BEYOND LAPLACE AND FOURIER TRANSFORMS
Challenges and Future Prospects

by

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Laplace and Fourier transforms are widely used independently in engineering for linear differential equations including fractional differential equations. Here we introduce a generalized integral transform, which is a generalization of the Fourier transform, Laplace transform, and other transforms, e.g., Sumudu transform, Aboodh transform, Pourreza transform, and Mohand transform, making the new transform much attractive and promising. Its basic properties are elucidated, and its applications to initial value problems and integral equations are illustrated, when coupled with the homotopy perturbation, it can be used for various non-linear problems, opening a new window for non-linear science.

Key words: *Laplace transform, Fourier transform, Volterra integral equations, system of ODE, initial value problems*

Introduction

Researchers have developed several mathematical methods that are being employed in numerous fields of science, technology, and engineering in order to better understand nature. Particularly, the idea of integral transformation was put forth and has since been discovered to be a practical mathematical tool for addressing a variety of issues in both pure and applied mathematics [1-3]. It is important to recall that a mathematical operator is referred to as an integral transform if it transfers a function by means of an integral from its original function to another function space. Open literature demonstrates that there are numerous probability applications that are related to integral transformations, such as the price kernel, also known as the stochastic discount factor [4]. The application of these mathematical operators in control theory [5] is another significant area.

Since roughly 200 years ago, integral transforms were appeared in literature, among which Fourier and Laplace transforms are the most famous ones. Other than the Laplace transform, a number of alternative integrals have been proposed in recent years and have been discovered to share certain intriguing Laplace transform-like features. Elzaki transform [6], Su-

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mudu transform [7], Aboodh transform [8], Natural transform [9], Mohand transform [10], Pourreza transform [11], Kamal transform [12], Sawi transform [13], and Emad-Sara transform [14] are in the list. These transforms have been crucial in resolving differential equations of both integer and non-integer orders. Coupled with the analytical methods like the homotopy perturbation method [15] or the variational iteration method [16, 17], these transforms can be extended to non-linear problems and fractal/fractional equations [18-24].

Hossein [25] introduced a generalized transform and deduced that every integral transform in the class of Laplace transform is actually a special case of its generalized integral transform, however, this transform did not preserve the properties of Fourier transform. Recently, Khan and Khalid [26] proposed the Fareeha transform, which, however, does not contain many integral transforms falling under the Laplace transform category. In this article, we are being proposed a new generalized integral transform that remove the aforementioned disadvantages. The suggested integral transform not only includes many integral transforms falling under the Laplace transform category but also holds the properties of the Fourier transform as the special case. This unification offers a totally new window for wide applications.

Background

The background of integral transforms can be traced back to the development of integral calculus and the need to solve complex problems involving differential equations and other mathematical operations [27]. Generally, an integral transform of an input function $f(x)$ defined in $a \le x \le b$ can be expressed:

$$\mathcal{I}\left\{f(x)\right\} = F(k) = \int_a^b K(x,k) f(x) \mathrm{d}x \tag{1}$$

where $K(x, k)$ is the kernel of the transformation, $\mathcal{I}$ – the integral transform operator, $F(k)$ – the image of $f(x)$, and k – the transform variable. In order to find $f(x)$ from given $F(k)$, we introduce the inverse operator $\mathcal{I}^{-1}$:

$$\mathcal{I}^{-1}\left\{F(k)\right\} = f(x) \tag{2}$$

Based on eq. (1), the Fourier and the Laplace transforms of a function can be written, respectively:

$$\mathcal{F}\left\{f(x)\right\} = F(k) = \int_{-\infty}^{\infty} \mathrm{e}^{-ikx} f(x) \mathrm{d}x \tag{3}$$

and

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_0^{\infty} \mathrm{e}^{-st} f(t) \mathrm{d}t \tag{4}$$

Table 1 lists a few integral transformations from the category of the Laplace transform. Different types of integral equations, as well as ordinary, partial and fractional equations, have all been solved using these transformations [28-31]. Additionally, these types of transforms have been used in conjunction with other semi-analytical techniques, including the homotopy perturbation method, the variational iteration method, Adomian decomposition, differential transform methods, to solve a variety of ordinary, partial and fractional equations [32-36]. Numerous applications of the integral transformations mentioned in tab. 1 can be found in science and engineering, including solitary waves, mechanics, finance, economics, and chemistry.

