# RIEMANNIAN SUBMERSIONS ENDOWED WITH A NEW TYPE OF SEMI-SYMMETRIC NON-METRIC CONNECTION

by

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Original scientific paper
https://doi.org/10.2298/TSCI2304393K

*In this paper we study relations for the covariant derivative of O'Neill's tensor fields, Riemannian curvature, Ricci curvature and scalar curvature of the Riemannian submersion from a Riemannian manifold with respect to a new type of semi-symmetric non-metric connection to a Riemannian manifold, respectively, and demonstrate the relationship between them.*

Key words: *Riemannian manifold, Riemannian submersion, curvature tensors a new type semi-symmetric non-metric connection*

## Introduction

Defining smooth mappings from one manifold to another is a common method of comparing two manifolds. Submersion is one such map, with a rank equal to the target manifold dimension. The concept of submersion has enabled many important geometric studies. O'Neill [1] and Gray [2] introduced Riemannian submersion between Riemannian submanifolds. Also, for other studies on Riemannian submersion, [3-8].

On the other hand, the concept of semi-symmetric linear connection on a manifold $M$ was introduced by Friedman and Schouten [9]. Also, the semi-symmetric metric connection was defined and studied by Hayden [10]. Later, Yano [11] researched a Riemannian manifold with a new connection, known as a semi-symmetric metric connection. Also, in 1992, Agashe and Chafle [12] introduced a new class of the semi-symmetric connection, called the semi-symmetric non-metric connection, on a Riemannian manifold and studied some of its geometric properties. After, Sengupta *et al.* [13] defined a new type of semi-symmetric non- metric connection on Riemannian manifold. Using these studies, Chaubey and Yildiz [14] have defined and demonstrated the existence of a new type of semi-symmetric non-metric connection on a Riemannian manifold. Many author have studied this type of connection [15-19].

Motivated by this studies in the present paper we consider Riemannian submersions from a Riemanian manifold with respect to a new type of semi-symmetric non-metric connection to a Riemannian manifold.

## Preliminaries

Let $(M, g_M)$ and $(N, g_N)$ be $C^\infty -$ Riemannian manifolds of dimension $m$ and $n$, respectively, Riemannian manifolds. A Riemannian submersion $f : M \to N$ is a mapping of $M$

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on $N$ satisfies axioms: i. $f$ has maximal rank and ii. $f_*$ preserves the lengths of horizontal vectors. If a surjective $C^\infty-$ manifold $f$: $M \to N$ has maximal rank at any point $M$, it is a $C^\infty-$ submersion. We obtain an integrable distribution $\nu$ that corresponds to the foliation of $M$ by putting $\nu_p = \ker f_{*p}$ for any $p \in M$. Each $\nu_p$ is defined the vertical space at $p$, $\nu$ is the vertical distribution, the sections of $\nu$ are the known vertical vector fields and determine a Lie subalgebra
$\chi^\nu(M)$ of $\chi(M)$. Let $H$ represent, the complementary distribution of $\nu$ that Riemannian metric $g$ produces. Therefore, the orthogonal decomposition $T_pM = \nu_p \oplus \mathcal{H}_p$ at any $p \in M$ is called the horizontal space at $p$. The sections of the horizontal distribution $\mathcal{H}$ are the horizontal vector fields. They establish a subspace $\chi^h(M)$ of $\chi(M)$. The $\upsilon$E and $hE$ stand for the vertical and horizontal components of $E$, respectively, for any $E \in \chi(M)$ [20]. A horizontal vector field $X$ on $M$ is said to be basic if $X$ is $f-$ related to a vector field $X'$ on $N$. A Riemannian submersion determines O'Neill tensor fields with the type (1, 2). For any $E, F \in \chi(M)$, the fundamental tensor fields are:

$$\mathcal{T}_E F = h\nabla_{vE} vF + v\nabla_{vE} hF, \quad \mathcal{A}_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{1}$$

where $v$: $\chi(M) \to \chi^\nu(M)$ and h: $\chi(M) \to \chi^h(M)$ are vertical and horizontal projections, respectively.

For any $X, Y \in \chi^h(M)$ and $U, V \in \chi^\nu(M)$, from eq. (1), we can obtain:

$$\nabla_U V = \mathcal{T}_U V + v\nabla_U V, \quad \nabla_U X = \mathcal{T}_U X + h\nabla_U X, \quad \mathcal{T}_U V = \mathcal{T}_V U \tag{2}$$

$$\nabla_X U = \mathcal{A}_X U + v\nabla_X U, \quad \nabla_X Y = \mathcal{A}_X Y + h\nabla_X Y, \quad \mathcal{A}_X Y = -\mathcal{A}_Y X \tag{3}$$

So, if $X$ is basic vector field then, $h\nabla_U X = h\nabla_X U = \mathcal{A}_X U$. Also, for any $E, F, H \in \chi(M)$ the covariant derivatives of $A$ and $T$ are given by [4]:

$$(\nabla_E \mathcal{A})_F H = \nabla_E(\mathcal{A}_F H) - \mathcal{A}_{\nabla_E F}(H) - \mathcal{A}_F(\nabla_E H) \tag{4}$$

$$(\nabla_E \mathcal{T})_F H = \nabla_E(\mathcal{T}_F H) - \mathcal{T}_{\nabla_E F}(H) - \mathcal{T}_F(\nabla_E H) \tag{5}$$

