NEW SOLITON SOLUTIONS FOR THE LOCAL FRACTIONAL VAKHNENKO-PARKES EQUATION

by

Zhi-Yong FAN *

Institute of Applied Mathematics, Jiaozuo Normal College, Jiaozuo, Henan, China

Original scientific paper
https://doi.org/10.2298/TSCI2305877F

In this paper, we mainly consider the local fractional Vakhnenko-Parkes equation with the local fractional derivative for the first time. Some new soliton solutions of local fractional Vakhnenko-Parkes equation are derived by using local fractional wave method. These obtained soliton solutions suggest that this proposed approach is effective, simple and reliable. Finally, the physical characteristics of these new soliton solutions are described through 3-D figures.

Key words*: local fractional derivative, local fractional wave method local fractional Vakhnenko-Parkes equation*

Introduction

Various physical and natural phenomena in fluid mechanics, ocean engineering, space science, biological science, plasma physics, new materials science and polymer chemistry are modeled by nonlinear evolution equations [1-3]. With the development of non-linear science, directly searching the exact travelling wave solutions of non-linear evolution equations has become a very hot topic. There are many powerful methods for calculating the travelling wave solutions of non-linear evolution equations in current literature, such as Hirota's method [4], Functional variable method [5], (m+G'/G)-expansion method [6], exp-function method [7], modified auxiliary equation method [8], two-scale wave method [9, 10], Adams-Bashforth-Moulton method [11], Laplace variational method [12], *etc*. [13-19].

The Vakhnenko-Parkes equation (VPE) is an important non-linear evolution equation in physics, which is utilized to elaborate the propagation of waves in a relaxing medium. The (3+1)-D VPE is given as [20]:

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial t}+w\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial z}\right)w+w=0 \tag{1}$$

which can be rewritten into the following form:

$$\frac{\partial^3 w}{\partial x^2\partial t}+\frac{\partial^3 w}{\partial x^2\partial z}+\frac{\partial^3 w}{\partial x^2\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}+\frac{\partial w}{\partial x}=0 \tag{2}$$

* Author's e-mail: zyongfan@163.com

So far, the travelling wave solution of VPE has been derived by many scholars via using different mathematical methods. Ozkan, *et al.* [20] used the exp-function technique to gain the N-soliton solutions of the (3+1)-D VPE. Baskonus, *et al.* [21] obtained the travelling wave solition of VPE by utilizing the sine-Gordon expansion method (SGEM). Wazwaz [22] investigated the travelling wave solutions of VPE and modified VPE via the simplified Hirota's method (SHM), and gained their multiple complex soliton solutions with great ease.

Because of the propagation of waves with unsmooth boundaries or in microgravity [23], the VPE has to be modified by using the local fractional derivative. The local fractional derivative is a powerful mathematical tool to describe complex natural phenomena [24, 25].

The local fractional VPE is the following form:

$$\begin{aligned} &{}_L D^{2\alpha,\alpha}_{x,t} w + {}_L D^{2\alpha,\alpha}_{x,z} w + {}_L D^{2\alpha,\alpha}_{x,y} w + \left({}_L D^{\alpha}_{x} w\right)\left({}_L D^{\alpha}_{t} w\right) + \\ &+\left({}_L D^{\alpha}_{x} w\right)\left({}_L D^{\alpha}_{z} w\right) + \left({}_L D^{\alpha}_{x} w\right)\left({}_L D^{\alpha}_{y} w\right) + {}_L D^{\alpha}_{x} w = 0 \end{aligned} \tag{3}$$

where ${}_L D^{\alpha}_{x}$ is local fractional derivative [14, 25].

The VPE is defined by employing the local fractional derivative sense for the first time, which is called local fractional Vakhnenko-Parkes equation (LFVPE). In this paper, we mainly investigate the LFVPE. We successfully establish an efficient and simple scheme to acquire the soliton solutions of LFVPE. The new mathematical scheme is constructed based on the travelling wave transform, which is local fractional wave method (LFWM). The LFWM has the advantage of being direct and easy to operate, and then new soliton solutions of LFVPE are obtained. These new soliton solutions have not appeared in the existing literature. Furthermore, the dynamical behaviors of these new soliton solutions are shown through 3-D figures.

