A NEW 3-D SIXTH-ORDER BOUSSINESQ MODEL IN SHALLOW WATER WAVE

by

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In this article, the surface wave in inviscid fluid was analyzed. Based on the Euler equation and mass conservation equation, and coupled with a set of boundary conditions, the (2+1)-dimensional sixth-order Boussinesq equation is derived for the first time. According to double-series perturbation analysis and scale transformation, the one soliton solution is obtained with (G'/G)-expansion method. Finally, the effects of amplitude parameter and shallowness parameter on the amplitude of surface wave are analyzed.

Key words: *3-D sixth-order Boussinesq equation, soliton solution, double-series perturbation analysis*

Introduction

A shallow water wave is a wave with a water depth less than the wavelength [1, 2]. In 1872, the French scientist Boussinesq assumed that the water depth was constant, and the vertical velocity was distributed linearly along the water depth, and then obtained a set of non-linear equations with horizontal 1-D weak dispersion. With the deep research of the water wave theory and application of Boussinesq equation, a great number of modified Boussinesq equations have been obtained, such as a 2-D Boussinesq equation describing the propagation of gravity waves on the surface of water [3], the classical fourth order Boussinesq equation [4], the 2-D sixth-order non-linear Boussinesq equation [5] and a new (3+1)-Boussinesq equation implemented for the first time to this model [6]. The model describing water waves gradually develops from low to high dimensions and orders. The propagation law of water waves will be explored by solving the model.

The Boussinesq equation applies to many physical models [7]. Theoretical solution of partial differential systems is a priority method to consider. Many methods have been used to solve fractional order equations, such as Hirota bilinear method [8], the exp-function method [9], local fractional derivative [10, 11], the Galerkin finite element method [12], (G'/G) expansion method [13], and so on. These solutions of fractional equations studied in these theories have made great contributions to the theory of natural science, allowing future generations to better understand natural science. In this paper, we will use (G'/G) expansion method to solve the equation.

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In this paper, the 3-D sixth-order Boussinesq equation in the water wave equation is derived by using double-series perturbation analysis and scale transformation, and the soliton solution of the equation is obtained based on the (G'/G)-expansion method. Finally, some conclusions are given.

Derivation of the 3-D sixth-order Boussinesq model

According to the Euler equation of motion, we give the governing equations [14]:

$$\begin{gathered} u_t + uu_x + vu_y + wu_z = -\frac{1}{\rho}P_x \\ v_t + uv_x + vv_y + wv_z = -\frac{1}{\rho}P_y \\ w_t + uw_x + vw_y + ww_z = -\frac{1}{\rho}P_z - \mathrm{g} \\ u_x + v_y + w_z = 0 \end{gathered} \tag{1}$$

where (u, v, w) is the velocity in the direction (x, y, z) in Cartesian co-ordinate system, g – the gravity, and P – the pressure. Due to the incompressibility of inviscid fluid，fluid density ρ = constant. The kinematic condition，dynamic condition and bottom condition of the water wave problem satisfied:

$$\begin{gathered} w = h_t + uh_x + vh_y, \quad \text{on} \quad z = h(x, y, t) \\ P = P_a - \frac{\Gamma}{R}, \quad \text{on} \quad z = h(x, y, t) \\ w = 0, \quad \text{on} \quad z = 0 \end{gathered} \tag{2}$$

where $z = h(x, y, z)$ is the free surface, P_a – the pressure of fluid surface, Γ – the coefficient of surface tension, and Γ/R – the pressure difference:

$$\frac{1}{R} = \frac{(1+h_y^2)h_{xx} + (1+h_x^2)h_{yy} - 2h_x h_y h_{xy}}{(1+h_x^2+h_y^2)^{3/2}} \tag{3}$$