Table 1. Integral transforms from the class of Laplace transform

Integral formula	Transform name
$S\{f(t)\}=\dfrac{1}{s}\int\limits_0^\infty \mathrm{e}^{-t/s}f(t)\mathrm{d}t$	Sumudu transform [7, 37]
$A\{f(t)\}=\dfrac{1}{s}\int\limits_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$	Aboodh transform [8]
$N\{f(t)\}=s\int\limits_0^\infty \mathrm{e}^{-st}f(ut)\mathrm{d}t$	Natural transform [9, 38]
$P\{f(t)\}=s\int\limits_0^\infty \mathrm{e}^{-s^2t}f(ut)\mathrm{d}t$	Pourreza transform [11]
$E\{f(t)\}=s\int\limits_0^\infty \mathrm{e}^{-t/s}f(t)\mathrm{d}t$	Elzaki transform [6]
$M\{f(t)\}=s^2\int\limits_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$	Mohand transform [10]
$Sa\{f(t)\}=\dfrac{1}{s^2}\int\limits_0^\infty \mathrm{e}^{-t/s}f(t)\mathrm{d}t$	Sawi transform [13]
$K\{f(t)\}=\int\limits_0^\infty \mathrm{e}^{-t/s}f(t)\mathrm{d}t$	Kamal transform [12]
$ES\{f(t)\}=\dfrac{1}{s^2}\int\limits_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$	Emad-Sara transform [14]
$EF\{f(t)\}=\dfrac{1}{s}\int\limits_0^\infty \mathrm{e}^{-s^2t}f(t)\mathrm{d}t$	Emad-Falih transform [39]

Yang *et al.* [19] investigated the 1-D fractal heat-conduction problem in a fractal semi-infinite bar with local fractional calculus and the Yang-Fourier transform approach. The outcome demonstrates the correctness and dependability of the results. Nazari-Golshan *et al.* [18] examined a method by introducing He's polynomials into the homotopy perturbation method coupled with the Fourier transform for the Lane-Emden problem. He and Zhang [6] proposed an iterative transformation technique that combines the Elzaki transform and iterative approaches to resolve fractional order linear Klein-Gordon and Hirota-Satsuma-linked KdV equations. Manimegalai *et al.* [8] explored the Aboodh transform-based homotopy perturbation method to solve a generalized oscillatory differential equation and concluded that the coupling gave much better results than many existed methods. Akgul *et al.* [37] investigated a few alternative financial/economic theories based on market equilibrium and option pricing using three different fractional derivatives, and obtained the fundamental solutions of the models using the Sumudu transform and the Laplace transform. Ahmadi *et al.* [11] studied the Pourreza integral transform, which is useful for solving both Laguerre and Hermite differential equations used in quantum mechanics. Nadeem *et al.* [10] employed the Mohand transform with the homotopy perturbation method for the fractional order Newell-Whitehead-Segel equation. Higazy and Aggarwal [13] used the Sawi transformation to solve a system of ODE to calculate the concentration of chemical reactants in a series of chemical reactions.

The proposed generalized integral transform

This section is devoted to present the generalized integral transform that includes many integral transforms falling under the Laplace transform category and the properties of the Fourier transform as the special case.

Definition 1: Let $f(t)$ be an integrable function defined for $t \geq 0$, $p(s) \neq 0$ and s is from the complex domain, *i.e.*, $s = x + iy$. We define the generalized integral transform $\mathcal{H}(s)$ of $f(t)$:

$$\mathrm{H}\{f(t)\} = \mathcal{H}(s) = p(s)\int_0^\infty \mathrm{e}^{-s^n t} f(t)\mathrm{d}t \tag{5}$$

presuming that the integral exists for some s^n where $n \in Z$. Table 2 displays the generalized integral transform of some basic functions.

Table 2. New generalized integral transform of some elementary functions

$f(t)$	1	t	t^α, $\alpha > 0$	e^{at}	sinbt	cosbt	sinhbt	coshbt
$\mathcal{H}(s)$	$\dfrac{p(s)}{s^n}$	$\dfrac{p(s)}{s^{2n}}$	$\dfrac{p(s)\Gamma(\alpha+1)}{s^{n(\alpha+1)}}$	$\dfrac{p(s)}{s^n - a}$	$\dfrac{bp(s)}{s^{2n}+b^2}$	$\dfrac{p(s)s^n}{s^{2n}+b^2}$	$\dfrac{bp(s)}{s^{2n}-b^2}$	$\dfrac{p(s)s^n}{s^{2n}-b^2}$

Theorem 1. (Existence Theorem): Let $f(t)$ is a piecewise continuous function of exponential order for all $t \geq 0$, then $\mathcal{H}(s)$ exists for all $s^n > k$.

