A NEW (2+1)-D MZK-BURGERS MODEL FOR NON-LINEAR ROSSBY WAVES ASWELL AS THE ANALYTICAL SOLUTION

by

***Wei WANG*[a,b] *and Bao-Jun ZHAO*[a,c*]**

[a]Yangzhou Polytechnic Institute, Yangzhou, Jiangsu, China
[b]College of Economics and Management, Nanjing University of Aeronautics and Astronautics, Nanjing, Jiangsu, China
[c]Key Laboratory of Ministry of Education for Coastal Disaster and Protection, Hohai University, Nanjing, Jiangsu, China

Original scientific paper
https://doi.org/10.2298/TSCI2305883W

In the paper, based on the quasi-geostrophic potential vorticity equation with topography effect, we derived a modified ZakharovKuznetsor (mZK)-Burgers equation by employing multiscale analysis and perturbation method. The model can be described the propagation of the nonlinear long wave and solitary eddy. The exact solutions are given by virtue of the (G'/G)-expansion method to analyze wave propagation characteristics.

Key words*: mZK-Burgers equation, (G'/G)-expansion method, topography effect*

Introduction

The Rossby wave is a large-scale wave with a long lifespan generated by the combined effect of spherical effect and Earth rotation. Its horizontal scale can be compared to the radius of the Earth. One of the nonlinear wave structures has similar characteristics to stable large amplitude solitary waves, and many problems in non-linear atmospheric and oceanic dynamics can be attributed to the study of the evolution of large amplitude nonlinear Rossby waves [1]. In the early stages, the KdV and the modified KdV (mKdV) [2, 3] were derived to describe non-linear wave amplitudes. Afterwards, Boussinesq equation [4], Schrodinger equation [5], and Boussinesq-BO equation [6] are more suitable due to the multi-dimensional nature of fluid motion. Over the years, Yang *et al*. [7] has derived a new ZK-BO equation for 3-D algebraic Rossby solitary waves. The model is gradually developing from low dimensional to high dimensional, and the influencing factors considered are becoming more comprehensive.

Many scholars have devoted themselves to finding analytical solutions to explain the properties of wave motion, such as Homogeneous Balance Method [8], Backlund transformation [9, 10], Jacobi elliptic function expansion method [11, 12], Hirota's method [13], and extended tanh-function method and the (G'/G)-expansion method [14]. Different partial

*Corresponding author, e-mail: break200@163.com

differential modes need to be matched with appropriate methods. In this article, we solve the equation using the (G'/G)-expansion methodaccording to the new model.

Derivation of forced (2+1)-D mZK-Burgers equation

According to Earth's fluid dynamics, the dimensionless quasi-geostrophic vortex equation can be written [15]:

$$\left(\frac{\partial}{\partial T}+\frac{\partial\Psi}{\partial X}\frac{\partial}{\partial Y}-\frac{\partial\Psi}{\partial Y}\frac{\partial}{\partial X}\right)[\nabla^2\Psi+f_0+\beta Y+Kh(y)]=0 \quad (1)$$

where Ψ is the total stream function, $\nabla^2=(\partial^2/\partial X^2)+(\partial^2/\partial Y^2)$ is the Laplace operator, f_0 and β are called Coriolis parameter and Rossby parameter, $h(y)$ is the topography effect, and $K=(f_0L/HU)D$.

The boundary condition is:

$$\frac{\partial\Psi}{\partial X}=0,\ Y=0,\ 1 \quad (2)$$

We assume the total stream function:

$$\Psi(X,Y,T)=-\int_0^Y[U(s)-c_0+\delta\gamma]\mathrm{d}s+\varphi(X,Y,T) \quad (3)$$

where φ is disturbed stream function, $U(Y)$ is latitude shear flow, c_0 is wave velocity, $\delta(\delta << 1)$ is a parameter, γ is a detuning parameter with $O(\gamma)=1$.

