THE ANALYSIS AND APPLICATION OF A NEW INTEGRAL TRANSFORM W Transform

by

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The main purpose of this paper is to introduce a new integral transform named the W transform. We have been obtained some important results about the W transform. At the same time, the relation between the W transform and other transforms has been established. In order to prove the efficiency of this transform, we have solved the differential equations and integral equations.

Key words: *W transform, integral transform, differential equation, integral equation*

Introduction

It is well known that integral transform plays an important role in solving many problems in engineering science, applied mathematics and mathematical physics. Integral transform is valuable for its simplification, especially when we are dealing with differential equations and integral equations [1-3]. During the past two decades, many integral transforms similar to Laplace transform [4] were introduced, such as Sumudu [5], natural [6], Elzaki [7], Mohand [8], Aboodh [9], Sawi [10], Yang [11], ARA [12], Jafari [13], Emad-Falih [14], Fareeha [15], and Wang *et al.* [16-18] transform.

In this paper, we plan to proclaim a new integral transform, which is named the W transform. This transform is a powerful and versatile generalization, which have unified some variants of the classical Laplace transform. Although Yang [1] and Jafari [13] covered all class of Laplace transform, they defined the domain as a positive real function. We consider that $s \in C$ and introduce the definition and some properties of the W transform. The relation between the W transform and other transforms and the values of the W transform for some special functions will be presented. We also plan to apply the W transform in some differential equations and integral equations to show its efficiency and accuracy through applications.

Definitions and properties of the W transform

Definition 1. The W transform defined for function of exponential order we consider functions in the set A defined:

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$$A=\left\{f(t):\exists M>0,K>0,|f(t)|<M\mathrm{e}^{kt},t\in[0,\infty)\right\}$$

the new general integral transform

$$W\{f(t)\}=s^m\int_0^\infty \mathrm{e}^{-s^n t}f(t)\mathrm{d}t=P(s)$$

where $s=\alpha+\beta i$.

Definition 2. The inverse W transform of the function $f(t)$ is given:

$$W^{-1}\{P(s)\}=\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^m}\mathrm{e}^{s^n t}P(s)\mathrm{d}s=f(t)$$

Theorem 1. (Existence theorem) Let $f(t)$ be a piecewise continuous function on every finite interval in the range $t\geq 0$and satisfies $f(t)\leq M\mathrm{e}^{kt}$, then $P(s)$ exists for all $\mathrm{Re}(s^n)>k$.

Proof. By using the definition of W transform, we have:

$$|W\{f(t)\}|=|s^m\int_0^\infty \mathrm{e}^{-s^n t}f(t)\mathrm{d}t|\leq s^m\int_0^\infty \mathrm{e}^{-s^n t}|f(t)|\mathrm{d}t\leq Ms^m\int_0^\infty \mathrm{e}^{-s^n t}e^{kt}f(t)\mathrm{d}t=\frac{Ms^m}{k-s^n}$$

The relations among the W transform and other transforms

The special cases of the W transform are given:

If $m=0$ and $n=1$, then this new transform gives the Laplace transform [4]:

$$L\{f(t)\}=\int_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$$

If $m=-1$ and $n=-1$, then this new transform gives the Sumudu transform [5]:

$$S\{f(t)\}=\frac{1}{s}\int_0^\infty \mathrm{e}^{-\frac{1}{s}t}f(t)\mathrm{d}t$$

If $m=1$ and $n=1$, then this new transform gives the Natural transform [6]:

$$N\{f(t)\}=s\int_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$$

If $m=1$ and $n=-1$, then this new transform gives the Elzaki transform [7]:

$$E\{f(t)\}=s\int_0^\infty \mathrm{e}^{-\frac{1}{s}t}f(t)\mathrm{d}t$$

If $m=2$ and $n=1$, then this new transform gives the Mohand transform [8]:

$$M\{f(t)\}=s^2\int_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$$

If $m=-1$ and $n=1$, then this new transform gives the Aboodh transform [9]:

$$A\{f(t)\}=\frac{1}{s}\int_0^\infty \mathrm{e}^{-st}f(t)\mathrm{d}t$$

If $m=-2$ and $n=-1$, then this new transform gives the Sawi transform [10]:

$$Sa\{f(t)\}=\frac{1}{s^2}\int_0^\infty \mathrm{e}^{-\frac{1}{s}t}f(t)\mathrm{d}t$$