*Lemma 1.* Let $f$: $M \to N$ be Riemannian submersion and $X$, $Y$ basic vector fields on $M$, $f-$ related to $X'$ and $Y'$ on $N$, then we have the following properties:

i. $h[X, Y]$ is a basic vector field and $f_* h[X, Y] = [X', Y'] \circ f$,
ii. $h(\nabla_X Y)$ is a basic vector field $f-$ related to $(\nabla'_{X'} Y')$, where $\nabla$ and $\nabla'$ are the Levi-Civita connection on $M$ and $N$ respectively,
iii. For $\forall$U $\in \chi^\nu(M)$ and $\forall$E $\in \chi(M)$, $[E, U] \in \Gamma(\mathrm{v})$ [20].

*Definition 1.* Let $(M, g)$ be a Riemannian manifold, $p \in$ M and $K_p$ sectional curvature at $p$. Then the function $K_p$ given by:

$$K_p = \frac{gM[R(X,Y)Y,X]}{\|X\|^2\|Y\|^2 - g_M(X,Y)^2}$$

is called the sectional curvature at $p$. This curvature is usually used in the in the form of $K(X, Y)$ for $K$ ($\mathrm{span}_{\mathrm{R}}\{$X, Y$\}$).The Ricci curvature:

$$\mathrm{Ric}: C_2^\infty(TM) \to C_0^\infty(TM) \quad \text{by} \quad \mathrm{Ric}(X,Y) = \sum_{i=1}^m g_M[R(X,e_i)e_i, Y]$$

Also scalar curvature $\tau \in C^{\infty}(M)$ by:

$$\tau = \sum_{i=1}^{m} R_i(e_j, e_j) = \sum_{j=1}^{m}\sum_{i=1}^{m} g_M[R(e_i, e_j)e_j, e_i]$$

where $\{e_1, e_2, ..., e_m\}$ is any local orthonormal frame for the tangent bundle. We denote Ric($X$, $Y$) by $S(X, Y)$ in this paper [21].

*Definition 2.* Let $(M, g_M)$ be Riemannian manifold. We call $f-$ adapted a local orthonormal frame $\{Xi, U_j\}_{1 \le i \le n,\ 1 \le j \le r}$ on $M$, such that each $X_i$ is horizontal and each $U_j$ is vertical [20].

*Lemma 2.* Let $f$: $(M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Then we get:

$$\sum_{i=1}^{n} g_M(\mathcal{T}_U X_i, \mathcal{T}_v X_j) = \sum_{j=1}^{r} g_M(\mathcal{T}_U U_j, \mathcal{T}_v U_j) \tag{6}$$

$$\sum_{i=1}^{n} g_M(\mathcal{A}_X X_i, \mathcal{A}_Y X_i) = \sum_{j=1}^{r} g_M(\mathcal{A}_X U_j, \mathcal{A}_Y U_j) \tag{7}$$

$$\sum_{i=1}^{n} g_M(\mathcal{A}_X X_i, \mathcal{T}_U X_i) = \sum_{j=1}^{r} g_M(\mathcal{A}_X U_j, \mathcal{T}_U U_j) \tag{8}$$

where $X, Y \in \chi^h(M)$, $U, V \in \chi^v(M)$, and $\{Xi, U_j\}$ is $f-$ adaptable frame on $(M, g_M)$ [20].

*Definition 3.* Let $(M, g_M)$ be a Riemannian manifold and $V$ be the local orthonormal frame of the vertical distribution. Then we define horizontal vector field $\mathcal{N}$ on $(M, g_M)$ [20]:

$$\mathcal{N} = \sum_{j=1}^{r} \mathcal{T}_{U_j} U_j$$

*Lemma 3.* Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Let $\{U_j\}_{1 \le j \le r}$ be a local orthonormal frame of $V$. In this case for any $E \in \chi(M)$, we obtain [20]:

$$g_M(\nabla_E \mathcal{N}, X) = \sum_{j=1}^{r} g_M[(\nabla_E \mathcal{T})_{U_j} U_j, X]$$

Chaubey and Yildiz [14] introduce a new type of semi-symmetric non-metric connection on a Riemannian manifold as:

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}[\eta(Y)X - \eta(X)Y] \tag{9}$$

for arbitrary vector fields $X$ and $Y$ on $M$ and $\eta$ is a $1-$ form.

Now, we construct an example for Riemannian submersion.

