Beyond Laplace and Fourier Transforms: Challenges and Future Prospects

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Abstract: Laplace and Fourier transforms are widely used independently in engineering for linear differential equations including fractional differential equations. Here we introduce a generalized integral transform, which is a generalization of the Fourier transform, Laplace transform and other transforms, e.g., Elzaki Sumudu transform, Aboodh transform, Pourreza transform and Mohand transform, making the new transform much attractive and promising. Its basic properties are elucidated, and its applications to initial value problems and integral equations are illustrated, when coupled with the homotopy perturbation, it can be used for various nonlinear problems, opening a new window for nonlinear science.

Keywords: Laplace transform, Fourier transform, Initial value problems, System of ordinary differential equations, Volterra integral equations

1. Introduction

Researchers have developed several mathematical methods that are being employed in numerous fields of science, technology, and engineering in order to better understand nature. Particularly, the idea of integral transformation was put forth and has since been discovered to be a practical mathematical tool for addressing a variety of issues in both pure and applied mathematics [1-3]. It is important to recall that a mathematical operator is referred to as an integral transform if it transfers a function by means of an integral from its original function to another function space. Open literature demonstrates that there are numerous probability applications that are related to integral transformations, such as the price kernel, also known as the stochastic discount factor [4]. The application of these mathematical operators in control theory [5] is another significant area.

Since roughly 200 years ago, integral transforms were appeared in literature, among which Fourier and Laplace transforms are the most famous ones. Other than the Laplace transform, a number of alternative integrals have been proposed in recent years and have been discovered to share certain intriguing Laplace transform-like features. Elzaki transform [6], Sumudu transform [7], Aboodh transform [8], Natural transform [9], Mohand transform [10], Pourreza transform [11], Kamal transform [12], Sawi transform [13] and Emad–Sara transform

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are in the list. These transforms have been crucial in resolving differential equations of both integer and non-integer orders. Coupled with the analytical methods like the homotopy perturbation method [15] or the variational iteration method [16-17], these transforms can be extended to nonlinear problems and fractal/fractional equations [18-24].

Jafari [25] introduced a generalized transform and deduced that every integral transform in the class of Laplace transform is actually a special case of its generalized integral transform, however, this transform didn’t preserve the properties of Fourier transform. Recently, Khan and Khalid [26] proposed the Fareeha transform, which, however, doesn’t contain many integral transforms falling under the Laplace transform category. In this article, we are being proposed a new generalized integral transform that remove the aforementioned disadvantages. The suggested integral transform not only includes many integral transforms falling under the Laplace transform category but also holds the properties of the Fourier transform as the special case. This unification offers a totally new window for wide applications.

2. Background

The background of integral transforms can be traced back to the development of integral calculus and the need to solve complex problems involving differential equations and other mathematical operations [27]. Generally, an integral transform of an input function \( f(x) \) defined in \( a \leq x \leq b \) can be expressed as:

\[
I \{ f(x) \} = F(k) = \int_{a}^{b} K(x,k)f(x)dx
\]

where \( K(x,k) \) is the kernel of the transformation, \( I \) is the integral transform operator, \( F(k) \) is the image of \( f(x) \), and \( k \) is the transform variable. In order to find \( f(x) \) from given \( F(k) \), we introduce the inverse operator \( I^{-1} \), such that

\[
I^{-1}\{ F(k) \} = f(x)
\]

Based on Eq. (1), the Fourier and the Laplace transforms of a function can be written, respectively, as:

\[
F \{ f(x) \} = F(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx
\]

and

\[
L \{ f(t) \} = F(s) = \int_{0}^{\infty} e^{-st} f(t)dt.
\]

Table 1 lists a few integral transformations from the category of the Laplace transform. Different types of integral equations, as well as ordinary, partial and fractional equations, have all been solved using these transformations [28-31]. Additionally, these types of transforms have been used in conjunction with other semi-analytical techniques, including the homotopy perturbation method, the variational iteration method, Adomian decomposition, differential transform methods, to solve a variety of ordinary, partial and fractional equations [32-36]. Numerous applications of the integral transformations mentioned in Table 1 can be found in science and engineering, including solitary waves, mechanics, finance, economics, and chemistry.
Table 1: Integral transforms from the class of Laplace transform