We take a simplified perturbation approach, and assume that:

$$x=\delta^{\frac{1}{2}}X,\ y=\delta^{\frac{1}{2}}Y,\ t=\delta^{\frac{3}{2}}T \quad (4)$$

another expression of eq. (4):

$$\frac{\partial}{\partial X}=\delta^{\frac{1}{2}}\frac{\partial}{\partial x},\ \frac{\partial}{\partial Y}=\frac{\partial}{\partial Y}+\delta^{\frac{1}{2}}\frac{\partial}{\partial y},\ \frac{\partial}{\partial T}=\delta^{\frac{3}{2}}\frac{\partial}{\partial t} \quad (5)$$

Substituting eqs. (3) and (5) into eq. (1), we obtain:

$$\delta^{\frac{5}{2}}\frac{\partial^3\Psi}{\partial x^2\partial t}+\delta^{\frac{3}{2}}\left(\frac{\partial^3\Psi}{\partial Y^2\partial t}+2\delta^{\frac{1}{2}}\frac{\partial^3\Psi}{\partial Y\partial y\partial t}+\delta\frac{\partial^3\Psi}{\partial y^2\partial t}\right)+$$

$$+\delta^{\frac{1}{2}}\frac{\partial\Psi}{\partial x}\begin{pmatrix}\delta^2\frac{\partial^3\Psi}{\partial Y\partial x^2}+\delta^{\frac{1}{2}}\frac{\partial^3\Psi}{\partial y\partial x^2}\\ -\frac{\mathrm{d}^2U}{\mathrm{d}Y^2}+\frac{\partial^3\Psi}{\partial Y^3}+2\delta^{\frac{1}{2}}\frac{\partial^3\Psi}{\partial Y^2\partial y}+\delta\frac{\partial^3\Psi}{\partial Y\partial y^2}+\delta^{\frac{1}{2}}\frac{\partial^3\Psi}{\partial Y^2\partial y}+2\delta^{\frac{1}{2}}\frac{\partial^3\Psi}{\partial Y\partial y^2}+\delta\frac{\partial^3\Psi}{\partial y^3}+\beta\\ +K\frac{\partial}{\partial Y}h(y,Y)\end{pmatrix}+ \quad (6)$$

$$+(U-c_0+\delta\gamma)\left(\delta^{\frac{1}{2}}\frac{\partial^3\Psi}{\partial Y^2\partial x}+2\delta^{\frac{1}{2}}\frac{\partial^3\Psi}{\partial Y\partial y\partial x}+\delta\frac{\partial^3\Psi}{\partial y^2\partial x}+\delta^{\frac{3}{2}}\frac{\partial^3\Psi}{\partial x^3}\right)=0$$

$$\frac{\partial\varphi}{\partial x}=0,\ Y=0,1 \quad (7)$$

Adopting the perturbation expansion method, we assume:

$$\varphi(x,Y,y,t)=\delta\Psi_1+\delta^{\frac{3}{2}}\Psi_2+\delta^2\Psi_3+\cdots \tag{8}$$

Substituting eq. (8) into eqs. (6) and (7), specifically, we discuss the large topography effect, where the magnitude of K is $O(K) = 1$ we derive the multi order expression of δ and make the expression to zero:

$$O(\delta^{\frac{3}{2}}):\begin{cases}(U-c_0)\left(\dfrac{\partial^3\Psi_1}{\partial Y^2\partial x}\right)+\left(\beta+K\dfrac{\partial}{\partial Y}h(y,Y)-\dfrac{\mathrm{d}^2U}{\mathrm{d}Y^2}\right)\left(\dfrac{\partial\Psi_1}{\partial x}\right)=0\\ \dfrac{\partial\Psi_1}{\partial x}=0,\quad Y=0,1\end{cases} \tag{9}$$

$$O(\delta^2):\begin{cases}(U-c_0)\left(\dfrac{\partial^3\Psi_2}{\partial Y^2\partial x}\right)+\left(\beta+K\dfrac{\partial}{\partial Y}h(y,Y)-\dfrac{\mathrm{d}^2U}{\mathrm{d}Y^2}\right)\left(\dfrac{\partial\Psi_2}{\partial x}\right)=-2(U-c_0)\left(\dfrac{\partial^3\Psi_1}{\partial Y\partial x\partial y}\right)\\ \dfrac{\partial\Psi_2}{\partial x}=0,\quad Y=0,1\end{cases} \tag{10}$$