*Example 1.* Let $\mathbb{R}^5$ and $\mathbb{R}^4$ be Riemannian manifolds endowed with $g_M$ and $g_N$ standard inner product metrics, where $x_1, x_2, x_3, x_4, x_5$ and $y_1, y_2, y_3$ canonical coordinates on $\mathbb{R}^5$ and $\mathbb{R}^4$ respectively $f : (\mathbb{R}^5, g_M) \to (\mathbb{R}^4, g_N)$ be submersion defined by:

$$f(x_1, x_2, x_3, x_4, x_5) = \left(x_1, \frac{x_2 + x_5}{\sqrt{2}}, x_3, x_4\right)$$

Then the Jacobian matrix of $f$ is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

A straight computations gives:

$$\ker f_* = \operatorname{span}\left\{ Z_1 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5} \right\},$$

$$(\ker f_*)^{\perp} = \operatorname{span}\left\{ T_1 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_5}, T_2 = \frac{\partial}{\partial x_1}, T_3 = \frac{\partial}{\partial x_3}, T_4 = \frac{\partial}{\partial x_4}, \right\}$$

Also by direct computations yields:

$$f_*(T_1) = \sqrt{2}\frac{\partial}{\partial y_2}, \quad f_*(T_2) = \frac{\partial}{\partial y_1}, \quad f_*(T_3) = \frac{\partial}{\partial y_3}, \quad f_*(T_4) = \frac{\partial}{\partial y_4}$$

$f$ is a Riemannian submersion from the following equations:

$$g_M(T_1,T_1) = g_N[f_*(T_1), f_*(T_1)], \quad g_M(T_2,T_2) = g_N[f_*(T_2), f_*(T_2)],$$
$$g_M(T_3,T_3) = g_N[f_*(T_3), f_*(T_3)], \quad g_M(T_4,T_4) = g_N[f_*(T_4), f_*(T_4)]$$

## Riemannian submersions endowed with a new type of semi-symmetric non-metric connection

Let $M$ be a Riemannian manifold with metric $g_M$, $f$ be a Riemannian submersion from $M$ onto a Riemannian manifold $N$ with metric $g_N$ and $E, F \in \chi(M)$. Then we have:

$$\tilde{\mathcal{T}}_E F = h\tilde{\nabla}_{vE} vF + v\tilde{\nabla}_{vE} hF = \mathcal{T}_E F + \frac{1}{2}\eta(hF)vE \tag{10}$$

$$\tilde{\mathcal{A}}_E F = v\tilde{\nabla}_{hE} hF + h\tilde{\nabla}_{hE} vF = \mathcal{A}_E F + \frac{1}{2}\eta(vF)hE \tag{11}$$

where $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{A}}$ are tensor fields on $M$ admitting to a new type of semi-symmetric non-metric connection $\tilde{\nabla}$.

Proposition 1 Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Then, $\tilde{\mathcal{T}}$ is symmetric on the vertical distribution and $\tilde{\mathcal{A}}$ - is antisymmetric on the horizontal distribution.

*Proof 1.* Since $\tilde{\mathcal{T}}_U V = \mathcal{T}_U V$ for $U,V \in \mathcal{X}^v(M)$ and $\mathcal{T}$ is symmetric on the vertical distribution in Riemannian submersion, we get $\tilde{\mathcal{T}}_U V = \tilde{\mathcal{T}}_U V$. Similarly, because of $\tilde{\mathcal{A}}_X Y = \mathcal{A}_X Y$ for $X,Y \in \mathcal{X}^h(M)$ and $\mathcal{A}$ is anti-symmetric on the horizontal distribution in Rimannian submersion, we obtain $\tilde{\mathcal{A}}_X Y = \tilde{\mathcal{A}}_Y Y + 2\mathcal{A}_X Y$.

Thus, the proof is completed.

*Lemma 4.* Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$, $(N, g_N)$ and $E, F, G \in \mathcal{X}(M)$. Then we obtain,

i. $g_M(\tilde{\mathcal{T}}_E F, G) = g_M(\tilde{\mathcal{T}}_E G, F) + 2g_M(\mathcal{T}_E F, G) + \frac{1}{2}[\eta(hF)g_M(vE, vG) - \eta(hG)g_M(vE, vF)]$

ii. $g_M(\tilde{\mathcal{A}}_E F, G) = g_M(\tilde{\mathcal{A}}_E G, F) + 2g_M(\mathcal{A}_E F, G) + \frac{1}{2}[\eta(vF)g_M(hE, hG) - \eta(vG)g_M(hE, hF)]$

*Proof 2.* i) For any $E, F, G \in \mathcal{X}(M)$, by using (10) we obtain:

$$g_M(\tilde{\mathcal{T}}_E F, G) = g_M(\mathcal{T}_E F, vG) + \frac{1}{2}\eta(hF)g_M(vE, vG) + g_M(\mathcal{T}_E F, vG)$$

$$g_M(\tilde{\mathcal{T}}_E F, G) = g_M(\mathcal{T}_E F, G) + \frac{1}{2}\eta(hF)g_M(vE, hG) \tag{12}$$

Similarly we get:

$$g_M(\tilde{\mathcal{T}}_E G, F) = g_M(\mathcal{T}_E G, F) + \frac{1}{2}\eta(hF)g_M(vE, vG) \tag{13}$$

If eq. (12) is subtracted from eq. (13), i. comes. By using same way, it can see easily ii. Thus the proof is completed.

*Lemma 5.* Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$.Then we obtain:

$$\tilde{\nabla}_V X = \mathcal{T}_V X + h\tilde{\nabla}_V X + \frac{1}{2}\eta(X)V, \quad \tilde{\nabla}_V W = \mathcal{T}_V W + \tilde{\nabla}_V W \tag{14}$$

$$\tilde{\nabla}_X V = \mathcal{A}_X V + v\tilde{\nabla}_X V + \frac{1}{2}\eta(V)X, \quad \tilde{\nabla}_X Y = \mathcal{A}_X Y + h\tilde{\nabla}_X Y \tag{15}$$

for $V, W \in \mathcal{X}^v$ (M) and $X, Y \in \mathcal{X}^h$ (M), where $\tilde{\nabla}_V W = v\tilde{\nabla}_V W$.