Dimensionless and variable scaling are given:

$$\begin{gathered} x \to \lambda x, \ y \to \lambda y, \ z \to h_0 z, \ t \to \left(\frac{\lambda}{\sqrt{\mathrm{g}h_0}}\right)t \\ u \to \sqrt{\mathrm{g}h_0}u, \ v \to \sqrt{\mathrm{g}h_0}v, \ w \to \left(\frac{h_0\sqrt{\mathrm{g}h_0}}{\lambda}\right)w \\ h \to h_0 + a\varphi, \ P \to P_a + \rho \mathrm{g}(h_0 - z) + \rho \mathrm{g}h_0 p \end{gathered} \tag{4}$$

and

$$\begin{gathered} h \to h_0 + a\varphi, \ P \to P_a + \rho \mathrm{g}(h_0 - z) + \rho \mathrm{g}h_0 p \\ p \to \alpha p, \ w \to \alpha w, \ u \to \alpha u, \ v \to \alpha v \end{gathered} \tag{5}$$

Through dimensionless and variable scaling, the equations are obtained:

$$\begin{gathered} u_t + \alpha(uu_x + vu_y + wu_z) = -p_x \\ v_t + \alpha(uv_x + vv_y + wv_z) = -p_y \\ \beta[w_t + \alpha(uw_x + vw_y + ww_z)] = -p_z \\ u_x + v_y + w_z = 0 \end{gathered} \tag{6}$$

and boundary conditions are:

$$\begin{aligned}
&w=0, \quad \text{at} \quad z=0 \\
&w+\alpha\varphi w_z+\frac{\alpha^2\varphi^2}{2}w_{zz}=\varphi_t+\alpha\varphi_x(u+\alpha\varphi u_z)+\alpha\varphi_y(v+\alpha\varphi v_z)+o(\alpha^3), \quad \text{on} \quad z=1 \\
&p+\alpha\varphi p_z+\frac{\alpha^2\varphi^2}{2}p_{zz}=\varphi-\beta\tau\varphi_{xx}-\beta\tau\varphi_{yy}+o(\alpha^3,\alpha^2\beta^2), \quad \text{on} \quad z=1
\end{aligned} \tag{7}$$

where the amplitude parameter $\alpha = a/h_0$, the shallowness parameter $\beta = h_0/\lambda$ and the Weber number $\tau = \Gamma/\rho g h_0$. By further analyzing the boundary conditions, we expand the Taylor series of u, v, w, p at $z = 0$, and $z = 1$, and obtain:

$$\begin{aligned}
&w=0, \quad \text{at} \quad z=0 \\
&w+\alpha\varphi w_z+\frac{\alpha^2\varphi^2}{2}w_{zz}=\varphi_t+\alpha\varphi_x(u+\alpha\varphi u_z)+\alpha\varphi_y(v+\alpha\varphi v_z)+o(\alpha^3), \quad \text{on} \quad z=1 \\
&p+\alpha\varphi p_z+\frac{\alpha^2\varphi^2}{2}p_{zz}=\varphi-\beta\tau\varphi_{xx}-\beta\tau\varphi_{yy}+o(\alpha^3,\alpha^2\beta^2), \quad \text{on} \quad z=1
\end{aligned} \tag{8}$$

Introducing transformation the Y-direction on the space scale:

$$X=x, \quad y=\beta^{1/2}Y, \quad Z=z, \quad T=t \tag{9}$$

that is:

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y}=\beta^{1/2}\frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial z}=\frac{\partial}{\partial Z}, \quad \frac{\partial}{\partial t}=\frac{\partial}{\partial T} \tag{10}$$

We use the perturbation expansion in the form of two parameters (α and β) for the variable u, v, w, p, and φ:

$$\begin{aligned}
&u=u_{(0,0)}+\alpha u_{(1,0)}+\alpha^2u_{(2,0)}+\beta u_{(0,1)}+\beta^2u_{(0,2)}+\alpha\beta u_{(1,1)}+\cdots \\
&w=w_{(0,0)}+\alpha w_{(1,0)}+\alpha^2w_{(2,0)}+\beta w_{(0,1)}+\beta^2w_{(0,2)}+\alpha\beta w_{(1,1)}+\cdots \\
&p=p_{(0,0)}+\alpha p_{(1,0)}+\alpha^2p_{(2,0)}+\beta p_{(0,1)}+\beta^2p_{(0,2)}+\alpha\beta p_{(1,1)}+\cdots \\
&\varphi=\varphi_{(0,0)}+\alpha\varphi_{(1,0)}+\alpha^2\varphi_{(2,0)}+\beta\varphi_{(0,1)}+\beta^2\varphi_{(0,2)}+\alpha\beta\varphi_{(1,1)}+\cdots \\
&v=\beta^{\frac{3}{2}}v_{(0,1)}+\beta^{\frac{5}{2}}v_{(0,2)}+\cdots
\end{aligned} \tag{11}$$