Proof: Given that $f(t)$ is a piecewise continuous function of exponential order so it satisfies $|f(t)| \leq M\mathrm{e}^{kt}$, where M is positive constant and k is the order of the function. Since:

$$|\mathcal{H}(s)| = |\mathrm{H}\{f(t)\}| = \left|p(s)\int_0^\infty f(t)\mathrm{e}^{-s^n t}\mathrm{d}t\right| \leq p(s)\int_0^\infty |f(t)|\mathrm{e}^{-s^n t}\mathrm{d}t \leq p(s)\int_0^\infty M\mathrm{e}^{kt}\mathrm{e}^{-s^n t}\mathrm{d}t$$

equivalently

$$|\mathcal{H}(s)| \leq p(s)M\int_0^\infty \mathrm{e}^{-(s^n-k)t}\mathrm{d}t = \frac{p(s)M}{s^n - k},\ \ s^n > k \tag{6}$$

Thus the statement is correct.

Theorem 2. (Linearity Theorem): For any constants β and γ and any two functions $f(t)$ and $g(t)$ whose transforms exist individually, H satisfies:

$$\mathrm{H}\{\beta f(t) + \gamma g(t)\} = \beta\mathrm{H}\{f(t)\} + \gamma\mathrm{H}\{g(t)\} \tag{7}$$

Proof: By applying the definition (5), we have:

$$\mathrm{H}\{\beta f(t) + \gamma g(t)\} = p(s)\int_0^\infty \mathrm{e}^{-s^n t}\left[\beta f(t) + \gamma g(t)\right]\mathrm{d}t =$$

$$= \beta p(s)\int_0^\infty \mathrm{e}^{-s^n t} f(t)\mathrm{d}t + \gamma p(s)\int_0^\infty \mathrm{e}^{-s^n t} g(t)\mathrm{d}t = \beta\mathrm{H}\{f(t)\} + \gamma\mathrm{H}\{g(t)\}$$

Hence it is proved.

Theorem 3. (Differentiation Theorem): Let $f(t)$is differentiable for $t \geq 0$, $p(s) \neq 0$ and s is from the complex domain, *i.e.*, $s = x + iy$, then:

$$\text{(a)} \quad \mathrm{H}\{f'(t)\} = s^n \mathcal{H}(s) - p(s) f(0) \tag{8}$$

$$\text{(b)} \quad \mathrm{H}\{f''(t)\} = s^{2n} \mathcal{H}(s) - s^n p(s) f(0) - p(s) f'(0) \tag{9}$$

$$\text{(c)} \quad \mathrm{H}\left\{f^m(t)\right\} = s^{nm} \mathcal{H}(s) - p(s) \sum_{k=0}^{m-1} s^{nm-n-nk} f^{(k)}(0) \tag{10}$$

Proof. (a): Using the definition (5), we have:

$$\mathrm{H}\{f'(t)\} = p(s) \int_0^\infty \mathrm{e}^{-s^n t} f'(t) \mathrm{d}t = p(s) \left[\mathrm{e}^{-s^n t} f(t) \Big|_0^\infty + s^n \int_0^\infty \mathrm{e}^{-s^n t} f(t) \mathrm{d}t \right]$$

$$\mathrm{H}\{f'(t)\} = p(s) \mathrm{H}_f \{f(t)\} - p(s) f(0)$$

Proof. (b): we assume $h(t) = f'(t)$ so $f''(t) = h'(t)$, so we have:

$$\mathrm{H}\{h'(t)\} = p(s) \int_0^\infty h'(t) \mathrm{e}^{-s^n t} \mathrm{d}t = s^n \mathrm{H}\{h(t)\} - p(s) h(0) = s^n \mathrm{H}\{f'(t)\} - p(s) f'(0) =$$

$$= s^n \left[s^n \mathrm{H}\{f(t)\} - p(s) f(0) \right] - p(s) f'(0)$$

$$\mathrm{H}\{h'(t)\} = s^{2n} \mathrm{H}\{f(t)\} - s^n p(s) f(0) - p(s) f'(0)$$

By principle of mathematical induction, we can *Proof* (c).

Theorem 4. (Convolution Theorem): Let

$$\mathrm{H}\{f_1(t)\} = \mathcal{H}_1(s) \text{ and } \mathrm{H}\{f_2(t)\} = \mathcal{H}_2(s) \text{ then}$$

$$\mathrm{H}\{f_1(t) * f_2(t)\} = \mathrm{H}\{f_1(t)\} \mathrm{H}\{f_2(t)\} = \frac{1}{p(s)} \mathcal{H}_1(s) \mathcal{H}_2(s) \tag{11}$$

where $f_1(t)^* f_2(t)$ is called the convolution of $f_1(t)$ and $f_2(t)$ and is expressed:

$$f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau) \mathrm{d}\tau$$

Proof. Using the definition (5), we have:

$$\mathrm{H}\{f_1(t) * f_2(t)\} = p(s) \int_0^\infty \mathrm{e}^{-s^n t} \left(\int_0^t f_1(t-\tau) f_2(\tau) \mathrm{d}\tau \right) \mathrm{d}t$$

$$= p(s) \int_0^\infty \mathrm{e}^{-s^n t} \mathrm{d}t \int_0^t f_1(t-\tau) f_2(\tau) \mathrm{d}\tau$$

