THE METHOD FOR THE HEAT-DIFFUSION EQUATION IN ANALYTIC NUMBER THEORY

by

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This paper addresses the entire functions via theory of the tempered xi function. The zeros of the entire functions by the Fourier sine and cosine integrals are studied in detail. The method for the heat-diffusion equation is proposed to structure the connection between Fourier analysis and heat-diffusion equation show the open problems in analytic number theory.

Key words: *tempered xi function, Fourier analysis, heat-diffusion equation, open problems, analytic number theory*

Introduction

The theory of the tempered xi (TX) function represents one of the newest developments in mathematics [1, 2]. It has a deep connection with the number theory, complex analysis, Fourier analysis, and partial differential equations in mathematical physics [3, 4]. It has become one of important branches of analytic number theory.

The TX function, first proposed in 2022 by Yang [1], is an entire function of complex variable in analytic number theory. Let s be a complex variable. The TX function $\Phi(s)$ is defined by the integral [1]

$$\Phi(s) = 4\int_1^\infty \wp(l)\sinh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)\ln l\right]\mathrm{d}l, \tag{1}$$

where

$$\wp(l) = l^{-\frac{1}{4}}\frac{\mathrm{d}}{\mathrm{d}l}\left[l^{3/2}\frac{\mathrm{d}}{\mathrm{d}l}\left(\sum_{n=1}^{\infty}\mathrm{e}^{-n^2\pi\ell}\right)\right] \tag{2}$$

is the Riemann function [4].

In order to study the zeros of (1), we consider two entire function of complex variable as follows. The entire functions $\mathbb{A}(s)$ is defined by the integral [1]

$$\mathbb{A}(s) = 2\int_1^\infty \wp(l)\,l^{1/2(s-1/2)}\mathrm{d}l. \tag{3}$$

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The entire functions $\mathbb{B}(s)$ is defined by the integral [1]

$$\mathbb{B}(s) = 2\int_1^\infty \wp(l) l^{-1/2(s-1/2)} \mathrm{d}l. \tag{4}$$

We also write eqs. (3) and (4) as

$$\mathbb{A}(s) = 2\int_1^\infty \wp(l) \mathrm{e}^{1/2(s-1/2)\ln l} \mathrm{d}l \tag{5}$$

and

$$\mathbb{B}(s) = 2\int_1^\infty \wp(l) \mathrm{e}^{-1/2(s-1/2)\ln l} \mathrm{d}l, \tag{6}$$

respectively.

By eqs. (5) and (6), eq. (1) can be expressed as [1]

$$\Phi(s) = \mathbb{A}(s) - \mathbb{B}(s). \tag{7}$$

The distribution of the zeros of the TX function is one of the important and interesting topics in theory of the TX function. Let $\Re(s)$ be the real part of the complex variable s. There is a new challenge in the field of mathematics [1, 2].

Conjecture 1. All zeros of the function $\Phi(s)$ lie on the critical line $\Re(s) = 1/2$, see [1, 2].

The target of this paper is to study the new representation for the tempered xi function and to suggest two open problems.

Entire functions in analytic number theory

At first, we rewrite eq. (3) as [1]

$$\mathbb{A}(s) = 2\int_1^\infty l^{-1/4} \frac{\mathrm{d}}{\mathrm{d}l}\left[l^{3/2} \frac{\mathrm{d}\psi(l)}{\mathrm{d}l} \right] l^{1/2(s-1/2)} \mathrm{d}l, \tag{8}$$

where

$$\wp(l) = l^{-1/4} \frac{\mathrm{d}}{\mathrm{d}l}\left[l^{3/2} \frac{\mathrm{d}\psi(l)}{\mathrm{d}l} \right] = \frac{3}{2} l^{1/4} \frac{\mathrm{d}\psi(l)}{\mathrm{d}l} + l^{-1/4} \frac{\mathrm{d}^2\psi(l)}{\mathrm{d}l^2}$$

is the Riemann function [5], and

$$\psi(l) = \sum_{n=1}^\infty \mathrm{e}^{-n^2\pi\ell}$$

is the Jacobi function [5].

