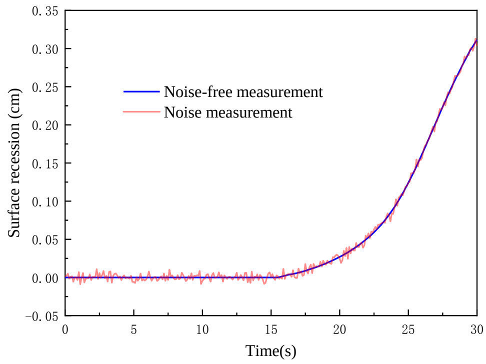


Fig. 9 Variation of surface ablation recession with time without error and with error.

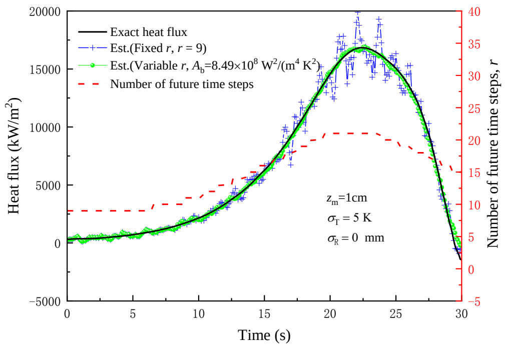


Fig. 10 Inversion results of variable and fixed future-time-steps method based on the discrepancy principle for Case #1.

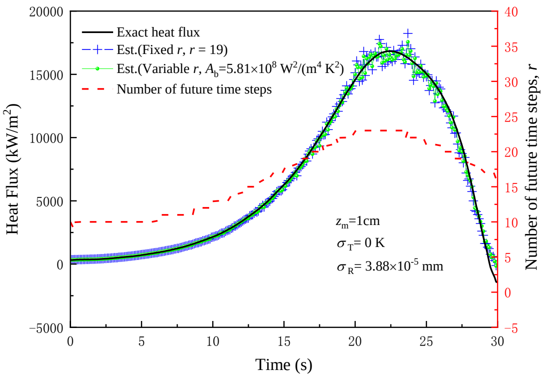


Fig. 11 Inversion results of variable and fixed future-time-steps method based on the criterion of minimum relative error of heat flux for Case #2.

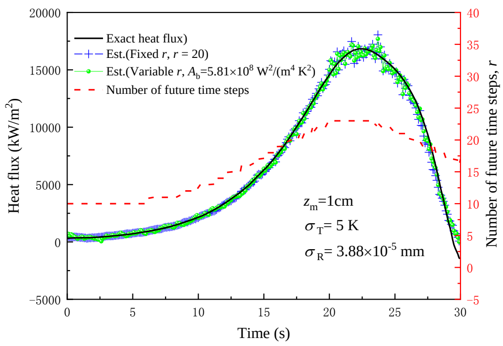


Fig. 12 Inversion results of variable and fixed future-time-steps method based on the criterion of minimum relative error of heat flux for Case #3.

Table 2 Comparison of inversion results of variable and fixed future-time-steps method.

Item	Noisy temperature measurement (#1)		Noisy position Measurement (#2)	Full noisy data (#3)
	Discrepancy principle	Minimize S_q		
S_q of variable r (%)	2.99		4.20	4.47
A_b [$W^2/(m^4 K^2)$]	8.49×10^8		5.81×10^8	5.81×10^8
S_q of fixed r (%)	8.75	3.00	5.00	5.14
r for fixed r	9	15	19	20

The new method is slightly better than the traditional fixed future-time-steps method, whose future time steps are determined through the criterion of minimum relative error of heat flux. There are two main reasons: First, the new method adopts the variable future-time-steps scheme, so that the error is more evenly distributed in the time domain. A small number of future time steps are adopted in the time interval with large sensitivity coefficients to reduce the deterministic error; in the time interval with small sensitivity coefficients, a large number of future time steps are used to reduce the variance. Therefore, the total error of estimated heat flux is smaller. Second, when the traditional method determine the number of future time steps, the hyperparameters, r, can only be an integer. The benchmark error amplification coefficient of the new method can be a continuous real number. Therefore, the new method can tune the inversion results more finely.

In addition, it can also be observed from Tab.2 that the error of inversion results with only position measurement error (Case #2) is greater than that with only temperature measurement error (Case #1). The error of inversion results (Case #3) is maximum when there are position measurement error and temperature measurement error.

8. Conclusions

For a class of nonlinear unsteady inverse heat conduction problems that the inversion solution oscillates violently in the whole-time domain due to the drastic change of the sensitivity coefficients, a new sequential function specification method with variable future-time-steps is proposed in this paper. The future time steps are adjusted by the error amplification coefficients which are defined as the reciprocal of the square sum of the sensitivity coefficients in the article. When the error amplification coefficients are small, a small number of future time steps is used to reduce the deterministic error; while in the period with large error amplification coefficient, a large number of future time steps is used to reduce stochastic error. Finally, the total error of estimated heat flux is reduced. Avoid the sharp fluctuation of estimated heat flux in time domain due to the drastic change of sensitivity coefficients.

The variable future-time-steps method is applied to the estimation of heat flux of nonlinear unsteady inverse heat conduction problems without ablation and with ablation. Numerical experiments show that the new method is obviously superior to the fixed-future-time steps method based on the discrepancy principle. The new method is slightly better than fixed-future-time steps method based on the criterion of minimum relative error of heat flux. This is due to the strategy of variable future-time-steps and the continuously adjustable characteristic of benchmark error amplification coefficient.

The variable future-time-steps method proposed in this article has the potential to be applied to the temporal-spatial distribution parameter estimation topics, which can be further studied in the future.

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