<table>
<thead>
<tr>
<th>Integral formula</th>
<th>Transform name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S{f(t)} = \frac{1}{s} \int_0^\infty e^{-\frac{t}{s}} f(t) dt )</td>
<td>Sumudu transform [7,37]</td>
</tr>
<tr>
<td>( A{f(t)} = \frac{1}{s} \int_0^\infty e^{-st} f(t) dt )</td>
<td>Aboodh transform [8]</td>
</tr>
<tr>
<td>( N{f(t)} = s \int_0^\infty e^{-ut} f(ut) dt )</td>
<td>Natural transform [9, 38]</td>
</tr>
<tr>
<td>( P{f(t)} = s \int_0^\infty e^{-\frac{t}{s^2}} f(ut) dt )</td>
<td>Pourreza transform [11]</td>
</tr>
<tr>
<td>( E{f(t)} = s \int_0^\infty e^{-s^2t} f(t) dt )</td>
<td>Elzaki transform [6]</td>
</tr>
<tr>
<td>( M{f(t)} = s^2 \int_0^\infty e^{-ut} f(t) dt )</td>
<td>Mohand transform [10]</td>
</tr>
<tr>
<td>( Sa{f(t)} = \frac{1}{s^2} \int_0^\infty e^{-\frac{t}{s^2}} f(t) dt )</td>
<td>Sawi transform [13]</td>
</tr>
<tr>
<td>( K{f(t)} = \int_0^\infty e^{-s^2t} f(t) dt )</td>
<td>Kamal transform [12]</td>
</tr>
<tr>
<td>( ES{f(t)} = \frac{1}{s^3} \int_0^\infty e^{-s^2t} f(t) dt )</td>
<td>Emad–Sara transform [14]</td>
</tr>
<tr>
<td>( EF{f(t)} = \frac{1}{s} \int_0^\infty e^{-s^2t} f(t) dt )</td>
<td>Emad–Falih transform [39]</td>
</tr>
</tbody>
</table>

Yang et al. [19] investigated the 1-D fractal heat-conduction problem in a fractal semi-infinite bar with local fractional calculus and the Yang-Fourier transform approach, the outcome demonstrates the correctness and dependability of the results. Nazari-Golshan, et al. [18] examined a method by introducing He's polynomials into the homotopy perturbation method coupled with the Fourier transform for the Lane-Emden problem. He and Zhang [6] proposed an iterative transformation technique that combines the Elzaki transform and iterative approaches to resolve fractional order linear Klein-Gordon and Hirota-Satsuma-linked KdV equations. Manimegalai et al. [8] explored the Aboodh transform-based homotopy perturbation method to solve a generalized oscillatory differential equation and concluded that the coupling gave much better results than many existed methods. Akgul et al. [37] investigated a few alternative financial/economic theories based on market equilibrium and option pricing using three different fractional derivatives, and obtained the fundamental solutions of the models using the Sumudu transform and the Laplace transform. Ahmadi, et al. [11] studied the Pourreza integral transform, which is useful for solving both Laguerre and Hermite differential equations used in quantum mechanics. Nadeem, et al. [10] employed the Mohand transform with the
homotopy perturbation method for the fractional order Newell-Whitehead-Segel equation. Higazy and Aggarwal [13] used the Sawi transformation to solve a system of ordinary differential equations to calculate the concentration of chemical reactants in a series of chemical reactions.

3. The proposed generalized integral transform

This section is devoted to present the generalized integral transform that includes many integral transforms falling under the Laplace transform category and the properties of the Fourier transform as the special case.

3.1 Definition: Let \( f(t) \) be an integrable function defined for \( t \geq 0, p(s) \neq 0 \) and \( s \) is from the complex domain, i.e., \( s = x + iy \); we define the generalized integral transform \( H(s) \) of \( f(t) \) by the formula

\[
H\{f(t)\} = H(s) = p(s) \int_0^\infty e^{-st} f(t) dt,
\]

presuming that the integral exists for some \( s^n \) where \( n \in \mathbb{Z} \). Table 2 displays the generalized integral transform of some basic functions.

Table 2: New generalized integral transform of some elementary functions

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( H(s) )</th>
<th>( \frac{p(s)}{s^n} )</th>
<th>( \frac{p(s)}{s^{2n}} )</th>
<th>( \frac{p(s)(\alpha + 1)}{s^{\alpha+1}} )</th>
<th>( \frac{p(s)}{s^n - a} )</th>
<th>( \frac{bp(s)}{s^{2n} + b^2} )</th>
<th>( \frac{p(s)s^n}{s^{2n} + b^2} )</th>
<th>( \frac{bp(s)}{s^{2n} - b^2} )</th>
<th>( \frac{p(s)s^n}{s^{2n} - b^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(t) )</td>
<td>( 1 )</td>
<td>( t )</td>
<td>( t^\alpha, \alpha &gt; 0 )</td>
<td>( e^{at} )</td>
<td>( \sin bt )</td>
<td>( \cos bt )</td>
<td>( \sinh bt )</td>
<td>( \cosh bt )</td>
<td></td>
</tr>
</tbody>
</table>

3.2 Theorem 1 (Existence Theorem): Let \( f(t) \) is a piecewise continuous function of exponential order for all \( t \geq 0 \), then \( H(s) \) exists for all \( s^n > k \).