By eq. (3) and eq. (8), we present [1]

$$\mathbb{A}(s) = 2\int_1^\infty \wp(l) l^{1/2(s-1/2)} \mathrm{d}l = 2\int_1^\infty l^{-1/4} \frac{\mathrm{d}}{\mathrm{d}l}\left[l^{\frac{3}{2}} \frac{\mathrm{d}\psi(l)}{\mathrm{d}l} \right] l^{1/2(s-1/2)} \mathrm{d}l. \tag{9}$$

In a similar manner, we rewrite eq. (4) as [1]

$$\mathbb{B}(s) = 2\int_1^\infty l^{-1/4} \frac{\mathrm{d}}{\mathrm{d}l}\left[l^{3/3} \frac{\mathrm{d}\psi(l)}{\mathrm{d}l} \right] l^{-1/2(s-1/2)} \mathrm{d}l. \tag{10}$$

So, there is [1]

$$\mathbb{B}(s)=2\int_1^\infty \wp(l)l^{-1/2(s-1/2)}\mathrm{d}l=2\int_1^\infty l^{-1/4}\frac{\mathrm{d}}{\mathrm{d}l}\left[l^{3/2}\frac{\mathrm{d}\psi(l)}{\mathrm{d}l}\right]l^{-1/2(s-1/2)}\mathrm{d}l. \tag{11}$$

In order to investigate the new representations for $\mathbb{A}(s)$ and $\mathbb{B}(s)$, we now suggest two entire functions as follows.

The entire function $\Lambda(s)$ is defined as

$$\Lambda(s)=-\frac{s(1-s)}{2}\int_1^\infty \psi(l)l^{s/2}\frac{\mathrm{d}l}{l}. \tag{12}$$

The entire function $\Omega(s)$ is defined as

$$\Omega(s)=-\frac{s(1-s)}{2}\int_1^\infty \psi(l)l^{\frac{1-s}{2}}\frac{\mathrm{d}l}{l}. \tag{13}$$

Following Edwards [6], we show

$$\xi(s)=\frac{1}{2}+\left(\Lambda(s)+\Omega(s)\right),$$

where $\xi(s)$ is the Riemann entire zeta function [7].

One of authors [1] suggested that

$$\xi(s)=\mathbb{A}(s)+\mathbb{B}(s).$$

We now set up the connections with the entire functions $\mathbb{A}(s)$ and $\Lambda(s)$.

Making use of the integral

$$\int_1^\infty \psi(x)l^{s/2}\frac{\mathrm{d}l}{l}+\int_1^\infty \frac{\mathrm{d}\psi(l)}{\mathrm{d}l}\left(\frac{2}{s}l^{s/2}\right)\mathrm{d}x=\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{s}\psi(x)l^{s/2}\right)\mathrm{d}x, \tag{14}$$

eq.(12) can be expressed as

$$\Lambda(s)=-\frac{s(1-s)}{2}\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{2}{s}\psi(l)l^{s/2}\right)\mathrm{d}l+\frac{s(1-s)}{2}\int_1^\infty \frac{\mathrm{d}\psi(l)}{\mathrm{d}l}\left(\frac{2}{s}l^{s/2}\right)\mathrm{d}l. \tag{15}$$

The second term of eq.(15) is equal to

$$-\frac{s(1-s)}{2}\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{2}{s}\psi(l)l^{s/2}\right)\mathrm{d}l=(1-s)\psi(1), \tag{16}$$

where

$$\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}l}\left(\frac{2}{s}\psi(l)l^{s/2}\right)\mathrm{d}l=\frac{2}{s}\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}l}\left(\psi(l)l^{s/2}\right)\mathrm{d}l=-\frac{2}{s}\psi(1).$$

With eqs. (15) and (16), we may get

$$\Lambda(s)=(1-s)\psi(1)+\frac{s(1-s)}{2}\int_1^\infty \frac{\mathrm{d}\psi(l)}{\mathrm{d}l}\left(\frac{2}{s}l^{s/2}\right)\mathrm{d}l=(1-s)\psi(1)+(1-s)\int_1^\infty l^{s/2}\frac{\mathrm{d}\psi(l)}{\mathrm{d}l}\mathrm{d}l. \tag{17}$$

The second term of eq. (17) is equal to

$$(1-s)\int_1^\infty \psi^{(1)}(l)l^{s/2}\mathrm{d}l=\int_1^\infty (1-s)\psi^{(1)}(l)l^{s/2}\mathrm{d}l=\int_1^\infty l^{3/2}\psi^{(1)}(l)\frac{\mathrm{d}}{\mathrm{d}l}\left(-2l^{\frac{s-1}{2}}\right)\mathrm{d}l. \tag{18}$$

