NUMERICAL SOLUTIONS OF THE VISCOELASTIC PLATE OF FRACTIONAL VARIABLE ORDER

by

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The fractional variable-order constitutive model for the viscoelastic plate is analyzed. The fractional variable-order constitutive equations for the plates are solved numerically using the shifted Bernstein polynomials directly in time domain. Numerical displacement of Polyurea and HDPE viscoelastic plates at a variety of loads was investigated. The results show that Polyurea has better bending resistance than HDPE viscoelastic plates, which verifies the practicability of the algorithm.

Key words*: viscoelastic plate, the shifted Bernstein polynomial, fractional variable-order model, numerical calculation*

Introduction

The viscoelastic material is a kind of material between viscosity and elasticity, which has good noise and vibration reduction performance, and a wide range of applications in aerospace, civil machinery and many fields. The Polyurea and HDPE materials have good material properties, which have attracted the attention of many researchers [1]. With the gradual development of the viscoelastic materials, the fractional-order models have been extensively studied in [2]. Compared with fractional order models, fractional variable-order models can more effectively describe the dynamic viscoelastic behaviors when materials are deformed. Hao *et al.* [3] established the equation of a viscoelastic polymer cantilever beam with the use of a fractional variable-order model. Meng *et al.* [4] studied the effect of fractional variable-order viscoelastic models on the performance of the parameters of the order function. Cao *et al.* [5] solved a mathematical analysis of a viscoelastic column based on a fractional order-variable rheological model to solve for the deformation and stress of the column under the same loading conditions.

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The polynomial algorithms for the mathematical solutions of fractional variable-order differential equations have been relatively mature in the field of numerical computation, including Legendre polynomial method, Chebyshev wavelet method, Bernoulli wavelet method, etc. Dang *et al.* [6] used the shifted Chebyshev wavelet method for mathematical calculation of viscoelastic plates. Sun *et al.* [7] proposed using the shifted Legendre algorithm to solve the trigonometric unknown function in viscoelastic plate equation. The main aim of this article is that the governing equations for a fractional variable-order viscoelastic plate is directly resolved in time domain with the use of the shifted Bernstein polynomial method.

Basic knowledge

The Caputo fractional differential operator $D^{r(t)}$ of variable order $r(t)$ is expressed by [2, 8]:

$$D^{r(t)} f(t) = \frac{1}{\Gamma\left(1-r(t)\right)} \int_0^t \frac{f'(\kappa)}{(t-\kappa)^{r(t)}} \mathrm{d}\kappa \tag{1}$$

where $0 < r(t) \le 1$ and $f(t)$ is a continuous function.

Remark. The different fractional derivative operators were reported in [8]. Recently, new fractional derivative operators have been developed with the different kernels, *e.g.*, exponential [9, 10], fractional-exponential [11, 12], and other [13, 14].

The Bernstein polynomials on the interval [0,1] is defined by:

$$\bar{b}_{i,k} = \binom{k}{i} x^i (1-x)^{k-i}, \quad 0 \le i \le k \tag{2}$$

If we extend x to the range $[0,R]$, the common term formulation for the shifted Bernstein polynomial is:

$$\bar{B}_{i,k} = \binom{k}{i} \frac{x^i (R-x)^{k-i}}{R^k}, \quad i = 0,1,...,k \tag{3}$$

The binomial theorem gives:

$$\bar{B}_{i,k} = \sum_{m=0}^{k-i} (-1)^m \binom{k}{i} \binom{k-i}{m} \left(\frac{x}{R}\right)^{i+m} \tag{4}$$

The matrix form is represented by a sequence of the shifted Bernstein polynomials:

$$\Phi(x) = \left[\bar{B}_{0,k}(x), \bar{B}_{1,k}(x), ..., \bar{B}_{k,k}(x)\right]^T = QG_k(x) \tag{5}$$

where k is the number of terms,

$$Q = \left[q_{i,j}\right]_{i,j=0}^{k}, \quad G_k(x) = \left[1, x, ..., x^k\right]^T,$$

Q is invertible, $G_k(x) = Q^{-1}\Phi(x)$, and

$$q_{i,j} = \begin{cases} \dfrac{(-1)^{j-i} k!}{i!(k-j)!(j-i)!R^j}, & j \ge i \\ 0, & j < i \end{cases}.$$

The governing equations

We now consider a viscoelastic plate subjected to an external load $f(x,y,t)$ perpendicular to the horizontal direction of the plate, fig. 1. Assume that M_x and M_y are bending moments, M_{xy} is twisting moment, σ_x and σ_y are normal stresses, τ_{xy} is shear stress, m is mass per unit area, and $u(x,y,t)$ is the displacement of the plate along the direction of external load. The dynamic equation of the viscoelastic plate is:

$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}-m\frac{\partial^2 u(x,y,t)}{\partial t^2}+Eu(x,y,t)-\eta\frac{\partial u(x,y,t)}{\partial t}=-f(x,y,t) \tag{6}$$