Substitute eqs. (10) and (11) into eqs. (7) and (8), according to the different orders of parameters α and β, and the following relations are obtained: $O(1)$, $O(\alpha)$, $O(\alpha^2)$ and $O(\beta)$, $O(\beta^2)$, $O(\alpha\beta)$ order equations and the boundary conditions. Through substitution and complex calculations [15], finally we get:

$$\begin{aligned}
&\varphi_{TT}-\varphi_{XX}-\alpha\left[\frac{1}{2}\left(\varphi^2\right)+\left(\int_{-\infty}^{X}\varphi_T\,\mathrm{d}X\right)^2\right]_{XX}+\alpha^2\left[\varphi\left(\int_{-\infty}^{X}\varphi_T\,\mathrm{d}X\right)^2\right]_{XX}- \\
&-\beta\left(\frac{1}{3}-\tau\right)\varphi_{XXXX}-\beta^2\left[\left(\frac{1}{3}-2\tau\right)\varphi_{XXYY}+\left(\frac{2}{15}-\frac{1}{3}\tau\right)\varphi_{XXXXXX}\right]+ \\
&+\alpha\beta\left[\tau\left(\varphi\varphi_{XXX}\right)_X-\frac{2}{3}\left(\varphi_T^2\right)_{XX}-\left(\varphi\varphi_{XX}\right)_{XX}\right]=0
\end{aligned} \tag{12}$$

where $\varphi = \varphi_{(0,\,0)}$. It describes the two-way propagation of small amplitude and long capillary gravity waves on the surface of shallow water. Let's consider the case where the surface tension τ is close to 1/3. That is to say, we can get $|(1/3) - \tau| = K_1\beta$, $\beta \rightarrow 0$. In order to balance the non-linear and dispersion relations, we make $O(\alpha) = O(\beta^3)$, $\beta \rightarrow 0$, that is $\alpha = K_2\beta^3$.

Finally, we get:

$$\varphi_{TT} - \varphi_{XX} - \alpha\left[\frac{1}{2}(\varphi^2) + \left(\int_{-\infty}^{X} \varphi_T \mathrm{d}X\right)^2\right]_{XX} - K_1\left(\frac{\alpha}{K_2}\right)^{2/3}\varphi_{XXXX} -$$
$$-\left[\frac{2K_1}{K_2}\alpha - \frac{1}{3}\left(\frac{\alpha}{K_2}\right)^{2/3}\right]\varphi_{XXYY} - \left[\frac{K_1}{3K_2}\alpha + \frac{1}{45}\left(\frac{\alpha}{K_2}\right)^{2/3}\right]\varphi_{XXXXXX} = 0 \tag{13}$$

All aforementioned equations ignore the higher-order term of $O(\beta^3)$, and then we take the co-ordinate transformation:

$$X' = X + \alpha\int_{-\infty}^{X} \varphi(X,Y,T)\mathrm{d}X$$
$$T' = T,\ Y' = Y \tag{14}$$
$$\eta = \varphi - \alpha\varphi^2$$

Calculate and omit the upper right corner apostrophe, ignore the higher-order item of $O(\alpha^2)$, and we get:

$$\eta_{TT} - \eta_{XX} - a_1(\eta^2)_{XX} - a_2\eta_{XXXX} - a_3\eta_{XXYY} - a_4\eta_{XXXXXX} = 0 \tag{15}$$

where

$$a_1 = \frac{3}{2}\alpha,\ a_2 = \frac{K_1}{(K_2)^{2/3}}\alpha^{2/3},\ a_3 = \frac{2K_1}{K_2}\alpha - \frac{1}{3}\left(\frac{\alpha}{K_2}\right)^{2/3}$$
$$a_4 = \frac{1}{3}\frac{K_1}{K_2}\alpha + \frac{1}{45}\left(\frac{\alpha}{K_2}\right)^{2/3} \tag{16}$$

Equation (15) is the (2+1)-D sixth-order Boussinesq equation, which describes the gravity shallow water waves. If $a_4 = 0$, eq. (15) is the 3-D fourth order Boussinesq equation. If $a_3 = a_4 = 0$, eq. (15) degenerates to the classical Boussinesq equation.