Proof: Given that \( f(t) \) is a piecewise continuous function of exponential order so it satisfies \( |f(t)| \leq Me^{kt} \), where \( M \) is positive constant and \( k \) is the order of the function. Since

\[
|H(s)| = |H\{f(t)\}| = \left| p(s) \int_0^\infty f(t)e^{-st} dt \right| \\
\leq p(s) \int_0^\infty |f(t)|e^{-st} dt \leq p(s) \int_0^\infty Me^{kt}e^{-st} dt.
\]

Equivalently

\[
|H(s)| \leq p(s) M \int_0^\infty e^{-(s^n - k)t} dt = \frac{p(s)M}{s^n - k}, \quad s^n > k.
\]

Thus the statement is correct.

3.3 Theorem 2 (Linearity Theorem): For any constants \( \beta \) and \( \gamma \) and any two functions \( f(t) \) and \( g(t) \) whose transforms exist individually, \( H \) satisfies:

\[
H\{\beta f(t) + \gamma g(t)\} = \beta H\{f(t)\} + \gamma H\{g(t)\}.
\]

Proof: By applying the definition (5), we have
\[ H\{\beta f(t) + \gamma g(t)\} = p(s) \int_0^\infty e^{-st} [\beta f(t) + \gamma g(t)] \, dt \]

\[ = \beta p(s) \int_0^\infty e^{-st} f(t) \, dt + \gamma p(s) \int_0^\infty e^{-st} g(t) \, dt = \beta H\{f(t)\} + \gamma H\{g(t)\}. \]

Hence it is proved.

### 3.4 Theorem 3 (Differentiation Theorem):

Let \( f(t) \) is differentiable for \( t \geq 0, p(s) \neq 0 \) and \( s \) is from the complex domain, i.e., \( s = x + iy \), then

a) \[ H\{f'(t)\} = sH\{f(t)\} - p(s)f(0), \quad (8) \]

b) \[ H\{f''(t)\} = s^2H\{f(t)\} - s p(s)f(0) - p(s)f'(0), \quad (9) \]

c) \[ H\{f^{(m)}(t)\} = s^mH\{f(t)\} - p(s) \sum_{k=0}^{m-1} s^{m-n-k} f^{(k)}(0). \quad (10) \]

**Proof.** (a). Using the definition (5), we have

\[ H\{f'(t)\} = p(s) \int_0^\infty e^{-st} f'(t) \, dt = p(s) \left[ e^{-st} f(t) \bigg|_0^\infty + s \int_0^\infty e^{-st} f(t) \, dt \right] \]

\[ = H\{f'(t)\} = p(s)H\{f(t)\} - p(s)f(0). \]

To proof (b), we assume \( h(t) = f'(t) \) so \( f''(t) = h'(t) \), so we have

\[ H\{h'(t)\} = p(s) \int_0^\infty e^{-st} h'(t) \, dt = sH\{h(t)\} - p(s)h(0) = sH\{f'(t)\} - p(s)f'(0) \]

\[ H\{h(t)\} = s^2H\{f(t)\} - s p(s)f(0) - p(s)f'(0). \]

By principle of mathematical induction, we can prove (c).

### 3.5 Theorem 4 (Convolution Theorem):

Let \( H\{f_1(t)\} = H_1(s) \) and \( H\{f_2(t)\} = H_2(s) \) then

\[ H\{f_1(t) * f_2(t)\} = H_1(s)H_2(s), \quad (11) \]

where \( f_1(t) * f_2(t) \) is called the convolution of \( f_1(t) \) and \( f_2(t) \) and is expressed as:

\[ f_1(t) * f_2(t) = \int_0^t f_1(t - \tau) f_2(\tau) \, d\tau. \]