*Proof 3*. Since $\nabla$ is a Levi-Civita connection, using eq. (9), we obtain:

$$\tilde{\nabla}_V X = v\left\{\nabla_V X + \frac{1}{2}[\eta(X)V - \eta(V)X]\right\} + h\tilde{\nabla}_V X$$

Also, from (10), we obtain $\mathcal{T}_V X + v\nabla_V X$. Thus, we can easily get the first eq. (14). The other equations can be obtained similarly.

We define following expression by using eqs. (4) and (5) for a new type semi-symmetric non-metric connections.

*Definition 4.* Let $f : M \to N$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g$N). The covariant derivatives of $\mathcal{T}$ and $\mathcal{A}$ are:

$$(\tilde{\nabla}_E \tilde{\mathcal{T}})_F H = \tilde{\nabla}_E(\mathcal{T}_F H) - \mathcal{T}_{\tilde{\nabla}_E F} H - \mathcal{T}_F(\tilde{\nabla}_E H) \tag{16}$$

$$(\tilde{\nabla}_E \mathcal{A})_F H = \tilde{\nabla}_E(\mathcal{A}_F H) - \mathcal{A}_{\tilde{\nabla}_E F} H - \mathcal{A}_F(\tilde{\nabla}_E H) \tag{17}$$

where $E, F, G$ arbitrary vector fields on $\chi(M)$.

*Lemma 6.* Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Then we obtain,

$$(\tilde{\nabla}_X \mathcal{A})_W = -\mathcal{A}_{\mathcal{A}_X W + \frac{1}{2}\eta(W)X}, \quad (\tilde{\nabla}_V \mathcal{T})_Y = -\mathcal{T}_{\mathcal{T}_V Y + \frac{1}{2}\eta(Y)V},$$

$$(\tilde{\nabla}_V \mathcal{A})_W = -\mathcal{A}_{\mathcal{T}_V W}, \quad (\tilde{\nabla}_X \mathcal{T})_Y = \mathcal{T}_{\mathcal{A}_X Y} \tag{18}$$

for any $X, Y \in \chi^h(M)$ and $V, W \in \chi^v(M)$.

*Proof 4.* Here, we will only give the proof of the first equation in (18). The proof of other equations in (18) can be done in a similar way. Let $F$ be an arbitrary vector field on $M$. If eq. (4) is used for a new type of semi-symmetric non-metric connections, then we obtain:

$$(\tilde{\nabla}_X \mathcal{A})_W F = \tilde{\nabla}_X (\mathcal{A}_W F) - \mathcal{A}_{\tilde{\nabla}_X W} F - \mathcal{A}_W \tilde{\nabla}_X F \tag{19}$$

Since $A$ is horizontal, we see that $\mathcal{A}_W = \mathcal{A}_{hW} = 0$. Also, if we use the first eq. (15), we get:

$$(\tilde{\nabla}_X \mathcal{A})_W F = -\mathcal{A}_{\tilde{\nabla}_X W} F = -\mathcal{A}_{\mathcal{A}_X W + \mathrm{v}\tilde{\nabla}_X W + \frac{1}{2}\eta(W)X} F = -\mathcal{A}_{\mathcal{A}_X W + \frac{1}{2}\eta(W)X} F$$

Thus the proof is completed.

## Curvature relations with respect to a new type of semi-symmetric non- metric connection

In this section, we will obtain the Riemannian curvatures of the Riemannian manifold with respect to a new type of semi-symmetric non-metric connection according to the O'Neill tensor fields and their covariant derivatives.

*Theorem 1.* Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Riemannian curvatures of $M$ with respect to a new type of semi-symmetric non-metric connection $\tilde{\nabla}$ are given by:

$$\begin{aligned}\tilde{R}(X,Y)V &= \mathcal{A}_X \mathcal{A}_Y V + h\tilde{\nabla}_X \mathcal{A}_Y V + \mathcal{A}_X v\tilde{\nabla}_Y V + v\tilde{\nabla}_X v\tilde{\nabla}_Y V - \mathcal{T}_{[X,Y]}V - \\ &- \mathcal{A}_Y \mathcal{A}_X V - h\tilde{\nabla}_Y \mathcal{A}_X V - \mathcal{A}_Y v\tilde{\nabla}_X V - v\tilde{\nabla}_Y v\tilde{\nabla}_X V - \tilde{\nabla}_{[X,Y]}V + \\ &+ \frac{1}{2}\eta(V)[(\mathcal{A}_X hY + h\tilde{\nabla}_X hY) - (\mathcal{A}_Y hX + h\tilde{\nabla}_Y hX)]\end{aligned} \tag{20}$$