Local fractional derivative

Definition 1. The local fractional derivative of $w(x,y,z,t)$ of α at $x = x_0$ is defined as [14, 25]:

$$ {}_L D^{\alpha}_{x} w = \lim_{\varepsilon\to 0} \frac{\Delta^{\alpha}\left(w(x,y,z,t) - w(x_0,y,z,t)\right)}{\left(x - x_0\right)^{\alpha}} \tag{4}$$

where:

$$\Delta^{\alpha}\left(w\left(x,y,z,t\right) - w\left(x_0,y,z,t\right)\right) \cong \Gamma\left(1+\alpha\right)\Delta\left(w\left(x,y,z,t\right) - w\left(x_0,y,z,t\right)\right) \tag{5}$$

Definition 2. The local Mittag-Leffler function on the Cantor sets is defined as [25]:

$$ML_{\alpha}(x^{\alpha}) = \sum_{m=0}^{\infty} \frac{x^{m\alpha}}{\Gamma(1+m\alpha)} \tag{6}$$

Some special functions are the following form [2, 25]:

$$\sin_{\alpha}(x^{\alpha}) = \frac{ML_{\alpha}(i^{\alpha}x^{\alpha}) - ML_{\alpha}(-i^{\alpha}x^{\alpha})}{2i^{\alpha}} \tag{7}$$

$$\csc_{\alpha}(x^{\alpha}) = \frac{2i^{\alpha}}{ML_{\alpha}(i^{\alpha}x^{\alpha}) - ML_{\alpha}(-i^{\alpha}x^{\alpha})} \tag{8}$$

$$\operatorname{sech}_\alpha(x^\alpha) = \frac{2}{ML_\alpha(x^\alpha) + ML_\alpha(-x^\alpha)} \tag{9}$$

A typical application

Consider the local fractional VPE as:

$$\begin{gathered} {}_L D^{2\alpha,\alpha}_{x,t} w + {}_L D^{2\alpha,\alpha}_{x,z} w + {}_L D^{2\alpha,\alpha}_{x,y} w + \left({}_L D^\alpha_x w\right)\left({}_L D^\alpha_t w\right) + \\ + \left({}_L D^\alpha_x w\right)\left({}_L D^\alpha_z w\right) + \left({}_L D^\alpha_x w\right)\left({}_L D^\alpha_y w\right) + {}_L D^\alpha_x w = 0 \end{gathered} \tag{10}$$

We now present the following fractional wave transformation:

$$W = w(\eta^\alpha), \quad \eta^\alpha = c^\alpha x^\alpha + d^\alpha y^\alpha + e^\alpha z^\alpha + f^\alpha t^\alpha \tag{11}$$

Consequently, eq. (10) is transformed into the following form:

$$\frac{d^{3\alpha} W}{d\eta^{3\alpha}} + \frac{1}{c^\alpha}\left(\frac{d^\alpha W}{d\eta^\alpha}\right)^2 + \frac{1}{\left(f^\alpha + e^\alpha + d^\alpha\right)}\frac{d^\alpha W}{d\eta^\alpha} = 0 \tag{12}$$

Assume that:

$$\phi = \frac{d^\alpha W}{d\eta^\alpha} \tag{13}$$

Then, we have:

$$\frac{d^{2\alpha}\phi}{d\eta^{2\alpha}} + \frac{1}{c^\alpha}\phi^2 + \frac{1}{f^\alpha + e^\alpha + d^\alpha}\phi = 0 \tag{14}$$

Multiplying $d^\alpha\phi/d\eta^\alpha$ in eq. (14), we have:

$$\frac{1}{2}\frac{d^\alpha}{d\eta^\alpha}\left(\frac{d^\alpha\phi}{d\eta^\alpha}\right)^2 + \frac{1}{2\left(f^\alpha + e^\alpha + d^\alpha\right)}\frac{d^\alpha\phi^2}{d\eta^\alpha} + \frac{1}{3c^\alpha}\frac{d^\alpha\phi^3}{d\eta^\alpha} = 0 \tag{15}$$

By calculating eq. (15), we get:

$$\left(\frac{d^\alpha\phi}{d\eta^\alpha}\right)^2 + \frac{1}{f^\alpha + e^\alpha + d^\alpha}\phi^2 + \frac{2}{3c^\alpha}\phi^3 = 0 \tag{16}$$

Assume that the solution of eq. (16) is the following form:

$$\varphi = A\operatorname{sec}h^2_\alpha(B\eta^\alpha) = \frac{4A}{\left\{ML_\alpha(B\eta^\alpha) + ML_\alpha(-B\eta^\alpha)\right\}^2} \tag{17}$$