**Proof.** Using the definition (5), we have

\[ H\{f_1(t) * f_2(t)\} = p(s) \int_0^\infty e^{-st} \left[ \int_0^t f_1(t - \tau) f_2(\tau) \, d\tau \right] \, dt \]

\[ = p(s) \int_0^\infty e^{-st} dt \int_0^t f_1(t - \tau) f_2(\tau) \, d\tau. \]

By changing the order of integration, we reach at:

\[ H\{f_1(t) * f_2(t)\} = p(s) \int_0^\infty f_2(\tau) d\tau \int_0^\infty e^{-st} f_1(t - \tau) dt. \]

By the change of variable \( t - \tau = u \), above equation leads to
\[
H\{f_1(t) \ast f_2(t)\} = p(s) \int_0^\infty f_2(\tau) d\tau \int_0^\infty e^{-s(\tau+\tau)} f_1(u) du \\
= p(s) \int_0^\infty e^{-s\tau} f_2(\tau) d\tau \int_0^\infty e^{-su} f_1(u) du = \frac{1}{p(s)} H_1(s) H_2(s). 
\]

This completes the proof.
Similarly, we can prove the first shifting theorem, second shifting theorem, scaling property and other concepts for this new generalized transform.

4. Analysis of Proposed Generalized Integral Transform

As discussed earlier that the proposed integral transform not only encompasses various transforms that belong to the Laplace transform category, but also exhibits the properties of the Fourier transform in specific instances. For \( p(s) = 1 \) and \( s = x + iy \), Eq. (5) becomes the Fareeha transform [27] that includes all the properties of Laplace and Fourier transform, and for positive real value of \( s^n \) where \( n \in \mathbb{Z} \), we can obtain an integral transform belonging to the Laplace transform class. Table 3 discusses about the integral transformations from the category of Laplace transform for various values of \( p(s) \) and \( n \).

Table 3: Integral transforms belongs to the class of Laplace transform for various values of \( p(s) \) and \( n \).

<table>
<thead>
<tr>
<th>( p(s) )</th>
<th>( n )</th>
<th>Integral formula</th>
<th>Transform name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>( \int_0^\infty e^{-st} f(t) dt )</td>
<td>Laplace transform [40,41]</td>
</tr>
<tr>
<td>( \frac{1}{s} )</td>
<td>-1</td>
<td>( \frac{1}{s} \int_0^\infty e^{-\frac{t}{s}} f(t) dt )</td>
<td>Sumudu transform [7,37]</td>
</tr>
<tr>
<td>( \frac{1}{s} )</td>
<td>1</td>
<td>( \frac{1}{s} \int_0^\infty e^{-st} f(t) dt )</td>
<td>Aboodh transform [8]</td>
</tr>
<tr>
<td>( s )</td>
<td>1</td>
<td>( s \int_0^\infty e^{-st} f(ut) dt )</td>
<td>Natural transform [9, 38]</td>
</tr>
<tr>
<td>( s )</td>
<td>2</td>
<td>( s \int_0^\infty e^{-st} f(ut) dt )</td>
<td>Pourreza transform [11]</td>
</tr>
<tr>
<td>( s )</td>
<td>-1</td>
<td>( s \int_0^\infty e^{-st} f(t) dt )</td>
<td>Elzaki transform [6]</td>
</tr>
<tr>
<td>( s^2 )</td>
<td>1</td>
<td>( s^2 \int_0^\infty e^{-st} f(t) dt )</td>
<td>Mohand transform [10]</td>
</tr>
<tr>
<td>( \frac{1}{s^2} )</td>
<td>-1</td>
<td>( \frac{1}{s^2} \int_0^\infty e^{-st} f(t) dt )</td>
<td>Sawi transform [13]</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>( \int_0^\infty e^{-st} f(t) dt )</td>
<td>Kamal transform [12]</td>
</tr>
</tbody>
</table>
\[
\begin{array}{|c|c|c|}
\hline
\frac{1}{s^2} & 1 & \frac{1}{s^2} \int_0^\infty e^{-st} f(t) \, dt \\
\frac{1}{s} & 2 & \frac{1}{s} \int_0^\infty e^{-st} f(t) \, dt \\
\hline
\end{array}
\]

Emad–Sara transform [14]

Emad–Falih transform [39]

The convergence of the generalized integral transform of a function \( f(t) \) depends on three factors. \( f(t) \) must be continuous, bounded by an exponential function and absolutely integrable over the real line. The generalized transform of a function may diverge and in this case it cannot be computed using the standard techniques.

5. Applications

This section is devoted to present methodologies stem from generalized integral transform to solve initial value problems (IVPs), system of first-order differential equations and Volterra integral equations.