$$\begin{aligned}\tilde{R}(U,V)W &= \mathcal{T}_U \mathcal{T}_V W + h\tilde{\nabla}_U \mathcal{T}_V W - h\tilde{\nabla}_{[U,V]}W + \mathcal{T}_U \hat{\nabla}_V W + \hat{\nabla}_U \hat{\nabla}_V W - \hat{\nabla}_{[U,V]}W - \\ &- \mathcal{T}_V \mathcal{T}_U W - h\tilde{\nabla}_V \mathcal{T}_U W + \frac{1}{2}[\eta(\mathcal{T}_V W)U - \eta(\mathcal{T}_U W)V] - \mathcal{T}_V \hat{\nabla}_U W - \hat{\nabla}_V \hat{\nabla}_U W\end{aligned} \tag{21}$$

$$\begin{aligned}\tilde{R}(X,Y)Z &= \nabla'_X \nabla'_Y Z - \nabla'_Y \nabla'_X Z + \mathcal{A}_X \mathcal{A}_Y Z - \mathcal{A}_Y \mathcal{A}_X Z - 2\mathcal{A}_Z \mathcal{A}_X Y - \\ &- 2\mathcal{T}_{\mathcal{A}_X Y} Z + v\tilde{\nabla}_X \mathcal{A}_Y Z - v\tilde{\nabla}_Y \mathcal{A}_X Z + \mathcal{A}_X \nabla'_Y Z - \mathcal{A}_Y \nabla'_X Z + \\ &+ \frac{1}{2}[\eta(\mathcal{A}_Y Z)X - \eta(\mathcal{A}_X Z)Y - \eta(Z)(X,Y)]\end{aligned} \tag{22}$$

$$\tilde{R}(X,V)Y = \mathcal{A}_X \mathcal{T}_V Y + v\tilde{\nabla}_X \mathcal{T}_V Y + \frac{1}{2}\eta(\mathcal{T}_V Y)X + \mathcal{A}_X h\tilde{\nabla}_V Y + h\tilde{\nabla}_X h\tilde{\nabla}_V Y + + \frac{1}{2}\eta(Y)\left[\mathcal{A}_X V + v\tilde{\nabla}_X V + \frac{1}{2}\eta(V)hX - (X,V)\right] - \mathcal{T}_V \mathcal{A}_X Y - \hat{\nabla}_V \mathcal{A}_X Y - -\mathcal{T}_V h\tilde{\nabla}_X Y - h\tilde{\nabla}_V \nabla'_X Y - \frac{1}{2}\eta(h\tilde{\nabla}_X Y)V - \mathcal{T}_{[X,V]}Y - h\tilde{\nabla}_{[X,V]}Y \tag{23}$$

where $U, V, W \in \chi^v(M)$ and $X, Y, Z \in \chi^h(M)$.

*Proof 5.* We will only give the proof of eq. (20). The proofs of eqs. (21), (22), and (23) can be obtained easily in a similar way. We define Rimannian curvature tensor of $M$ with respect to a new type of semi-symmetric non-metric connection $\tilde{\nabla}$ by:

$$\tilde{R}(X,Y)V = \tilde{\nabla}_X \tilde{\nabla}_Y V - \tilde{\nabla}_Y \tilde{\nabla}_X V - \tilde{\nabla}_{[X,Y]}V \tag{24}$$

From $[X, Y] \in \chi^v(M)$, eqs. (14), (15), and (24), we get:

$$\tilde{R}(X,Y)V = \tilde{\nabla}_X \mathcal{A}_Y V + \tilde{\nabla}_X v\tilde{\nabla}_Y V - \tilde{\nabla}_Y \mathcal{A}_X V - \tilde{\nabla}_Y \tilde{\nabla}_X V + +\frac{1}{2}\eta(V)\tilde{\nabla}_X hY - \frac{1}{2}\eta(V)\tilde{\nabla}_Y hX - \mathcal{T}_{[X,Y]}V - \hat{\nabla}_{[X,Y]}V$$

If the necessary straightforward computation is made in the last equation, we obtain eq. (20).

*Corollary 1.* Let $f : (M, g_M) \rightarrow (N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. The $\tilde{R}, R'$, and $\hat{R}$ denote the Rimannian curvatures of $M$, $N$, and $[f^{-1}(x), \hat{g}x]$ for $x \in N$ respectively. In this case, the following equations are obtained:

$$g_M[\tilde{R}(U,V)W,F] = g_M[\hat{R}(U,V)W,F] - g_M(\mathcal{T}_U F, \mathcal{T}_V W) + g_M(\mathcal{T}_V F, \mathcal{T}_U W) + +\frac{1}{2}[\eta(\mathcal{T}_V W)g_M(U,F) - \eta(\mathcal{T}_U W)gM(V,F)] \tag{25}$$

$$g_M[\tilde{R}(U,V)W,X] = g_M[(\tilde{\nabla}_U \mathcal{T})_V W, X] - g_M[(\tilde{\nabla}_V \mathcal{T})_U W, X] \tag{26}$$

$$g_M[\tilde{R}(X,Y)Z,H] = g_M[R'(X,Y)Z,H] - g_M(\mathcal{A}_X H, \mathcal{A}_Y Z) + g_M(\mathcal{A}_Y H, \mathcal{A}_X Z) + +2g_M(\mathcal{A}_Z H, \mathcal{A}_X Y) + \frac{1}{2}[\eta(\mathcal{A}_Y Z)g_M(X,H) - \eta(\mathcal{A}_X Z)g_M(Y,H)] \tag{27}$$