We now consider the integral

$$\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}l}\left[l^{3/2}\psi^{(1)}(l)\left(2l^{\frac{s-1}{2}}\right)\right]\mathrm{d}l=\int_1^\infty l^{3/2}\psi^{(1)}(l)\frac{\mathrm{d}}{\mathrm{d}l}\left(2l^{\frac{s-1}{2}}\right)\mathrm{d}l+2\int_1^\infty\left[\frac{\mathrm{d}}{\mathrm{d}l}\left(l^{\frac{3}{2}}\psi^{(1)}(l)\right)\right]l^{\frac{s-1}{2}}\mathrm{d}l. \tag{19}$$

The second term of eq. (19) is equal to

$$\int_1^\infty \frac{\mathrm{d}}{dl}\left[l^{3/2}\psi^{(1)}(l)\left(2l^{\frac{s-1}{2}}\right)\right]\mathrm{d}l=\int_1^\infty \frac{\mathrm{d}}{\mathrm{d}l}\left[\psi^{(1)}(l)\left(2l^{\frac{s-2}{2}}\right)\right]\mathrm{d}l=-2\psi^{(1)}(1). \tag{20}$$

Combining eqs. (19) and (20), we obtain

$$\int_1^\infty l^{3/2}\psi^{(1)}(l)\frac{\mathrm{d}}{\mathrm{d}l}\left(2l^{\frac{s-1}{2}}\right)\mathrm{d}l+2\int_1^\infty\left[\frac{\mathrm{d}}{\mathrm{d}l}\left(l^{3/2}\psi^{(1)}(l)\right)\right]l^{\frac{s-1}{2}}\mathrm{d}l=-2\psi^{(1)}(1). \tag{21}$$

By eq .(21), we rewrite eq.(18) as

$$(1-s)\int_1^\infty \psi^{(1)}(l)l^{s/2}\mathrm{d}l=-\int_1^\infty l^{3/2}\psi^{(1)}(l)\frac{\mathrm{d}}{\mathrm{d}l}\left(2l^{\frac{s-1}{2}}\right)\mathrm{d}l=2\psi^{(1)}(1)+2\int_1^\infty\left[\frac{\mathrm{d}}{\mathrm{d}l}\left(l^{3/2}\psi^{(1)}(l)\right)\right]l^{\frac{s-1}{2}}\mathrm{d}l. \tag{22}$$

Thus, we have from eqs. (17) and (22) that

$$\Lambda(s)=(1-s)\psi(1)+2\psi^{(1)}(1)+2\int_1^\infty\left[\frac{\mathrm{d}}{\mathrm{d}l}\left(l^{3/2}\psi^{(1)}(l)\right)\right]l^{\frac{s-1}{2}}\mathrm{d}l. \tag{23}$$

By using the relation

$$\mathbb{A}(s)=2\int_1^\infty l^{-1/4}\frac{\mathrm{d}}{\mathrm{d}l}\left[l^{3/2}\frac{\mathrm{d}\psi(l)}{\mathrm{d}l}\right]l^{1/2(s-1/2)}\mathrm{d}l=2\int_1^\infty\left[\frac{\mathrm{d}}{\mathrm{d}l}\left(l^{3/2}\psi^{(1)}(l)\right)\right]l^{\frac{s-1}{2}}\mathrm{d}l,$$

we have from eq. (23) that

$$\Lambda(s)=(1-s)\psi(1)+2\psi^{(1)}(1)+\mathbb{A}(s). \tag{24}$$

We now show the connections with the entire functions $\mathbb{B}(s)$ and $\Omega(s)$.
By eqs. (12) and (13), we show

$$\Lambda(s)=\Omega(1-s) \tag{25}$$

and

$$\Omega(s)=\Lambda(1-s). \tag{26}$$

By eqs. (5) and (6), we have

$$\mathbb{A}(s)=\mathbb{B}(1-s) \tag{27}$$

and

$$\mathbb{B}(s)=\mathbb{A}(1-s). \tag{28}$$

With the aid of eq. (24), we may get

$$\Lambda(1-s)=s\psi(1)+2\psi^{(1)}(1)+\mathbb{A}(1-s). \tag{29}$$

It implies from eqs. (26) and (28) that

$$\Omega(s)=s\psi(1)+2\psi^{(1)}(1)+\mathbb{B}(s). \tag{30}$$

There exists the functional equation

$$\Phi(s)=\left(\Lambda(s)-\Omega(s)\right)+2\left(s-\frac{1}{2}\right)\psi(1). \tag{31}$$