5.1. Solving initial value problem by new generalized transform

Consider the general form of an IVP as:

\[
z^{(m)}(t) + a_i z^{(m-1)}(t) + \ldots + a_m z(t) = g(t),
\]

\[
z(0) = z_0, z'(0) = z_1, \ldots, z^{(m-1)}(0) = z_{m-1}.
\]

Now we employ new generalized integral transform to each side of Eq. (11), then apply linearity and differentiation theorems, we have

\[
H\left\{z^{(m)}(t) + a_i z^{(m-1)}(t) + \ldots + a_m z(t)\right\} = H\{g(t)\},
\]

\[
H\left\{z^{(m)}(t) + a_i H\{z^{(m-1)}(t)\} + \ldots + H\{a_m z(t)\}\right\} = H\{g(t)\}.
\]

\[
s^{m-n}H(s) - p(s) \sum_{k=0}^{m-1} s^{m-n-k} z^{(k)}(0) + a_i s^{m-n}H(s) - p(s) \sum_{k=0}^{m-2} s^{m-2n-k} z^{(k)}(0) + \ldots + a_m H(s) = G(s).
\]

where \( G(s) = H\{g(t)\} \). By applying the initial conditions in Eq. (11) we have

\[
h(s)H(s) = G(s) + \Psi(s),
\]

where \( h(s) = \left(s^{m-n} + a_i s^{m-n} + \ldots + a_m\right) \) and

\[
\Psi(s) = p(s) \left( \sum_{k=0}^{m-1} s^{m-n-k} z_k + a_i \sum_{k=0}^{m-2} s^{m-2n-k} z_k + \ldots + z_0\right).
\]

From Eq. (14), we find \( H(s) \) as

\[
H(s) = \frac{G(s)}{h(s)} + \frac{\Psi(s)}{h(s)}.
\]

Lastly, we apply inverse generalized transform on each side of above equation to get the solution as:

\[
z(t) = H^{-1}\left[\frac{G(s)}{h(s)}\right] + H^{-1}\left[\frac{\Psi(s)}{h(s)}\right].
\]
Two examples will be solved by utilizing the aforementioned approach.

5.1.1 Example 1: Homogenous IVP

Consider the following second-order homogenous IVP

\[ z''(t) + z'(t) - 6z(t) = 0, \]
\[ z(0) = 1, \ z'(0) = 0. \]  

By employing \( H \) on each side of (17) yield

\[ s^{2n}H(s) - p(s) [s^n z(0) + s^n z'(0)] + s^n H(s) - p(s) z(0) - 6H(s) = 0. \]

By substituting the initial conditions, we reach at

\[ \left[ s^{2n} + s^n - 6 \right] H(s) = p(s)(s^n + 1). \]

After simplification, we have

\[ H(s) = \frac{(s^n + 1)p(s)}{s^{2n} + s^n - 6}. \]  

After simple operation, Eq.(19) becomes

\[ H(s) = \frac{3p(s)}{5(s^n - 2)} + \frac{2p(s)}{5(s^n + 3)}. \]  

Now applying \( H^{-1} \) on both sides of Eq. (20), we obtain the exact solution as

\[ z(t) = H^{-1} \left[ \frac{3p(s)}{5(s^n - 2)} + \frac{2p(s)}{5(s^n + 3)} \right] = \frac{3}{5} e^{2t} + \frac{2}{5} e^{-3t}. \]  

5.1.2 Example 2: Inhomogenous IVP

Consider the following third-order inhomogenous IVP

\[ z'''(t) + 2z'' + 2z' + 3z = \sin t + \cos t, \]
\[ z(0) = z''(0) = 0, \ z'(0) = 1. \]

Utilizing \( H \) on both sides of Eq. (22), we have

\[ s^{3n}H(s) - p(s) \left[ s^{2n}z(0) + s^n z'(0) + z''(0) \right] + 2 \left[ s^{2n}H(s) - p(s) \left[ s^n z(0) + z'(0) \right] \right] \]
\[ + 2 \left[ sH(s) - p(s) z(0) \right] + 3H(s) = \frac{p(s)}{s^{2n} + 1} + \frac{s^n p(s)}{s^{2n} + 1}, \]

By applying the initial conditions and then simplification of above equation, we obtain

\[ \left[ s^{3n} + 2s^{2n} + 2s^n + 3 \right] H(s) = \frac{p(s)}{s^{2n} + 1} + \frac{s^n p(s)}{s^{2n} + 1} + p(s)s^n + 2p(s), \]