$$g_M[\tilde{R}(X,Y)Z,V] = -2g_M(\mathcal{T}_V Z, \mathcal{A}_X Y) + g_M(\tilde{\nabla}_X \mathcal{A}_Y Z, V) - g_M(\tilde{\nabla}_Y \mathcal{A}_X Z, V) + +g_M(\mathcal{A}_X \nabla'_Y Z, V) - g_M(\mathcal{A}_Y \nabla'_X Z, V) - \frac{1}{2}\eta(Z)g_M[(X,Y),V] \tag{28}$$

$$g_M[\tilde{R}(X,Y)V,W] = \frac{1}{2}\{g_M[(\tilde{\nabla}_X \mathcal{A})(Y,V),W] - g_M[(\tilde{\nabla}_Y \mathcal{A})(X,V),W] - \eta(V)g_M(\mathcal{A}_X Y,W)\} + +g_M(\tilde{\nabla}_X v\tilde{\nabla}_Y V, W) - g_M(\tilde{\nabla}_Y v\tilde{\nabla}_X V, W) - g_M(\tilde{\nabla}_{[X,Y]}V, W) \tag{29}$$

$$g_M[\tilde{R}(X,Y)V,H] = g_M[(\tilde{\nabla}_X\mathcal{A})(Y,V),H] - g_M[(\tilde{\nabla}_Y\mathcal{A})(X,V),H] + \\ +\frac{1}{2}\eta(V)[g_M(\tilde{\nabla}_X Y,H) - g_M(\tilde{\nabla}_Y X,H)] + g_M(\mathcal{T}_V H,[X,Y]) \tag{30}$$

$$g_M[\tilde{R}(X,Y)Y,W] = g_M[(\tilde{\nabla}_X\mathcal{T})_V Y,W] - g_M[(\tilde{\nabla}_V\mathcal{A})_X Y,W] + g_M(\mathcal{A}_Y W,h\tilde{\nabla}_V X) + \\ +g_M(\mathcal{T}_W Y,v\tilde{\nabla}_V X) + \frac{1}{2}[\eta(Y)g_M(\tilde{\nabla}_V X,W) - \eta(h\tilde{\nabla}_X Y)g_M(V,W)] \tag{31}$$

$$g_M[\tilde{R}(X,V)Y,H] = g_M(\mathcal{T}_V H,\mathcal{A}_X Y) - g_M(\mathcal{A}_X H,\mathcal{T}_V Y) + \\ +g_M(h\tilde{\nabla}_X h\tilde{\nabla}_V Y,H) - g_M(h\tilde{\nabla}_V h\tilde{\nabla}_X Y,H) - g_M(h\tilde{\nabla}_{[X,V]}Y,H) + \\ +\frac{1}{2}\{\eta(\mathcal{T}_V Y)g_M(X,\mathrm{H}) + \eta(Y)[g_M(\mathcal{A}_X V,H) + \frac{1}{2}\eta(V)g_M(X,\mathrm{H})]\} \tag{32}$$

where $X, Y, Z, H \in \chi^h(M)$ and $U, V, W, F \in \chi^v(M)$.

*Proof 6.* Here we will investigate eq. (31). The proof of other equations can be obtained in a similar way. By using eqs. (14), (15), and (23) with straightforward computations, we have:

$$g_M[R(X,V)Y,H] = g_M(v\tilde{\nabla}_X\mathcal{T}_V Y,W) + g_M(\mathcal{A}_X h\tilde{\nabla}_V Y,W) + \frac{1}{2}\eta(Y)g_M(v\tilde{\nabla}_X V,W) - \\ -g_M(\tilde{\nabla}_V\mathcal{A}_X Y,W) - g_M(\mathcal{T}vh\tilde{\nabla}_X Y,W) - \frac{1}{2}\eta(Y)g_M([X,V],W) - \\ -\frac{1}{2}\eta(h\tilde{\nabla}_X Y)g_M(V,W) - g_M(\mathcal{T}_{v\tilde{\nabla}_{xv}}Y,W) + g_M(\mathcal{T}_{v\tilde{\nabla}_{vx}}Y,W) \tag{33}$$

where $X, Y \in \chi^h(M)$, $V, W \in \chi^v(M)$. On the other hand from eq. (16) we get:

$$g_M[(\tilde{\nabla}_X\mathcal{T})_V Y,W] = g_M(\tilde{\nabla}_X\mathcal{T}_V Y,W) + g_M(\mathcal{T}_{\tilde{\nabla}_x V}Y,W) - g_M(\mathcal{T}_V\tilde{\nabla}_X Y,W)$$

Additionally from eq. (15) we have:

$$g_M[(\tilde{\nabla}_X\mathcal{T})_V Y,W] = g_M(\mathcal{A}_X\mathcal{T}_V Y,W) - g_M(\tilde{\nabla}_X\mathcal{T}_V Y,W) + \frac{1}{2}\eta(\mathcal{T}_V Y)g_M(hX,W) - \\ -g_M(\mathcal{T}_{\mathcal{A}_x V}Y,W) - g_M(\mathcal{T}v_{\tilde{\nabla}_x}VY,W) - \frac{1}{2}\eta(V)g_M(\mathcal{T}_{hX}Y,W) - \\ -g_M(\mathcal{T}_V\mathcal{A}_X Y,W) - g_M(\mathcal{T}_V h\tilde{\nabla}_X Y,W)$$