$$O(\delta^{\frac{5}{2}}):\begin{cases}(U-c_0)\left(\dfrac{\partial^3\Psi_3}{\partial Y^2\partial x}\right)+\left(\beta+K\dfrac{\partial}{\partial Y}h(y,Y)-\dfrac{\mathrm{d}^2U}{\mathrm{d}Y^2}\right)\left(\dfrac{\partial\Psi_3}{\partial x}\right)=-G\\ \dfrac{\partial\Psi_3}{\partial x}=0,\ Y=0,\ 1\end{cases} \tag{11}$$

where

$$\begin{gathered}G=(U-c_0)\left(\frac{\partial^3\Psi_1}{\partial x^3}\right)+(U-c_0)\left(\frac{\partial^3\Psi_1}{\partial y^2\partial x}\right)+\\ +2(U-c_0)\left(\frac{\partial^3\Psi_2}{\partial Y\partial y\partial x}\right)+\left(\frac{\partial\Psi_1}{\partial x}\right)\left(\frac{\partial^3\Psi_1}{\partial Y^3}\right)+\frac{\partial^3\Psi_1}{\partial Y^2\partial t}+\left(\frac{\partial^3\Psi_1}{\partial Y^2\partial x}\right)\gamma\end{gathered} \tag{12}$$

We assume $\Psi_1 = A_1(x,y,t)\Phi_1(Y)$ and substitute it into eq. (13):

$$\Phi_1''+\frac{\beta+K\dfrac{\partial}{\partial Y}h(y,Y)-U''}{U-c_0}\Phi_1=0,\quad \Phi_1(0)=\Phi_1(1)=0 \tag{13}$$

In eq. (14), $U - c_0$ □0, we suppose $\Psi_2 = A_2(x,y,t)\Phi_2(Y)$ and substitute Ψ_2 into $O(\delta^2)$:

$$O(\delta^2):\begin{cases}\Phi_2''(Y)\dfrac{\partial A_2}{\partial x}+\dfrac{\beta+K\dfrac{\partial}{\partial Y}h(y,Y)-U''}{U-c_0}\Phi_2(Y)\dfrac{\partial A_2}{\partial x}=-2\dfrac{\partial^2A_1}{\partial x\partial y}\Phi_1'(Y)\\ \dfrac{\partial\Psi_2}{\partial x}=0,\quad Y=0,1\end{cases} \tag{14}$$

According to eqs. (14) and (15), we obtain:

$$\frac{\partial A_2}{\partial x}=\frac{\partial^2A_1}{\partial x\partial y}\rightarrow\frac{\partial^2A_2}{\partial x\partial y}=\frac{\partial^3A_1}{\partial x\partial y^2} \tag{15}$$

So

$$\Phi_2''(Y)+\frac{\beta+K\dfrac{\partial}{\partial Y}h(y,Y)-U''}{\left(U-c_0\right)}\Phi_2(Y)=-2\Phi_1'(Y),\quad \Phi_2(0)=\Phi_2(1)=0 \tag{16}$$

We suppose $\Psi_3 = A_3(x,y,t)\Phi_3(Y)$, because of $\Phi_1(0) = \Phi_1(1) = 0$, with an identity relationship:

$$\phi_1\frac{\partial^2\phi_3}{\partial y^2}=\frac{\partial}{\partial y}\left(\phi_1\frac{\partial\phi_3}{\partial \mathrm{y}}\right)-\frac{\partial}{\partial y}\left(\phi_3\frac{\partial\phi_1}{\partial \mathrm{y}}\right)+\phi_3\frac{\partial^2\phi_1}{\partial y^2} \tag{17}$$

For eqs. (12), (13), (18), the singularity elimination condition can be obtained

$$\int_0^1\frac{G}{U-c_0}\mathrm{d}Y=0$$

To simplify writing, let $A_1 = B$ after a series of calculations, and we finally obtained:

$$B_t+e_1B_xB+e_2B_{xxx}+e_3B_{xyy}+e_4B_x=0 \tag{18}$$

where

$$e_1=\frac{\int_0^1\Phi^2{}_1\Phi_1'''\mathrm{d}Y}{\int_0^1\Phi_1\Phi_2''(Y)\mathrm{d}Y},\quad e_2=\frac{\int_0^1\Phi^2{}_1\mathrm{d}Y}{\dfrac{1}{U-c_0}\int_0^1\Phi_1\Phi_2''(Y)\mathrm{d}Y}$$