By changing the order of integration:

$$\mathrm{H}\{f_1(t) * f_2(t)\} = p(s) \int_0^\infty f_2(\tau) d\tau \int_{t=\tau}^\infty \mathrm{e}^{-s^n t} f_1(t-\tau) \mathrm{d}t$$

Table 3. Integral transforms belongs to the class of Laplace transform for various values of $p(s)$ and n

$p(s)$	n	Integral formula	Transform name
1	1	$\int_0^\infty e^{-st} f(t)dt$	Laplace transform [40, 41]
$1/s$	–1	$\frac{1}{s}\int_0^\infty e^{-t/s} f(t)dt$	Sumudu transform [7, 37]
$1/s$	1	$\frac{1}{s}\int_0^\infty e^{-st} f(t)dt$	Aboodh transform [8]
s	1	$s\int_0^\infty e^{-st} f(ut)dt$	Natural transform [9, 38]
s	2	$s\int_0^\infty e^{-s^2 t} f(ut)dt$	Pourreza transform [11]
s	–1	$s\int_0^\infty e^{-t/s} f(t)dt$	Elzaki transform [6]
s^2	1	$s^2\int_0^\infty e^{-st} f(t)dt$	Mohand transform [10]
$1/s^2$	–1	$\frac{1}{s^2}\int_0^\infty e^{-t/s} f(t)dt$	Sawi transform [13]
1	–1	$\int_0^\infty e^{-t/s} f(t)dt$	Kamal transform [12]
$1/s^2$	1	$\frac{1}{s^2}\int_0^\infty e^{-st} f(t)dt$	Emad-Sara transform [14]
$1/s$	2	$\frac{1}{s}\int_0^\infty e^{-s^2 t} f(t)dt$	Emad-Falih transform [39]

By the change of variable $t - \tau = u$, previous equation leads to:

$$\mathrm{H}\left\{f_1(t) * f_2(t)\right\} = p(s)\int_0^\infty f_2(\tau)d\tau\int_0^\infty e^{-s^n(u+\tau)} f_1(u)du$$

$$= p(s)\int_0^\infty e^{-s^n\tau} f_2(\tau)d\tau\int_0^\infty e^{-s^n u} f_1(u)du = \frac{1}{p(s)}\mathcal{H}_1(s)\mathcal{H}_2(s).$$

This completes the proof.

Similarly, we can prove the first shifting theorem, second shifting theorem, scaling property and other concepts for this new generalized transform.

Analysis of proposed generalized integral transform

As discussed earlier that the proposed integral transform not only encompasses various transforms that belong to the Laplace transform category, but also exhibits the properties of the Fourier transform in specific instances. For $p(s) = 1$ and $s = x + iy$, eq. (5) becomes the Fareeha transform [27] that includes all the properties of Laplace and Fourier transform, and for positive real value of s^n where $n \in Z$, we can obtain an integral transform belonging to the Laplace transform class. Table 3 discusses about the integral transformations from the category of Laplace transform for various values of $p(s)$ and n.

The convergence of the generalized integral transform of a function $f(t)$ depends on three factors. The $f(t)$ must be continuous, bounded by an exponential function and absolutely integrable over the real line. The generalized transform of a function may diverge and in this case it cannot be computed using the standard techniques.

Applications

This section is devoted to present methodologies stem from generalized integral transform to solve initial value problems (IVP), system of first-order differential equations and Volterra integral equations.

Solving initial value problem by new generalized transform

Consider the general form of an IVP:

$$\begin{gathered} z^{(m)}(t)+a_1 z^{(m-1)}(t)+\ldots+a_m z(t)=g(t) \\ z(0)=z_0,\; z'(0)=z_1,\ldots,\; z^{(m-1)}(0)=z_{m-1} \end{gathered} \tag{12}$$

Now we employ new generalized integral transform to each side of eq. (12), then apply linearity and differentiation theorems, we have:

$$\begin{gathered} \mathrm{H}\left\{z^{(m)}(t)+a_1 z^{(m-1)}(t)+\ldots+a_m z(t)\right\}=H\{g(t)\} \\ \mathrm{H}\left\{z^{(m)}(t)\right\}+a_1 H\left\{z^{(m-1)}(t)\right\}+\ldots+H\left\{a_m z(t)\right\}=H\{g(t)\} \\ s^{nm}\mathcal{H}(s)-p(s)\sum_{k=0}^{m-1} s^{nm-n-nk}(s)z^{(k)}(0)+a_1 s^{nm-n}\mathcal{H}(s) \\ -p(s)\sum_{k=0}^{m-2} s^{nm-2n-nk} z^{(k)}(0)+\ldots+a_m\mathcal{H}(s)=G(s) \end{gathered} \tag{13}$$

where $G(s) = \mathrm{H}\{g(t)\}$. By applying the initial conditions in eq. (13):

$$h(s)\mathcal{H}(s)=G(s)+\Psi(s), \tag{14}$$

where

$$h(s)=\left(s^{nm}+a_1 s^{nm-n}+\ldots+a_m\right) \text{ and}$$

$$\Psi(s)=p(s)\left(\sum_{k=0}^{m-1} s^{nm-n-nk}(s)z_k+a_1\sum_{k=0}^{m-2} s^{nm-2n-nk}(s)z_k+\ldots+z_0\right)$$