Thus we obtain:

$$g_M[(\tilde{\nabla}_X\mathcal{T})_V Y,W] = g_M(v\tilde{\nabla}_X\mathcal{T}_V Y,W) - g_M(\mathcal{T}_{v\tilde{\nabla}_x V}Y,W) - g_M(\mathcal{T}_V h\tilde{\nabla}_X Y,W) \tag{34}$$

Similarly from eqs. (17) and (14) we have:

$$g_M[(\tilde{\nabla}_V\mathcal{A})_X Y,W] = g_M(\hat{\nabla}_V\mathcal{A}_X Y,W) - g_M(\mathcal{A}_{h\tilde{\nabla}_v X}Y,W) - g_M(\mathcal{A}_X h\tilde{\nabla}_V Y,W) \tag{35}$$

In that case, from eqs. (33), (34), and (35) we obtain eq. (31). Thus the proof is completed.

*Theorem 2.* Let $f : (M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Let $K'$ and $\hat{K}$ be sectional curvatures of $N$ and any fibre $(f^{-1}(x), \hat{g}_x)$, respectively. Sectional curvatures of $M$ with respect to a new type of semi-symmetric non-metric connection $\tilde{\nabla}$ are given by:

$$\tilde{K}(X,V) = g_M[(\tilde{\nabla}_V \mathcal{A})_X X,V] - g_M[(\tilde{\nabla}_X \mathcal{T})_V X,V) - g_M(\mathcal{A}_X V, h\tilde{\nabla}_V X) - \\ -g_M(\mathcal{T}_V X, v\tilde{\nabla}_V X) - \frac{1}{2}\eta(X) g_M(\tilde{\nabla}_V X,V) + \frac{1}{2}\eta(h\tilde{\nabla}_X X) \tag{36}$$

$$\tilde{K}(U,V) = \hat{K}(U,V) + \|\mathcal{T}_U V\|^2 - g_M(\mathcal{T}_U U, \mathcal{T}_V V) + \frac{1}{2}\eta(\mathcal{T}_V V) - \frac{1}{2}\eta(\mathcal{T}_U V) g_M(V,U) \tag{37}$$

$$\tilde{K}(X,Y) = K'(X',Y') - 3\|\mathcal{A}_X Y\|^2 - \frac{1}{2}\eta(\mathcal{A}_X Y) g_M(X,Y) \tag{38}$$

where $X$, $Y$ (and $U$, $V$) are orthonormal horizontal (and vertical) vector fields and $\{X_i, U_j\}$ is *f-adap*table frame on $(M, g_M)$.

*Proof 7.* Since $\tilde{K}(X,V) = g_M[\tilde{R}(X,V)V,X]$ changing the roles of $X$ and $V$ with $Y$ and $W$, respectively, in the eq. (31) we obtain:

$$\tilde{K}(X,V) = g_M[(\tilde{\nabla}_V \mathcal{A})_X X,V] - g_M[(\tilde{\nabla}_X \mathcal{T})_V X,V] - g_M(\mathcal{A}_X V, h\tilde{\nabla}_V X) - \\ -g_M(\mathcal{T}_V X, v\tilde{\nabla}_V X) - \frac{1}{2}\eta(X) g_M(\tilde{\nabla}_V X,V) + \frac{1}{2}\eta(h\tilde{\nabla}_X X) g_M(V,V)$$

Considering that the vector fields $U$ and $V$ are orthonormal vector fields, we get eq. (36).

The eqs. (37) and (38) can be obtained in a similar way.

*Theorem 3.* Let f : $(M, g_M) \to (N, g_N)$ be a Riemannian submersion between Riemannian manifolds. Let $S'$ and $\hat{S}$ be Ricci tensors of $N$ and any fibre $[f^{-1}(x), \hat{g}_x]$ for $x \in$ N, respectively. Ricci tensors of $M$ with respect to a new type of semi-symmetric non-metric connection $\tilde{\nabla}$ are given by

$$\tilde{S}(U,V) = \hat{S}(U,V) + g_M(N, \mathcal{T}_U V) - \frac{1}{2}\eta(\mathcal{T}_U V) + \sum_i [g_M[(\tilde{\nabla}_{X_i} \mathcal{T})_U X_i, V] + \\ + g_M(\mathcal{A}_{X_i} V, h\tilde{\nabla}_U X_i) + g_M(\mathcal{T}_V X_i, v\tilde{\nabla}_U X_i) + \frac{1}{2}\eta(X_i) g_M(\tilde{\nabla}_U X_i, V) - \\ - \frac{1}{2}\eta(h\tilde{\nabla}_{X_i} X_i) g_M(U,V)] - \sum_j [g_M(\mathcal{T}_U U_j \mathcal{T}_{V,} U_j) - \frac{1}{2}\eta(\mathcal{T}_{U_j} V) g_M(U,U_j)] \tag{39}$$