Analytical solutions of (2+1)-D sixth-order Boussinesq equation

According to the (G'/G)-expansion method [16], in order to provide the solitary wave solution of eq. (15), we can consider the traveling wave transformation:

$$\eta_{(X,Y,T)} = u_{(\xi)},\ \xi = kX + lY - cT \tag{17}$$

Substituting eq. (17) into eq. (15), we can obtain:

$$(c^2 - k^2)u^{(2)} - a_1k^2(u^2)^{(2)} - a_2k^4u^{(4)} - a_3k^2L^2u^{(4)} - a_4k^6u^{(6)} = 0 \tag{18}$$

Then we integrate eq. (18) twice:

$$(c^2 - k^2)u - a_1k^2u^2 - (a_2k^4 - a_3k^2L^2)u^2 - a_4k^6u^{(4)} = 0 \tag{19}$$

We give the form of the solution eq. (15):

$$u(\xi) = \sum_{i=0}^{n} b_i\left(\frac{G'}{G}\right)^i,\ G'' + \lambda G' + \mu G = 0 \tag{20}$$

with $G = G(\xi)$ satisfying the second order linear ODE:

$$G'' + \lambda G' + \mu G = 0 \tag{21}$$

where λ, μ are constants.

Taking the homogeneous balance between u^2 and $u_{\xi\xi\xi\xi}$ in eq. (19), we obtain $n = 4$:

$$\begin{aligned} u_{(\xi)} &= b_0 + b_1((G'(\xi),\xi)/G(\xi)) + b_2((G'(\xi),\xi)/G(\xi))^2 + \\ &+ b_3((G'(\xi),\xi)/G(\xi))^3 + b_4((G'(\xi),\xi)/G(\xi))^4 \end{aligned} \tag{22}$$

Now we substitute eq. (22) into eq. (15), collect the coefficients of $(G'/G)^r$, (r = 0, 1, 2, 3, 4), set each coefficient to zero, and we can obtain a set of algebraic equation for $b_i(i$ = 0. 1, 2, 3, 4), c, λ, and μ. With the help of MAPLE software, we solve the system of algebraic equations:

$$\begin{aligned} b_0 &= -\frac{12a_4k^4(3\lambda^4 - 24\lambda^2\mu - 22\mu^2)}{a_1},\ b_1 = -\frac{1680a_4k^4\lambda\mu}{a_1} \\ b_2 &= -\frac{840(\lambda^2 + 2\mu)a_4k^4}{a_1},\ b_3 = -\frac{1680a_4k^4\lambda}{a_1},\ b_4 = -\frac{840a_4k^4}{a_1} \end{aligned} \tag{23}$$

$$\begin{aligned} c &= \pm k\sqrt{1 + 36a_4k^4\lambda^4 - 288a_4k^4\lambda^2\mu + 576a_4k^4\mu^2} \\ l &= \pm k\sqrt{\frac{52a_4k^2\mu - 13a_4k^2\lambda^2 - a_2}{a_3}} \end{aligned} \tag{24}$$

Therefore, we obtained the value of the undetermined coefficient in eq. (22), and thus obtained the structural expression of the solution of eq. (17). As compared to the single soliton solution, we get more general and some new exact traveling wave solutions.

Conclusion

In this paper, starting from the Euler equation of motion and the mass equation, the 3-D sixth-order Boussinesq equation is derived by double-series perturbation analysis method. We obtain the soliton solution and periodic solution of Boussinesq equation from the (G'/G) expansion method. As a result, this prominent method might be more effectively used for solving a wide variety of non-linear partial differential equations that frequently arise in science, engineering and other technological fields.

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Nomenclature

t – time, [second]

x, y, z – co-ordinates, [m]

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