After simple calculation, we get
Employing $H^{-1}$ on each side of Eq. (24) yields the following analytic solution:

$$z(t) = H^{-1}\left\{\frac{p(s)}{s^{2n} + 1}\right\} = \sin t.$$  (25)

5.2. Solving system of first-order ODEs by new generalized transform

Consider the general form of a system of first-order ODEs as:

$$z_1'(t) = a_{11}z_1(t) + a_{12}z_2(t) + \ldots + a_{1\alpha}z_\alpha(t) + g_1(t),$$
$$z_2'(t) = a_{21}z_1(t) + a_{22}z_2(t) + \ldots + a_{2\alpha}z_\alpha(t) + g_2(t),$$
$$\vdots$$
$$z_\alpha'(t) = a_{\alpha1}z_1(t) + a_{\alpha2}z_2(t) + \ldots + a_{\alpha\alpha}z_\alpha(t) + g_\alpha(t),$$

with initial conditions

$$z_1(0) = z_1^{(0)}, \quad z_2(0) = z_2^{(0)}, \ldots, \quad z_\alpha(0) = z_\alpha^{(0)}.$$  (27)

Now we employ new generalized integral transform to both side of equation (26), then apply linearity and differentiation theorems, we have

$$H\left\{z_1'(t)\right\} = H\left\{a_{11}z_1(t) + a_{12}z_2(t) + \ldots + a_{1\alpha}z_\alpha(t) + g_1(t)\right\},$$
$$H\left\{z_2'(t)\right\} = H\left\{a_{21}z_1(t) + a_{22}z_2(t) + \ldots + a_{2\alpha}z_\alpha(t) + g_2(t)\right\},$$
$$\vdots$$
$$H\left\{z_\alpha'(t)\right\} = H\left\{a_{\alpha1}z_1(t) + a_{\alpha2}z_2(t) + \ldots + a_{\alpha\alpha}z_\alpha(t) + g_\alpha(t)\right\}.$$  (28)

Assuming now that $H_1(s), H_2(s), \ldots, H_\alpha(s)$ are the corresponding generalized transform of $z_1(t), z_2(t), \ldots, z_\alpha(t)$. Under these conditions

$$s^nH_1(s) - p(s)z_1(0) = a_{11}H_1(s) + a_{12}H_2(s) + \ldots + a_{1\alpha}H_\alpha(s) + G_1(s),$$
$$s^nH_2(s) - p(s)z_2(0) = a_{21}H_1(s) + a_{22}H_2(s) + \ldots + a_{2\alpha}H_\alpha(s) + G_2(s),$$
$$\vdots$$
$$s^nH_\alpha(s) - p(s)z_\alpha(0) = a_{\alpha1}H_1(s) + a_{\alpha2}H_2(s) + \ldots + a_{\alpha\alpha}H_\alpha(s) + G_\alpha(s).$$

Eq. (28) can be written an algebraic system of $\alpha$ linear equations as:

$$s^nH(s) = p(s)Z_0 + AH(s) + G,$$  (29)

where

$$H(s) = \begin{bmatrix} H_1(s) \\ H_2(s) \\ \vdots \\ H_\alpha(s) \end{bmatrix}, \quad Z_0 = \begin{bmatrix} z_1^{(0)} \\ z_2^{(0)} \\ \vdots \\ z_\alpha^{(0)} \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1\alpha} \\ a_{21} & a_{22} & \ldots & a_{2\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\alpha1} & a_{\alpha2} & \ldots & a_{\alpha\alpha} \end{bmatrix}, \quad \text{and} \quad G = \begin{bmatrix} G_1(s) \\ G_2(s) \\ \vdots \\ G_\alpha(s) \end{bmatrix}. $$
Once Eq. (29) is solved by means of any algebraic method or solved computationally, the inverse generalized transform is applied to the values of $H_1(s)$, $H_2(s)$, ..., $H_\alpha(s)$ in order to get the solution of $z_1(t)$, $z_2(t)$, ..., $z_\alpha(t)$.

The generalized integral transform method for the system of first-order ODEs will be illustrated by studying the following examples.