From eq. (14), we find $\mathcal{H}(s)$:

$$\mathcal{H}(s)=\frac{G(s)}{h(s)}+\frac{\Psi(s)}{h(s)} \tag{15}$$

Lastly, we apply inverse generalized transform on each side of previous equation get the solution:

$$z(t)=\mathrm{H}^{-1}\left\{\frac{G(s)}{h(s)}\right\}+\mathrm{H}^{-1}\left\{\frac{\Psi(s)}{h(s)}\right\} \tag{16}$$

Two examples will be solved by utilizing the aforementioned approach.

Example 1. Homogenous IVP

Consider the following second-order homogenous IVP:

$$z''(t)+z'(t)-6z(t)=0,\quad z(0)=1,\ z'(0)=0 \tag{17}$$

By employing H on each side of eq. (17) yield:

$$\begin{gathered}\mathrm{H}\{z''(t)\}+\mathrm{H}\{z'(t)\}-6\mathrm{H}\{z(t)\}=0\\ s^{2n}\mathcal{H}(s)-p(s)\left[s^{n}z(0)+s^{n}z'(0)\right]+s^{n}\mathcal{H}(s)-p(s)z(0)-6\mathcal{H}(s)=0\end{gathered} \tag{18}$$

By substituting the initial conditions, we reach:

$$\left[s^{2n}+s^{n}-6\right]\mathcal{H}(s)=p(s)(s^{n}+1)$$

After simplification:

$$\mathcal{H}(s)=\frac{(s^{n}+1)p(s)}{s^{2n}+s^{n}-6} \tag{19}$$

After simple operation, eq. (19) becomes:

$$\mathcal{H}(s)=\frac{3p(s)}{5(s^{n}-2)}+\frac{2p(s)}{5(s^{n}+3)} \tag{20}$$

Now applying H^{-1} on both sides of eq. (20), we obtain the exact solution:

$$z(t)=H^{-1}\left\{\frac{3p(s)}{5(s^{n}-2)}+\frac{2p(s)}{5(s^{n}+3)}\right\}=\frac{3}{5}\mathrm{e}^{2t}+\frac{2}{5}\mathrm{e}^{-3t} \tag{21}$$

Example 2. Inhomogenous IVP

Consider the following third-order inhomogenous IVP:

$$\begin{gathered}z'''(t)+2z''+2z'(t)+3z(t)=\sin t+\cos t\\ z(0)=z''(0)=0,\ z'(0)=1\end{gathered} \tag{22}$$

Utilizing H on both sides of eq. (22), we have

$$\begin{gathered}\mathrm{H}\{z'''(t)\}+2\mathrm{H}\{z''(t)\}+2\mathrm{H}\{z'(t)\}+3\mathrm{H}\{z(t)\}=\mathrm{H}\{\sin t\}+\mathrm{H}\{\cos t\}\\ s^{3n}\mathcal{H}(s)-p(s)\left[s^{2n}z(0)+s^{n}z'(0)+z''(0)\right]+2\left\{s^{2n}\mathcal{H}(s)-p(s)\left[s^{n}z(0)+z'(0)\right]\right\}+\\ +2\left\{s^{n}\mathcal{H}(s)-p(s)z(0)\right\}+3\mathcal{H}(s)=\frac{p(s)}{s^{2n}+1}+\frac{s^{n}p(s)}{s^{2n}+1}\end{gathered} \tag{23}$$

By applying the initial conditions and then simplification of aforementioned equation, we obtain:

$$\left[s^{3n}+2s^{2n}+2s^{n}+3\right]\mathcal{H}(s)=\frac{p(s)}{s^{2n}+1}+\frac{s^{n}p(s)}{s^{2n}+1}+p(s)s^{n}+2p(s)$$