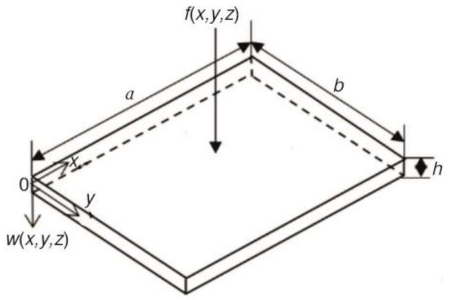

Figure 1. The model of the four-sided simple support plate

where

$$M_x=\int_{-h/2}^{h/2} z\sigma_x \mathrm{d}x,$$

$$M_y=\int_{-h/2}^{h/2} z\sigma_y \mathrm{d}y, \quad \text{and}$$

$$M_{xy}=\int_{-h/2}^{h/2} z\tau_{xy} \mathrm{dz}$$

The stress strain relation of viscoelastic plate is:

$$\sigma(x,y,t)=\left(E+\eta D_t^{r(t)}\right)\varepsilon(x,y,t) \tag{7}$$

The governing equation of the obtained plate is:

$$\begin{aligned}&D\left(\frac{\partial^4 u(x,y,t)}{\partial^4 x}+2\frac{\partial^4 u(x,y,t)}{\partial x^2\partial y^2}+\frac{\partial^4 u(x,y,t)}{\partial^4 y}\right)+\\&+\eta D D_t^r\left(\frac{\partial^4 u(x,y,t)}{\partial^4 x}+2\frac{\partial^4 u(x,y,t)}{\partial x^2\partial y^2}+\frac{\partial^4 u(x,y,t)}{\partial^4 y}\right)-m\frac{\partial^2 u}{\partial t^2}+Eu-\eta\frac{\partial u}{\partial t}=-f(x,y,t)\end{aligned} \tag{8}$$

The boundary conditions of the viscoelastic plate read:

$$u(x,y,t)\big|_{x=0,a}=\left.\frac{\partial^2 u(x,y,t)}{\partial x^2}\right|_{x=0,a}=0 \tag{9}$$

$$u(x,y,t)\big|_{x=0,b}=\left.\frac{\partial^2 u(x,y,t)}{\partial y^2}\right|_{x=0,b}=0 \tag{10}$$

The shifted Bernstein polynomial algorithm

Approximation of the displacement function

The displacement function $u(x,y,t)\in L^2([0,L]\times[0,T]\times[0,K])$ may be estimated by the shifted Bernstein polynomial:

$$u(x,y,t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}u_{ij}c_k\,\overline{B}_{i,n}(x)\overline{B}_{j,n}(y)\overline{B}_{k,n}(t)=\Phi^T(x)U\Phi(y)C^T\Phi(t) \tag{11}$$

where U and C are coefficient matrices.

Integer-order differential operator matrix

The matrix D with respect to x of the integer-order differential operators is given as:

$$\Phi'(x)=\left(QG_k(x)\right)'=Q\left(G_k(x)\right)'=QVG_k(x)=QVQ^{-1}\Phi(x)=D\Phi(x) \tag{12}$$

where $D = QVQ^{-1}$ and $V=\{v_{ij}\}_{i,j=0}^{k}$.

If we differentiate the left and right sides of eq. (12), then we have**:**

$$\Phi''(x)=D^2\Phi(x) \tag{13}$$

This implies that:

$$\Phi^{(k)}(x)=D^k\Phi(x),\;\;\Phi^{(k)}(y)=Z^k\Phi(y),\;\;\Phi^{(k)}(t)=W^k\Phi(t) \tag{14)-(16}$$

By eqs. (12) and (14), we show that:

$$\begin{aligned}\frac{\partial^4 u(x,y,t)}{\partial x^4}&=\frac{\partial^4\left(\Phi^T(x)U\Phi(y)C^T\Phi(t)\right)}{\partial x^4}=\frac{\partial^4\left(\Phi^T(x)\right)}{\partial x^4}U\Phi(y)C^T\Phi(t)\\&=D^4\Phi(x)U\Phi(y)C^T\Phi(t)\end{aligned} \tag{17}$$

Fractional differential operator matrix

The matrix N with respect to t of fractional-order differential operators is suggested as:

$$D^{r(t)}\Phi(t)=D^{r(t)}\left(QG_k(t)\right)=QD^{r(t)}\left(G_k(t)\right)=QMG_k(t)=QMQ^{-1}\Phi(t)=N\Phi(t) \tag{18}$$

where

$$N=QMQ^{-1},\;\;N=\left\{n_{ij}\right\}_{i,j=0}^{k},\;\;\text{and}\;\;n_{ij}=\begin{cases}\dfrac{\Gamma(i+1)}{\Gamma\left(i+1-r(t)\right)}t^{-r(t)}, & i=j\neq 0\\ 0, & \text{elso}\end{cases}$$

Combining eqs. (12), (14), and (17), we get:

$$\begin{aligned}D^{r(t)}\frac{\partial^4 u(x,y,t)}{\partial x^4}&=D^{r(t)}\frac{\partial^4\left(\Phi^T(x)U\Phi(y)C^T\Phi(t)\right)}{\partial x^4}=\frac{\partial^4\left(\Phi^T(x)\right)}{\partial x^4}U\Phi(y)C^TD^{r(t)}\Phi(t)\\&=\Phi^T(x)\left(D^T\right)^4U\Phi(y)C^TN\Phi(t)\end{aligned} \tag{19}$$