If $m = 0$ and $n = -1$, then this new transform gives the Yang transform [11]:

$$Y\{f(t)\} = \int_0^\infty e^{-\frac{1}{s}t} f(t)\mathrm{d}t$$

If $m = -1$ and $n = 2$, then this new transform gives the Emad-Falih transform [14]:

$$EF\{f(t)\} = \frac{1}{s}\int_0^\infty e^{-s^2 t} f(t)\mathrm{d}t$$

If $m = 0$ and $n = n$, and then this new transform gives the Fareeha transform [15]:

$$Fa\{f(t)\} = \int_0^\infty e^{-s^n t} f(t)\mathrm{d}t$$

For more general integral transforms and its applications, see [1, 19-21].

The W transform of some basic functions

The W transform for the functions is listed in tab. 1.

Table 1. The following table given the W transform of some basic functions

$f(t)$	$V\{f(t)\} = P(s)$	$f(t)$	$V\{f(t)\} = P(s)$
c	$c\dfrac{s^m}{s^n}$	$e^{at}\sin(kt)$	$\dfrac{ks^m}{(sn-a)^2+k^2}$
t	$\dfrac{s^m}{s^{2n}}$	$e^{at}\cos(kt)$	$\dfrac{(s^n-a)s^m}{(s^n-a)^2+k^2}$
t^a	$\dfrac{a!s^m}{(s^n)^{a+1}}$	$e^{at}\sinh(kt)$	$\dfrac{ks^m}{(s^n-a)^2-k^2}$
e^{at}	$\dfrac{s^m}{s^n-a}$	$e^{at}\cosh(kt)$	$\dfrac{(s^n-a)s^m}{(s^n-a)^2-k^2}$
$\sin(kt)$	$\dfrac{ks^m}{s^{2n}+k^2}$	$H(t-a)$	$e^{-as^n}\dfrac{s^m}{s^n}$
$\cos(kt)$	$\dfrac{s^{m+n}}{s^{2n}+k^2}$	$f(t-a)H(t-a)$	$e^{-as^n}P(s)$
$\sinh(kt)$	$\dfrac{ks^m}{s^{2n}-k^2}$	$f(t)g(t)$	$\dfrac{1}{s^m}P(s)G(s)$
$\cosh(kt)$	$\dfrac{s^{m+n}}{s^{2n}-k^2}$	$f^{(k)}(t)$	$s^{kn}P(s)-s^m\sum_{j=0}^{k-1}(s^n)^{k-j-1}f^{(k)}(0)$

In tab. 1, we denote that $f(t)g(t)$ is the convolution of the functions $f(t)$ and $g(t)$, defined:

$$f(t)g(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$$

Applications of the W transform

Example 1. Consider the following first-order ordinary differential equation:

$$f'(t)+3f(t)=0, \quad f(0)=7 \tag{1}$$

Applying the W transform to both side of eq. (1) and the condition, we get:

$$s^n P(s)-s^m f(0)+3P(s)=0$$

With use of the inverse W transform, then:

$$f(t)=7\mathrm{e}^{-3t}$$

Example 2. Consider the following differential equation:

$$f''(t)-3f'(t)+2f(t)=4\mathrm{e}^{3t}, \quad f(0)=-3 \text{ and } f'(0)=5 \tag{2}$$

Applying the W transform to both side of eq. (3) and the conditions, we get:

$$s^{2n}P(0)-s^{m+n}f(0)-s^m f'(0)-3s^n P(s)+3s^m f(0)+2P(s)=\frac{4s^m}{s^n-3}$$

Taking the inverse W transform, then:

$$f(t)=2\mathrm{e}^{3t}+4\mathrm{e}^{2t}-9\mathrm{e}^{t}$$

Example 3. Consider the following Volterra integral equation:

$$f(t)+\int_0^t (t-\tau)f(\tau)\mathrm{d}\tau=1 \tag{3}$$

Applying the W transform to both sides of eq. (4), we have:

$$P(s)+\frac{1}{s^{2n}}P(s)=\frac{s^m}{s^n}$$

Making use of the inverse W transform, then:

$$f(t)=\cos t$$

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Nomenclature

$W\{f(t)\}$ – W transform, [–]

$W^{-1}\{P(s)\}$ – inverse W transform, [–]

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