After simple calculation, we get:

$$\mathcal{H}(s)=\frac{p(s)}{s^{2n}+1} \tag{24}$$

Employing H^{-1} on each side of eq. (24) yields the following analytic solution:

$$z(t)=H^{-1}\left\{\frac{p(s)}{s^{2n}+1}\right\}=\sin t \tag{25}$$

Solving system of first-order ODE by new generalized transform

Consider the general form of a system of first-order ODE:

$$\begin{aligned}
z_1'(t)&=a_{11}z_1(t)+a_{12}z_2(t)+\ldots+a_{1\alpha}z_\alpha(t)+g_1(t)\\
z_2'(t)&=a_{21}z_1(t)+a_{22}z_2(t)+\ldots+a_{2\alpha}z_\alpha(t)+g_2(t)\\
&\vdots\\
z_\alpha'(t)&=a_{\alpha1}z_1(t)+a_{\alpha2}z_2(t)+\ldots+a_{\alpha\alpha}z_\alpha(t)+g_\alpha(t)
\end{aligned} \tag{26}$$

with initial conditions:

$$z_1(0)=z_1^{(0)},\ z_2(0)=z_2^{(0)},\ \ldots,\ z_\alpha(0)=z_\alpha^{(0)} \tag{27}$$

Now we employ new generalized integral transform to both side of eq. (26), then apply linearity and differentiation theorems:

$$\begin{aligned}
\mathrm{H}\left\{z_1'(t)\right\}&=\mathrm{H}\left\{a_{11}z_1(t)+a_{12}z_2(t)+\ldots+a_{1\alpha}z_\alpha(t)+g_1(t)\right\}\\
\mathrm{H}\left\{z_2'(t)\right\}&=\mathrm{H}\left\{a_{21}z_1(t)+a_{22}z_2(t)+\ldots+a_{2\alpha}z_\alpha(t)+g_2(t)\right\}\\
&\vdots\\
\mathrm{H}\left\{z_\alpha'(t)\right\}&=\mathrm{H}\left\{a_{\alpha1}z_1(t)+a_{\alpha2}z_2(t)+\ldots+a_{\alpha\alpha}z_\alpha(t)+g_\alpha(t)\right\}
\end{aligned}$$

Assuming now that $\mathcal{H}_1(s)$, $\mathcal{H}_2(s)$,..., $\mathcal{H}_\alpha(s)$ are the corresponding generalized transform of $z_1(t)$, $z_2(t)$..., $z_\alpha(t)$. Under these conditions:

$$\begin{aligned}
s^n\mathcal{H}_1(s)-p(s)z_1(0)&=a_{11}\mathcal{H}_1(s)+a_{12}\mathcal{H}_2(s)+\ldots+a_{1\alpha}\mathcal{H}_\alpha(s)+G_1(s)\\
s^n\mathcal{H}_2(s)-p(s)z_2(0)&=a_{21}\mathcal{H}_1(s)+a_{22}\mathcal{H}_2(s)+\ldots+a_{2\alpha}\mathcal{H}_\alpha(s)+G_2(s)\\
&\vdots\\
s^n\mathcal{H}_\alpha(s)-p(s)z_\alpha(0)&=a_{\alpha1}\mathcal{H}_1(s)+a_{\alpha2}\mathcal{H}_2(s)+\ldots+a_{\alpha\alpha}\mathcal{H}_\alpha(s)+G_\alpha(s)
\end{aligned} \tag{28}$$

Equation (28) can be written an algebraic system of α linear equations:

$$s^n\mathcal{H}(s)=p(s)Z_0+A\mathcal{H}(s)+G \tag{29}$$

where

$$\mathcal{H}(s)=\begin{bmatrix}\mathcal{H}_1(s)\\ \mathcal{H}_2(s)\\ \vdots\\ \mathcal{H}_\alpha(s)\end{bmatrix},\quad Z_0=\begin{bmatrix}z_1^{(0)}\\ z_2^{(0)}\\ \vdots\\ z_\alpha^{(0)}\end{bmatrix},\quad A=\begin{bmatrix}a_{11} & a_{12} & \dots & a_{1\alpha}\\ a_{21} & a_{22} & \dots & a_{2\alpha}\\ \vdots & \vdots & \ddots & \vdots\\ a_{\alpha 1} & a_{\alpha 2} & \dots & a_{\alpha\alpha}\end{bmatrix},\quad \text{and}\quad G=\begin{bmatrix}G_1(s)\\ G_2(s)\\ \vdots\\ G_\alpha(s)\end{bmatrix}$$

Once eq. (29) is solved by means of any algebraic method or solved computationally, the inverse generalized transform is applied to the values of $\mathcal{H}_1(s)$, $\mathcal{H}_2(s)$,..., $\mathcal{H}_\alpha(s)$ in order to get the solution of $z_1(t)$, $z_2(t)$..., $z_\alpha(t)$.

The generalized integral transform method for the system of first-order ODE will be illustrated by studying the following examples.