Then, we suggest the form of matrix product as:

$$\begin{aligned}
&D\Phi_1^T(x)D^4U\Phi_2(y)C^T\Phi_3(t)+D\Phi_1^T(x)UZ^4\Phi_2(y)C^T\Phi_3(t)+\\
&+2D\Phi_1^T(x)\left(D^T\right)^2UZ^2\Phi_2(y)C^T\Phi_3(t)+\\
&+\eta D\Phi_1^T(x)N^4U\Phi_2(y)C^TA\Phi_3(t)+2\eta D\Phi_1^T(x)\left(N^T\right)^2UM^2\Phi(y)C^TA\Phi(t)-\\
&-m\Phi_1^T(x)U\Phi_2(y)C^TW^2\Phi_3(t)+E\left(\Phi_1^T(x)U\Phi_2(y)C^T\Phi_3(t)\right)-\\
&-\eta\Phi_1^T(x)U\Phi_2(y)C^TW\Phi_3(t)=-f(x,y,t)
\end{aligned} \tag{20}$$

The method is different from the result in [15].

Numerical simulation

The dynamical properties of viscoelastic polymer plates at a variety of loads are investigated. The displacement of the viscoelastic plate is calculated using the numerical algorithm previously proposed. The geometrical material characteristics of the plates are listed in tab. 1, where $r(t)$ is gained by program.

Table.1 Optimal simulation parameters of Polyurea and HDPE

Materials	ρ	E	η	$r(t)$
Polyurea	1060	$1.2\cdot10^7$	0.0012	0.1 + 0.5t
HDPE	960	$8.5\cdot10^6$	0.0039	0.1 + 0.5t

The displacements of two polymer plates under the different uniform loads are solved by the proposed the shifted Bernstein polynomial algorithm. As illustrated in figs. 2 and 3, the uniform loads $F = 10N$, $F = 30N$, and $F = 50N$ are applied to Polyurea and HDPE plates, respectively. It is seen that the displacement solution of the board reaches the maximum value in the middle position, progressively reduces from the centre to the ends, and the displacement of the two ends is 0. When the same load is applied, the Polyurea plate always has less displacement than the HDPE plate. It implies that the bending performance of Polyurea material is much better than that of HDPE material. The numerical results agree well with the results of the experiments, proving the precision of the algorithm.

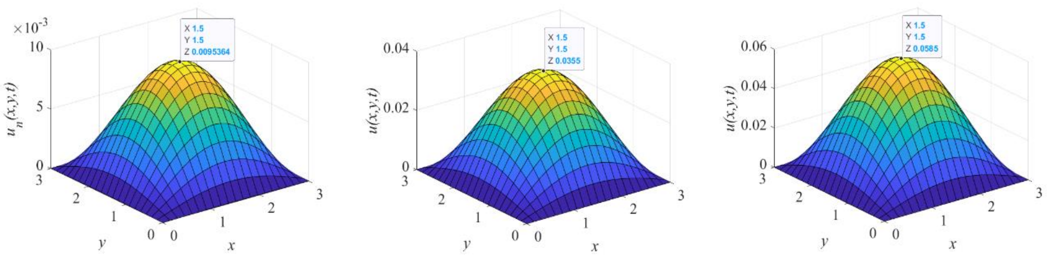

Figure 2. The displacement of Polyurea under the different uniform loads;
(a) $F = 10N$, (b) $F = 30N$, and (c) F = 50N

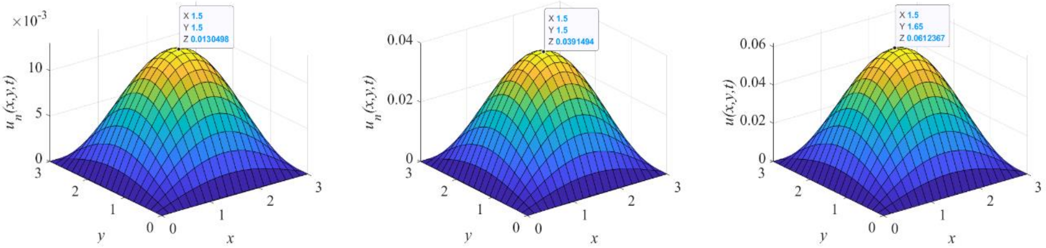

Figure 3. The displacement of HDPE under the different uniform loads; (a) $F = 10N$, (b) $F = 30N$, and (c) $F = 50N$

Conclusion

In the present work, the shifted Benstein polynomial was used to numerically analyze the fractional variable-order viscoelastic plate. The numerical consequences indicate that the method is effective for solving the viscoelastic differential equations. The displacement solutions of plates of the viscoelastic materials under the different uniform loads were computed and compared.

Nomenclature

t – time co-ordinate [s]

x,y – space co-ordinate, [m]

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