$$\tilde{S}(X,Y) = S'(X',Y')^\circ f - \frac{1}{2}\eta(\mathcal{A}_X Y) - \frac{1}{2}\eta(h\tilde{\nabla}_X Y) + \sum_i [3 g_M(\mathcal{A}_{X_i} X, \mathcal{A}_{X_i} Y) + \\ + \frac{1}{2}\eta(\mathcal{A}_{X_i} Y) g_M(X, X_i)] + \sum_j \{g_M[(\tilde{\nabla}_X \mathcal{T})_{U_j} Y, U_j) - g_M[(\tilde{\nabla}_{U_j} \mathcal{A})_X Y, U_j] + \\ + g_M(\mathcal{A}_Y U_j, h\tilde{\nabla}_{U_j} X) + g_M(\mathcal{T}_{U_j} Y, v\tilde{\nabla}_{U_j} X) + \frac{1}{2}\eta(Y) g_M(\tilde{\nabla}_{U_j} X, U_j)\} \tag{40}$$

$$\tilde{S}(U,X)=g_M(\tilde{\nabla}_U N,X)+\sum_i\{g_M[(\tilde{\nabla}_X\mathcal{A})(X_i,U),X_i]-g_M[(\tilde{\nabla}_{X_i}\mathcal{A})(X,U)X_i]+$$
$$+\frac{1}{2}\eta(U)g_M(\tilde{\nabla}_X X_i,X_i)-\frac{1}{2}\eta(U)g_M(\tilde{\nabla}_{X_i}X,X_i)+g_M(\mathcal{T}_U X_i,[X,X_i])\}-$$
$$-\sum_j\{g_M[(\tilde{\nabla}_{U_j}\mathcal{T})_U U_j,X]\} \tag{41}$$

where $U, V\in\chi^{v}(M)$; $X, Y\in\chi^{h}(M)$, and $\{X_i, U_j\}$ is $f$ - adaptable frame on $(M, g_M)$.

*Proof 8.* Let us first give the proof of (39). In this case, Ricci tensor of $M$ with respect to a new type of semi-symmetric non-metric connection $\tilde{\nabla}$ is defined by:

$$\tilde{S}(U,V)=\sum_i\tilde{R}(X_i,U,X_i,V)-\sum_j\tilde{R}(U_j,U,V,U_j) \tag{42}$$

From eq. (31), we have:

$$\sum_i\tilde{R}(X_i,U,X_i,V)=\sum_i\begin{cases}g_M[(\tilde{\nabla}_{X_i}\mathcal{T})_U X_i,V]+g_M(\mathcal{A}_{X_i}V,h\tilde{\nabla}_U X_i)+g_M(\mathcal{T}_V X_i,v\tilde{\nabla}_U X_i)+\\+\frac{1}{2}[\eta(X_i)g_M(\tilde{\nabla}_U X_i,V)-\eta(h\tilde{\nabla}_{X_i}X_i)g_M(U,V)]\end{cases} \tag{43}$$

On the other hand, from eq. (25), we get:

$$\sum_i g_M[\tilde{R}(U_j,U)V,U_j]=\sum_j\begin{Bmatrix}g_M[\hat{R}(U_j,U)V,U_j]-g_M(\mathcal{T}_{U_j}U_j\mathcal{T}_U V)+\\+g_M(\mathcal{T}_U U_j,\mathcal{T}_{U_j}V)-\frac{1}{2}\eta(\mathcal{T}_{U_j}V)g(U,U_j)\end{Bmatrix}+\frac{1}{2}\eta(\mathcal{T}_U V) \tag{44}$$

So, if eqs. (43) and (44) are substituted in eq. (42), the proof is complete.

Other equations can be obtained in a similar way.

*Corollary 2.* Let $f:(M, g_M)\to(N, g_N)$ be a Riemannian submersion between Riemannian manifolds $(M, g_M)$ and $(N, g_N)$. Let $\tau'$ and $\hat{\tau}$ be scalar curvatures of $N$ and any fibre $[f^{-1}(x),\hat{g}x]$ for $x\in N$, respectively. Scalar curvature of $M$ with respect to a new type of semi-symmetric non-metric connection $\tilde{\nabla}$ is given by:

$$\tilde{\tau}=\hat{\tau}+\tau'\circ f-\sum_i\eta(h\tilde{\nabla}_{X_i}U_jX_i)+\sum_{i,j}\{2g_{\mathrm{M}}[(\tilde{\nabla}_{X_i}\mathcal{T})_{U_j}X_i,U_j]+2g_{\mathrm{M}}(A_{X_i}U_j,h\nabla_{U_j}X_i)+$$
$$+2g_{\mathrm{M}}(\mathcal{T}_{U_j}X_i,v\tilde{\nabla}_{U_j}X_i)+\eta(X_i)g_M(\tilde{\nabla}_{U_j}X_i,U_j)+3g_M(\mathcal{A}_{X_i}U_j,\mathcal{A}_{X_i}U_j)\}-\eta(N)$$

where $X, Y$ (and $U, V$) are orthonormal horizontal (and vertical) vector fields and $\{X_i, U_j\}$ is *f-adap*table frame on $(M, g_M)$.

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Paper submitted: February 6, 2023
Paper revised: April 8, 2023
Paper accepted: June 10, 2023