New entire functions in analytic number theory

The entire function $\Delta(s)$ is defined as

$$\Delta(s)=\Lambda(s)-\Omega(s)=-\frac{s(1-s)}{2}\hat{\Delta}(s), \tag{32}$$

where

$$\hat{\Delta}(s)=\int_1^\infty l^{-1}\psi(l)\left(l^{s/2}-l^{\frac{1-s}{2}}\right)\mathrm{d}l. \tag{33}$$

It is easy to see that

$$\hat{\Delta}(s)=2\int_1^\infty l^{-3/4}\psi(l)\sinh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)\ln l\right]\mathrm{d}l. \tag{34}$$

Substituting $s=1/2+ix$ into eq. (34), we present

$$\hat{\Delta}\left(\frac{1}{2}+ix\right)=2i\int_1^\infty l^{-3/4}\psi(l)\sin\left(\frac{x\ln l}{2}\right)\mathrm{d}l. \tag{35}$$

The entire function $\aleph(s)$ is defined as

$$\aleph(s)=\Lambda(s)+\Omega(s)=-\frac{s(1-s)}{2}\hat{\aleph}(s), \tag{36}$$

where

$$\hat{\aleph}(s)=\int_1^\infty l^{-1}\psi(l)\left(l^{s/2}+l^{\frac{1-s}{2}}\right)\mathrm{d}l. \tag{37}$$

In view of eq. (37), we can get

$$\hat{\aleph}(s)=2\int_1^\infty l^{-3/4}\psi(l)\cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)\ln l\right]\mathrm{d}l. \tag{38}$$

Substituting $s = 1/2 + ix$ into eq. (38), we have

$$\hat{\aleph}\left(\frac{1}{2}+ix\right)=2\int_1^\infty l^{-3/4}\psi(l)\cos\left(\frac{x}{2}\ln l\right)\mathrm{d}l. \tag{39}$$

New entire functions in fourier analysis

To study the entire functions in analytic number theory, we give the following results. Set

$$\omega(x):=-i\hat{\Delta}\left(\frac{1}{2}+ix\right)=2\int_1^\infty l^{-3/4}\psi(l)\sin\left(\frac{x\ln l}{2}\right)\mathrm{d}l \tag{40}$$

and

$$\sigma(x)=\hat{\aleph}\left(\frac{1}{2}+ix\right)=2\int_1^\infty l^{-3/4}\psi(l)\cos\left(\frac{x}{2}\ln l\right)\mathrm{d}l. \tag{41}$$

Taking $\ell = e^{2\mu}$ into eqs. (40) and (41), we present

$$\omega(x) = \int_0^\infty \mathbb{S}(\mu)\sin(x\mu)\,\mathrm{d}\mu \tag{42}$$

and

$$\sigma(x) = \int_0^\infty \mathbb{S}(\mu)\cos(x\mu)\,\mathrm{d}\mu, \tag{43}$$

where

$$\mathbb{S}(\mu) = 4e^{1/2\mu}\sum_{n=1}^\infty e^{-n^2\pi e^{2\mu}}. \tag{44}$$

Thus, we obtain

$$\Phi\left(\frac{1}{2}+ix\right) = 2ix\psi(1) - i\left(\frac{1}{4}+x^2\right)\omega(x) \tag{45}$$

and

$$\xi\left(\frac{1}{2}+ix\right) = \frac{1}{2}+\left(\frac{1}{4}+x^2\right)\sigma(x). \tag{46}$$

We now study the special value of the entire function

$$\Phi\left(\frac{1}{2}+ix\right) - 2ix\psi(1) = 0. \tag{47}$$

In this case, we obtain the following result:

Conjecture 2. All zeros of the entire function $\omega(x)$ are purely real.