5.2.1. Example 1: Homogenous system of first-order ODEs

Consider the following first-order homogenous system of ODEs:

$$
\begin{align*}
    z_1'(t) + z_2(t) &= 0, \quad z_1(0) = 3, \\
    z_2'(t) + z_1(t) &= 0, \quad z_2(0) = 0.
\end{align*}
$$

(30)

By applying generalized transform operator $H$ on each side of (30) we have

$$
\begin{align*}
    s^nH_1(s) - 3p(s) + H_2(s) &= 0, \\
    s^nH_2(s) + H_1(s) &= 0,
\end{align*}
$$

that gives

$$
\begin{align*}
    s^nH_1(s) + H_2(s) &= 3p(s), \\
    s^nH_2(s) + H_1(s) &= 0,
\end{align*}
$$

(31)

Solving the above system for $H_1(s)$ and $H_2(s)$, we have

$$
\begin{align*}
    H_1(s) &= \frac{-3p(s)}{s^{2n} - 1}, \\
    H_2(s) &= \frac{-3s^n p(s)}{s^{2n} - 1},
\end{align*}
$$

(32)

By using $H^{-1}$ on aforementioned system, the exact solution can be obtained as

$$
\begin{align*}
    z_1(t) &= 3\cosh t, \\
    z_2(t) &= -3\sinh t.
\end{align*}
$$

(33)

5.2.2. Example 2: Inhomogenous system of first-order ODEs

Consider the following first-order inhomogenous system of ODEs:

$$
\begin{align*}
    z_1'(t) + 2z_2(t) &= 3t, \quad z_1(0) = 2 \\
    z_2'(t) - 2z_1(t) &= 4, \quad z_2(0) = 3
\end{align*}
$$

(34)

By utilizing generalized transform operator $H$ on both sides of (34), we have

$$
\begin{align*}
    s^nH_1(s) - 2p(s) + 2H_2(s) &= \frac{3p(s)}{s^{2n}}, \\
    s^nH_2(s) - 2p(s) + 2H_1(s) &= \frac{4p(s)}{s^n},
\end{align*}
$$

this in turn gives
\[ s^n H_1(s) + 2H_2(s) = \frac{3p(s)}{s^{2n}} + 2p(s), \]
\[ s^n H_2(s) + 2H_1(s) = \frac{4p(s)}{s^n} + 3p(s), \]

Solution of the above system can thus be expressed as
\[ H_1(s) = \frac{p(s)}{s^n} \left[ \frac{2s^{2n} - 6s^n - 5}{s^{2n} + 4} \right], \]
\[ H_2(s) = \frac{p(s)}{s^n} \left[ \frac{3s^{3n} + 8s^{2n} + 6}{s^{2n} + 4} \right]. \]

Partial fraction of the above system leads to
\[ H_1(s) = p(s) \left[ \frac{13s^n}{4(s^{2n} + 4)} - \frac{6}{s^{2n} + 4} - \frac{5}{4s^n} \right], \]
\[ H_2(s) = p(s) \left[ \frac{3s^n}{s^{2n} + 4} + \frac{13}{2(s^{2n} + 4)} + \frac{3}{2s^{2n}} \right]. \]

Taking \( H^{-1} \) gives us
\[ z_1(t) = \frac{13}{2} \cos 2t - 3 \sin 2t - \frac{5}{4}, \]
\[ z_2(t) = 3 \cos 2t + \frac{13}{4} \sin 2t + \frac{3}{2} t. \]

5.3. Solving integral equations by new generalized transform

In this section, with the help of generalized integral transform, we solve two linear and one nonlinear Volterra integral equations and exact solutions are found in all cases.

5.3.1. Example 1: Volterra integral equation of the second kind

Consider the equation
\[ z(t) - \int_0^t (1 + \tau) z(t - \tau) d\tau = 1 - \sinh t. \]  

By taking the generalized transform on each side of above equation and then using the convolution theorem, we have
\[ H(s) = \frac{1}{p(s)} \left[ H\{1 + t\} H\{z(t)\} \right] = \frac{p(s)}{s^n} - \frac{p(s)}{s^{2n} - 1}, \]
this is equivalent to
\[ H(s) - \left[ \frac{1}{s^n} + \frac{1}{s^{2n}} \right] H(s) = p(s) \left[ \frac{1}{s^n} - \frac{1}{s^{2n} - 1} \right]. \]

After simple calculation, we reach at
\[ H(s) = \frac{p(s)s^n}{s^{2n} - 1}. \]
By using $H^{-1}$, the solution can be expressed as 
$$z(t) = \cosh t.$$ (40)

### 5.3.2. Example 2: Volterra integral equation of the first kind

Suppose the Volterra integral equation of the first kind 
$$e^t - \sin t - \cos t = \int_0^t 2e^{-\tau} z(\tau) d\tau.$$ (41)