$$e_3=\frac{\int_0^1(\Phi^2{}_1+2\Phi_1\Phi_2'(Y))\mathrm{d}Y}{\dfrac{1}{U-c_0}\int_0^1\Phi_1\Phi_2''(Y)\mathrm{d}Y} \tag{19}$$

$$e_4=\frac{\gamma\left(\int_0^1\Phi_1\Phi_2''(Y)\mathrm{d}Y\right)}{\int_0^1\Phi_1\Phi_2''(Y)\mathrm{d}Y}$$

where B_xB represents nonlinear convection term, B_{xxx} and B_{xyy} represent the dispersive term. Equation (19) is called the (2+1)-D mZK-Burgers equation.

Analytical solutions of (2+1)-S mZK-Burgers equation

According to the (G'/G)-expansion method [16], we give the traveling wave transformation to obtain the solitary wave solution of the equation:

$$\eta_{(X,Y,T)}=u_{(\xi)},\xi=kX+lY-cT \tag{20}$$

Substituting eq. (20) into eq. (19), we can get:

$$B_t+e_1B_xB+e_2B_{xxx}+e_3B_{xyy}+e_4B_x=0 \tag{21}$$

Then we integrate eq. (21) twice:

$$(e_2k^3+e_3kl^2)B_{\xi\xi}+\frac{1}{2}e_1k(B^2)+(e_4l-c)B=0 \tag{22}$$

We give the solution to eq. (22) in the following form:

$$B(\xi)=\sum_{i=0}^{n} b_i (G'/G)^i, \quad G''+\lambda G'+\mu G=0 \tag{23}$$

where $G = G(\xi)$ satisfies the second order linear ODE:

$$G''+\lambda G'+\mu G=0 \tag{24}$$

where λ and μ are constants. Taking the homogeneous balance between $B_\xi B$ and $B_{\xi\xi\xi}$ in eq. (22), we obtain $n = 2$:

$$B_{(\xi)}=b_0+b_1\frac{G'(\xi)}{G(\xi)}+b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2 \tag{25}$$

We obtain the value of the parameter b_0, b_1, b_2, and l through Maple software:

$$b_0=\frac{48\mu\left(ce_3k\left(\frac{-\lambda^2}{4}+\mu\right)-\frac{e_4^2}{8}+\frac{e_4\sqrt{\left(-16k\left(4e_2k^3\left(\frac{-\lambda^2}{4}+\mu\right)+c\right)\left(\frac{-\lambda^2}{4}+\mu\right)e_3+e_4^2\right)}}{8}\right)}{e_1e_3k^2(\lambda^2-4\mu)^2} \tag{26}$$

$$b_1=\frac{6\lambda\left(-2ce_3\mathrm{k}(\lambda^2-4\mu)-e_4^2+e_4\sqrt{\left(-16k\left(4e_2k^3\left(\frac{-\lambda^2}{4}+\mu\right)+c\right)\left(\frac{-\lambda^2}{4}+\mu\right)e_3+e_4^2\right)}\right)}{e_1e_3k^2(\lambda^2-4\mu)^2} \tag{27}$$

$$b_2=\frac{48ce_3\mathrm{k}\left(\frac{-\lambda^2}{4}+\mu\right)-6e_4^2+6e_4\sqrt{\left(-16k\left(4e_2\left(\frac{-\lambda^2}{4}+\mu\right)k^3+c\right)\left(\frac{-\lambda^2}{4}+\mu\right)e_3+e_4^2\right)}}{e_1e_3k^2(\lambda^2-4\mu)^2} \tag{28}$$

and

$$l=\frac{-e_4+\sqrt{\left(-16k\left(4e_2\left(\frac{-\lambda^2}{4}+\mu\right)k^3+c\right)\left(\frac{-\lambda^2}{4}+\mu\right)e_3+e_4^2\right)}}{2e_3k\left(\lambda^2-4\mu\right)} \tag{29}$$

Finally, we obtain the solitary wave solution of eq. (19).