Therefore, we have the following results:

$$\begin{gathered} \left(\frac{d\phi}{d\eta^\alpha}\right)^2 = \frac{64A^2B^2\left\{ML_\alpha(B\eta^\alpha) - ML_\alpha(-B\eta^\alpha)\right\}^2}{\left\{ML_\alpha(B\eta^\alpha) + ML_\alpha(-B\eta^\alpha)\right\}^6} = \frac{64A^2B^2\left\{(ML_\alpha(B\xi^\alpha) + ML_\alpha(-B\xi^\alpha))^2 - 4\right\}}{\left\{ML_\alpha(B\eta^\alpha) + ML_\alpha(-B\eta^\alpha)\right\}^6} = \\ = \frac{64A^2B^2}{\left\{ML_\alpha(B\eta^\alpha) + ML_\alpha(-B\eta^\alpha)\right\}^4} - \frac{256A^2B^2}{\left\{ML_\alpha(B\xi^\alpha) + ML_\alpha(-B\xi^\alpha)\right\}^6} = 4B^2\phi^2 - \frac{4B^2}{A}\phi^3 \end{gathered} \tag{18}$$

By comparing the coefficients of eqs. (18) and (16), we may get:

$$4B^2 = \frac{1}{f^\alpha + e^\alpha + d^\alpha}, \quad -\frac{4B^2}{A} = \frac{2}{3c^\alpha} \tag{19}$$

Solving eq. (19), we obtain:

$$B = \sqrt{\frac{1}{4\left(f^\alpha + e^\alpha + d^\beta\right)}}, \ A = -\frac{3c^\alpha}{2\left(f^\alpha + e^\alpha + d^\alpha\right)} \tag{20}$$

Hence, the soliton solution of eq. (16) is obtained as:

$$\varphi = -\frac{3c^\alpha}{2\left(f^\alpha + e^\alpha + d^\alpha\right)} \operatorname{sech}_\alpha^2 \left(\sqrt{\frac{1}{4\left(f^\alpha + e^\alpha + d^\alpha\right)}} (c^\alpha x^\alpha + d^\alpha y^\alpha + e^\alpha z^\alpha + f^\alpha t^\alpha) \right) \tag{21}$$

The soliton solution of eq.(10) is derived as:

$$w = \frac{3c^\alpha \sinh_\alpha \left(\sqrt{\frac{1}{4\left(f^\alpha + e^\alpha + d^\alpha\right)}} (c^\alpha x^\alpha + d^\alpha y^\alpha + e^\alpha z^\alpha + f^\alpha t^\alpha) \right)}{\left(e^\alpha + f^\alpha + d^\alpha\right) \sqrt{\frac{1}{\left(e^\alpha + f^\alpha + d^\alpha\right)}} \cosh_\alpha \left(\sqrt{\frac{1}{4\left(f^\alpha + e^\alpha + d^\alpha\right)}} (c^\alpha x^\alpha + d^\alpha y^\alpha + e^\alpha z^\alpha + f^\alpha t^\alpha) \right)} \tag{22}$$

The corresponding graphs of eq. (21) are plotted in fig. 1 with fractal dimension α = ln2/in3 and fractal parameters $c^\alpha = 1$, $d^\alpha = 3$, $e^\alpha = 4$, $f^\alpha = -1$, at $y = 0$, $z = 0$.

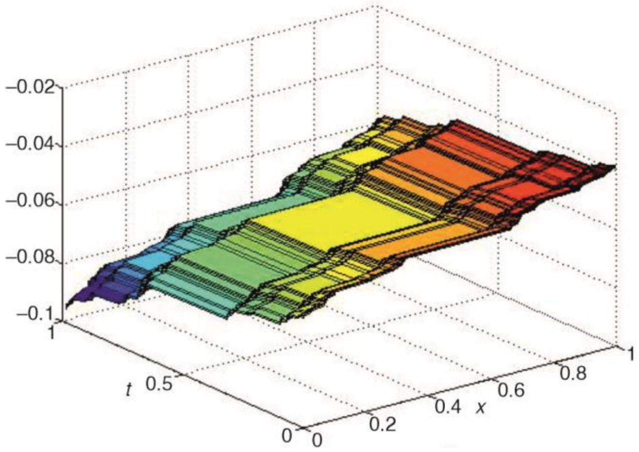

Figure 1. The corresponding graph of eq. (21) with fractal dimension α = ln2/ln3

Conclusion

In this paper, the local fractional derivative is employed to give the fractional Vakhnenko-Parkes model. We have successfully obtained the new soliton solutions of LFVPM by using LFWM. The properties of these new soliton solutions are analyzed through some 3-D graphs. The acquired results are very useful for studying the dynamics of soliton waves in complex situations in future.

Nomenclature

t – time co-ordinate, [s]

x,y,z – space co-ordinates, [m]

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Paper submitted: March 11, 2023
Paper revised: May 13, 2023
Paper accepted: June 10, 2023