Example 3. Homogenous system of first-order ODE

Consider the following first-order homogenous system of ODE:

$$\begin{aligned} z_1'(t)+z_2(t)=0,\quad z_1(0)=3\\ z_2'(t)+z_1(t)=0,\quad z_2(0)=0 \end{aligned} \tag{30}$$

By applying generalized transform operator H on each side of eq. (30) we have:

$$s^n\mathcal{H}_1(s)-3p(s)+\mathcal{H}_2(s)=0$$

$$s^n\mathcal{H}_2(s)+\mathcal{H}_1(s)=0$$

that give:

$$\begin{aligned} s^n\mathcal{H}_1(s)+\mathcal{H}_2(s)=3p(s)\\ s^n\mathcal{H}_2(s)+\mathcal{H}_1(s)=0 \end{aligned} \tag{31}$$

Solving the aforementioned system for $H_1(s)$ and H_2(s):

$$\begin{aligned} \mathcal{H}_1(s)=\frac{-3p(s)}{s^{2n}-1}\\ \mathcal{H}_2(s)=\frac{-3s^n p(s)}{s^{2n}-1} \end{aligned} \tag{32}$$

By using H^{-1} on aforementioned system, the exact solution can be obtained:

$$\begin{aligned} z_1(t)=3\cosh t\\ z_1(t)=-3\sinh t \end{aligned} \tag{33}$$

Example 4. Inhomogenous system of first-order ODE

Consider the following first-order inhomogenous system of ODE:

$$\begin{aligned} z_1'(t)+2z_2(t)=3t,\ \ z_1(0)=2\\ z_2'(t)-2z_1(t)=4,\ \ z_2(0)=3 \end{aligned} \tag{34}$$

By utilizing generalized transform operator H on both sides of eq. (34), we havwe:

$$s^n\mathcal{H}_1(s)-2p(s)+2\mathcal{H}_2(s)=\frac{3p(s)}{s^{2n}}$$

$$s^n\mathcal{H}_2(s)-2p(s)+2\mathcal{H}_1(s)=\frac{4p(s)}{s^n}$$

this in turn gives:

$$\begin{aligned} s^n\mathcal{H}_1(s)+2\mathcal{H}_2(s) &= \frac{3p(s)}{s^{2n}}+2p(s) \\ s^n\mathcal{H}_2(s)+2\mathcal{H}_1(s) &= \frac{4p(s)}{s^n}+3p(s) \end{aligned} \tag{35}$$

Solution of the previous system can thus be expressed:

$$\begin{aligned} \mathcal{H}_1(s) &= \frac{p(s)\left[2s^{2n}-6s^n-5\right]}{s^n\left(s^{2n}+4\right)} \\ \mathcal{H}_2(s) &= \frac{p(s)\left[3s^{3n}+8s^{2n}+6\right]}{s^{2n}\left(s^{2n}+4\right)} \end{aligned} \tag{36}$$

Partial fraction of the aforementioned system leads:

$$\begin{aligned} \mathcal{H}_1(s) &= p(s)\left[\frac{13s^n}{4\left(s^{2n}+4\right)}-\frac{6}{s^{2n}+4}-\frac{5}{4s^n}\right] \\ \mathcal{H}_2(s) &= p(s)\left[\frac{3s^n}{s^{2n}+4}+\frac{13}{2\left(s^{2n}+4\right)}+\frac{3}{2s^{2n}}\right] \end{aligned} \tag{37}$$

Taking H^{-1} gives:

$$\begin{aligned} z_1(t) &= \frac{13}{2}\cos 2t-3\sin 2t-\frac{5}{4} \\ z_1(t) &= 3\cos 2t+\frac{13}{4}\sin 2t+\frac{3}{2}t \end{aligned} \tag{38}$$

Solving integral equations by new generalized transform

In this section, with the help of generalized integral transform, we solve two linear and one non-linear Volterra integral equations and exact solutions are found in all cases.

Example 5. Volterra integral equation of the second kind

Consider:

$$z(t)-\int_0^t (1+\tau)z(t-\tau)\mathrm{d}\tau = 1-\sinh t \tag{39}$$

By taking the generalized transform on each side of previous equation and then using the convolution theorem, we have:

$$\mathcal{H}(s)-\frac{1}{p(s)}\Big[\mathrm{H}\{1+t\}\mathrm{H}\{z(t)\}\Big] = \frac{p(s)}{s^n}-\frac{p(s)}{s^{2n}-1}$$

this is equivalent to

$$\mathcal{H}(s)-\left[\frac{1}{s^n}+\frac{1}{s^{2n}}\right]\mathcal{H}(s) = p(s)\left[\frac{1}{s^n}-\frac{1}{s^{2n}-1}\right]$$