It is easy to see that $x = 0$ is a zero of the entire function $\omega(x)$. We also investigate the special value of the entire function

$$\xi\left(\frac{1}{2}+ix\right) - \frac{1}{2} = 0. \tag{48}$$

Similarly, we show the following result:

Conjecture 3. All zeros of the entire function $\sigma(x)$ are purely real.

Remark. The Fourier cosine integral of eq. (46) was presented by Jensen [8] in 1913.

The method of the heat-diffusion equation

We now structure two entire functions $\omega(t, x)$ and $\sigma(t, x)$ by the integrals

$$\omega(z,t) = \int_0^\infty e^{-\kappa t\mu^2}\mathbb{S}(\mu)\sin(z\mu)\,\mathrm{d}\mu \tag{49}$$

and

$$\sigma(z,t) = \int_0^\infty e^{-\kappa t\mu^2}\mathbb{S}(\mu)\cos(z\mu)\,\mathrm{d}\mu, \tag{50}$$

where κ is the thermal diffusivity.

Let us consider that

$$\frac{\partial\omega(z,t)}{\partial t} = -\kappa\int_0^\infty \mu^2\mathbb{S}(\mu)\sin(z\mu)\,\mathrm{d}\mu, \tag{51}$$

$$\frac{\partial^2\omega(z,t)}{\partial z^2}=-\int_0^\infty \mu^2\mathbb{S}(\mu)\sin(z\mu)\mathrm{d}\mu, \tag{52}$$

$$\frac{\partial\sigma(z,t)}{\partial t}=-\kappa\int_0^\infty \mu^2\mathbb{S}(\mu)\cos(z\mu)\mathrm{d}\mu, \tag{53}$$

$$\frac{\partial^2\sigma(z,t)}{\partial z^2}=-\int_0^\infty \mu^2\mathbb{S}(\mu)\cos(z\mu)\mathrm{d}\mu. \tag{54}$$

From eqs. (51) and (52) we have the heat-diffusion equation [9]

$$\frac{\partial\omega(z,t)}{\partial t}=\kappa\frac{\partial^2\omega(z,t)}{\partial z^2}, \tag{55}$$

subject to the initial value condition

$$\omega(z,t=0)=\int_0^\infty \mathbb{S}(\mu)\sin(z\mu)\mathrm{d}\mu \tag{56}$$

and the boundary value conditions

$$\omega(z=0,t)=0,\ \omega(z=1,t)=\int_0^\infty \mathrm{e}^{-\kappa t\mu^2}\mathbb{S}(\mu)\sin(\mu)\mathrm{d}\mu. \tag{57}$$

If the initial value condition for eq. (55) is given as

$$\omega(z,t=0)=\int_0^\infty \mathbb{S}(\mu)\sin(z\mu)\mathrm{d}\mu=0, \tag{58}$$

then it is *Conjecture 1*.

By eqs. (53) and (54), we present the heat-diffusion equation [9]

$$\frac{\partial\sigma(z,t)}{\partial t}=\kappa\frac{\partial^2\sigma(z,t)}{\partial z^2}, \tag{59}$$

subject to the initial value condition

$$\sigma(z,t=0)=\int_0^\infty \mathbb{S}(\mu)\cos(z\mu)\mathrm{d}\mu \tag{60}$$

and the boundary value conditions

$$\sigma(z=0,t)=\int_0^\infty \mathrm{e}^{-\kappa t\mu^2}\mathbb{S}(\mu)\mathrm{d}\mu,\ \sigma(z=1,t)=\int_0^\infty \mathrm{e}^{-\kappa t\mu^2}\mathbb{S}(\mu)\cos(\mu)\mathrm{d}\mu. \tag{61}$$

If the initial value condition for eq. (60) is given as

$$\sigma(z,t=0)=\int_0^\infty \mathbb{S}(\mu)\cos(z\mu)\mathrm{d}\mu=0, \tag{62}$$

then it is *Conjecture 2*.

Conclusion

In the present work, we had proposed the entire functions by the Fourier sine and cosine integrals. The method for the heat-diffusion equation had been proposed to structure the connection between Fourier analysis and heat-diffusion equation. The open problems in analytic number theory had been considered in detail.

Nomenclature

t - time co-ordinate, [second]

z – space co-ordinate, [m]

Greek symbol

κ – thermal diffusivity, [$Wm^{-1}K^{-1}$]

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