Conclusion

In our work, based on quasi-geostrophic potential vorticity model, we obtained a new mZK- Burgers model that characterizes nonlinear long waves in Earth's fluids through perturbation expansion and scale transformation. Different from the previous topography forcing effect, lower order models including topography affect spatial structure because of the large topography effect. The form of the theoretical solution reflects the characteristics of solitary waves.

Acknowledgment

This work is supported by Key Laboratory of Ministry of Education for Coastal Disaster and Protection, Hohai University (Grant No. 202201).

Nomenclature

t – time, [s]

x,y,z – co-ordinates, [m]

References

[1] Pedlosky, J., *Geophysical fluid dynamics*, Springer, New York, USA, 1979
[2] Hirota, R., Exact Solution of the Korteweg-de Vriesequation for Multiple Collisions of Solitons, *Physical Review Letters*, *27* (1971), 18, pp. 1192-1194
[3] Yang, X. J., *et al.*, On Exact Traveling-Wave Solutions for Local Fractional Korteweg-de Vries Equation, *Chaos*, *26* (2016), 8, pp. 1-6
[4] Yang, X. J., *et al.*, Exact Traveling-Wave Solution for Local Fractional Boussinesq Equation in Fractal Domain, *Fractals*, *25* (2017), 4, ID 1740006
[5] Liu, W., *et al.*, Some Generalized Coupled Nonlinear Schrödinger Equations and Conservation Laws, *Modern Physics Letters B*, *31* (2017), 32, ID 1750299
[6] Yang, H. Y., *et al.*, Time-Fractional Benjamin-Ono Equation for Algebraic Gravity Solitary Waves in Baroclinic Atmosphere and Exact Multi-Soliton Solution as well as Interaction, *Communications in Nonlinear Science and Numerical Simulation*, *71* (2019), 25, pp. 187-201
[7] Yang, H. W., *et al.*, A New ZK–BO Equation for Three-Dimensional Algebraic Rossby Solitary Waves and its Solution as well as Fission Property, *Nonlinear Dynamics*, *91* (2017), 3, pp. 2019-2032
[8] Fan, E., *et al.*, A Note on the Homogeneous Balance Method, *Physics Letters A*, *246* (1998), 5, pp. 403-406
[9] Zhang, Y., *et al.*, A Modified Bäcklund Transformation and Multi-Soliton Solution for the Boussinesq Equation, *Chaos, Solitons & Fractals*, *23* (2005), 1, pp. 175-181
[10] Yin, H. M., *et al.*, Solitons and Bilinear Bäcklund Transformations for a (3+1)-Dimensional Yu-Toda-Sasa-Fukuyama Equation in a Liquid or Lattice, *Applied Mathematics Letters*, *58* (2016), 5, pp. 178-183
[11] Chen, Y., *et al.*, Extended Jacobi Elliptic Function Rational Expansion Method and Abundant Families of Jacobi Elliptic Function Solutions to (1+1)-Dimensional Dispersive Long Wave Equation, *Chaos, Solitons & Fractals*, *24* (2005), 3, pp. 745-757
[12] Abdou, M. A., *et al.*, Construction of Periodic and Solitary Wave Solutions by the Extended Jacobi Elliptic Function Expansion Method, *Communications in Nonlinear Science & Numerical Simulation*, *12* (2007), 7, pp. 1229-1241
[13] Liu, W. J., Soliton Interaction in the Higher-Order Nonlinear Schrödinger Equation Investigated with Hirota's Bilinear Method, *Physical Review E*, *77* (2008), 6, ID 066605
[14] Wang, M., *et al.*, The (G'/G)-Expansion Method and Travelling Wave Solutions of Nonlinear Evolution Equations in Mathematical Physics, *Physics Letters A*, *372* (2008), 4, pp. 417-423
[15] Zhao, B. J., *et al.*, Forced Solitary Wave and Vorticity with Topography Effect in Quasi-Geostrophic Modelling, *Advances in Mechanical Engineering*, *15* (2023), 1, pp. 39-49
[16] Naher, H., *et al.*, The Basic (G'/G)-Expansion Method for the Fourth Order Boussinesq Equation, *Applied Mathematics*, *3* (2012), 10, pp. 1144-1152

Paper submitted: March 9, 2023
Paper revised: May 19, 2023
Paper accepted: June 20, 2023