After simple calculation, we reach:

$$\mathcal{H}(s)=\frac{p(s)s^n}{s^{2n}-1}$$

By using H^{-1}, the solution can be expressed:

$$z(t)=\cosh t \tag{40}$$

Example 6. Volterra integral equation of the first kind

Suppose the Volterra integral equation of the first kind:

$$\mathrm{e}^t-\sin t-\cos t=\int_0^t 2\mathrm{e}^{t-\tau}z(\tau)\mathrm{d}\tau \tag{41}$$

By using the generalized transform and the convolution theorem, we have:

$$\frac{p(s)}{s^n-1}-\frac{p(s)}{s^{2n}+1}-\frac{p(s)s^n}{s^{2n}-1}=\frac{2}{p(s)}\left[\mathrm{H}\left\{\mathrm{e}^t\right\}\mathrm{H}\left\{z(t)\right\}\right]$$

this in turn gives

$$p(s)\left[\frac{1}{s^n-1}-\frac{1}{s^{2n}+1}-\frac{s^n}{s^{2n}-1}\right]=\mathcal{H}(s)\left[\frac{2}{s^n-1}\right]$$

Simple calculation yields:

$$\mathcal{H}(s)=\frac{p(s)}{s^{2n}+1}$$

By utilizing H^{-1}, the solution can be depicted:

$$z(t)=\sin t \tag{42}$$

Example 7. Non-linear Volterra integral equation of the first kind

Consider the non-linear Volterra integral equation of the first kind:

$$\frac{1}{4}\mathrm{e}^{2t}-\frac{1}{2}t-\frac{1}{4}=\int_0^t (t-\tau)z^2(\tau)\mathrm{d}\tau \tag{43}$$

We first let:

$$v(t)=z^2(t),\quad z(t)=\pm\sqrt{v(t)}$$

Equation (43) gets the form:

$$\frac{1}{4}\mathrm{e}^{2t}-\frac{1}{2}t-\frac{1}{4}=\int_0^t (t-\tau)v(\tau)\mathrm{d}\tau$$

With the help of generalized transform and convolution theorem:

$$\frac{p(s)}{4\left(s^n-2\right)}-\frac{p(s)}{2s^{2n}}-\frac{p(s)}{s^n}=\frac{1}{p(s)}\left[\mathrm{H}\{t\}\mathrm{H}\left\{v(t)\right\}\right]$$

or equivalently

$$\frac{p(s)}{4}\left[\frac{1}{s^n-2}-\frac{2}{s^{2n}}-\frac{1}{s^n}\right]=\mathcal{H}(s)\left[\frac{1}{s^{2n}}\right]$$

Simple calculation results:

$$\mathcal{H}(s)=\frac{p(s)}{s^n-2}$$

By operating H^{-1}, we have:

$$v(t)=\mathrm{e}^{2t}$$

The exact solutions are therefore, can be expressed:

$$z(t)=\pm\mathrm{e}^{t} \tag{44}$$

It is important to note that there were two solutions found since equation is a non-linear and there may not be a single solution.

Discussion and conclusion

In this article, we introduce a new generalized integral transform unifying the Laplace, Fourier and many other integral transforms belonging to the family of Laplace transform as special cases as shown in tab. 3. This unification has offered many opportunities for engineering applications.

A methodology is developed for ODE with constant coefficients, and after that, this strategy is used to resolve two different systems. It is concluded that Laplace and all of the integral transforms from its family result in the same solution. In other words, the solution is unchanged by the choice of integral transform method for the case of ODE with constant coefficients. This property is advantageous because it allows mathematicians and engineers to choose the integral transform that is best applicable to a particular scenario or that simplifies the arithmetic while still being confident in the final solution.

We have also constructed a technique for the system of ODE and established the fact that the Laplace transform is recognized as the best method for resolving these systems with constant coefficients due to its efficiency in terms of computational requirements. This transform facilitates the manipulation of the equations, eliminates the need for recurrent differentiation, and frequently facilitates the derivation of closed-form solutions because there are many interrelate equations in this circumstance.

In the last part, we have solved linear and non-linear Volterra integral equations and noted that a linear equation has a unique solution while there may not be a single solution for the non-linear case. Non-linear equations commonly cause mathematical complexities, demanding for specialized techniques and methodologies and this generalized transform has done remarkably well for the said case. In addition complicating the problem, more than one solution gained by the proposed transform helps us understand the underlying system by shedding light on a variety of possible outcomes or scenarios.

We have studies all problems with constant coefficients, but for variable coefficients, whole scenarios will be changed and the choice of transform becomes much importance. We will develop methodologies based on generalized integral transform that will solve ODE with variable coefficients, such as those with polynomials, exponential and trigonometric coefficients in the future work. Consequently, when coupled with the homotopy perturbation method [15], the variational iteration method [16], and the Adomian decomposition method [41], the new integral transform becomes a promising tool to non-linear problems and fractal/fractional differential equations, and it is transformative for research because the transform has the same essential qualities as those for Laplace and Fourier transform.

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