By using the generalized transform and the convolution theorem, we have 
$$\frac{p(s)}{s^{n} - 1} - \frac{p(s)}{s^{2n} + 1} - \frac{p(s)s^n}{s^{2n} - 1} = \frac{2}{p(s)} \left[ H\{e^t\} H\{z(t)\} \right],$$
which in turn gives 
$$p(s) \left[ \frac{1}{s^{n} - 1} - \frac{1}{s^{2n} + 1} - \frac{s^n}{s^{2n} - 1} \right] = H(s) \left[ \frac{2}{s^n - 1} \right].$$

Simple calculation yields 
$$H(s) = \frac{p(s)}{s^{2n} + 1}.$$ (42)

By utilizing $H^{-1}$, the solution can be depicted as 
$$z(t) = \sin t$$

### 5.3.3. Example 3: Nonlinear Volterra integral equation of the first kind

Consider the nonlinear Volterra integral equation of the first kind 
$$\frac{1}{4} e^{2t} - \frac{1}{2} t - \frac{1}{4} = \int_0^t (t-\tau) z^2(\tau) d\tau.$$ (43)

We first let $v(t) = z^2(t)$, $z(t) = \pm \sqrt{v(t)}$. Eq. (43) gets the form 
$$\frac{1}{4} e^{2t} - \frac{1}{2} t - \frac{1}{4} = \int_0^t (t-\tau) v(\tau) d\tau.$$ (44)

With the help of generalized transform and convolution theorem, we have 
$$\frac{p(s)}{4(s^n - 2)} - \frac{p(s)}{2s^{2n}} - \frac{p(s)s^n}{s^{2n}} = \frac{1}{p(s)} \left[ H\{t\} H\{v(t)\} \right],$$
or equivalently 
$$\frac{p(s)}{4} \left[ \frac{1}{s^n - 2} - \frac{2}{s^{2n}} - \frac{1}{s^n} \right] = H(s) \left[ \frac{1}{s^{2n}} \right].$$ (45)

Simple calculation results in 
$$H(s) = \frac{p(s)}{s^n - 2}.$$ (46)

By operating $H^{-1}$, we have 
$$v(t) = e^{2t}.$$ (47)

The exact solutions are therefore can be expressed 
$$z(t) = \pm e^t.$$ (48)

It is important to note that there were two solutions found since equation is a nonlinear equation and there may not be a single solution.
6. Discussion and Conclusion

In this article, we introduce a new generalized integral transform unifying the Laplace, Fourier and many other integral transforms belonging to the family of Laplace transform as special cases as shown in Table 3. This unification has offered many opportunities for engineering applications.

A methodology is developed for ordinary differential equations (ODEs) with constant coefficients, and after that, this strategy is used to resolve two different systems. It is concluded that Laplace and all of the integral transforms from its family result in the same solution. In other words, the solution is unchanged by the choice of integral transform method for the case of ODEs with constant coefficients. This property is advantageous because it allows mathematicians and engineers to choose the integral transform that is best applicable to a particular scenario or that simplifies the arithmetic while still being confident in the final solution.

We have also constructed a technique for the system of ODEs and established the fact that the Laplace transform is recognized as the best method for resolving these systems with constant coefficients due to its efficiency in terms of computational requirements. This transform facilitates the manipulation of the equations, eliminates the need for recurrent differentiation, and frequently facilitates the derivation of closed-form solutions because there are many interrelate equations in this circumstance.

In the last part, we have solved linear and nonlinear Volterra integral equations and noted that a linear equation has a unique solution while there may not be a single solution for the nonlinear case. Nonlinear equations commonly cause mathematical complexities, demanding for specialized techniques and methodologies and this generalized transform has done remarkably well for the said case. In addition to complicating the problem, more than one solution gained by the proposed transform helps us understand the underlying system by shedding light on a variety of possible outcomes or scenarios.

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We have studies all problems with constant coefficients, but for variable coefficients, whole scenarios will be changed and the choice of transform becomes much importance. We will develop methodologies based on generalized integral transform that will solve ODEs with variable coefficients, such as those with polynomials, exponential and trigonometric coefficients in the future work. Consequently, when coupled with the homotopy perturbation method [15], the variational iteration method [16], and the Adomian decomposition method [41], the new integral transform becomes a promising tool to nonlinear problems and fractal/fractional differential equations, and it is transformative for research because the transform has the same essential qualities as those for Laplace and Fourier transform.

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Received: 4.8.2023.
Revised: 15.9.2023.
Accepted: 11.10.2023